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Characterization of Stieltjes transforms of vector measures and an application to spectral theory

By Werner RICKER

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Abstract

The classical result of D. V. Widder characterizing those complex-valued functions on $(0, \infty)$ which are the Stieltjes transform of a complex measure on $[0, \infty)$, is generalized to functions with values in a quasi-complete locally convex space. This result is then used to establish a criterion for operators with spectrum in $[0, \infty)$ to be scalar-type spectral operators.

Introduction

Let M and D respectively denote the formal operators of multiplication $M: f(t)\mapsto tf(t)$ and differentiation $D: f\to f'$. The (formal) Widder differential operators L_k are given by

$$L_k = c_k M^{k-1} D^{2k-1} M^k, \qquad k=1, 2, \cdots, \tag{1}$$

where $c_1=1$ and $c_k=(-1)^k[k!(k-2)!]^{-1}$ for $k\geq 2$.

It is known that a complex-valued function f on $(0, \infty)$ can be characterized as a Stieltjes transform in terms of the maps $L_k(f)$, $k=1, 2, \cdots$. Namely, there exists a (unique) regular complex Borel measure m on $[0, \infty)$ such that

$$f(t) = \hat{m}(t) = \int_0^\infty (s+t)^{-1} dm(s), \qquad t\in(0, \infty), \tag{2}$$

if and only if f has derivatives of all orders in $(0, \infty)$ and there exists a constant K such that

$$\int_0^\infty \left| L_k(f)(t)\right| dt \leq K, \qquad k=1, 2, \cdots, \tag{3}$$

(see [8], VIII Theorem 16 or [4], p. 165).

Let C_0 denote the space of all continuous complex-valued functions on $[0, \infty)$ which vanish at infinity, equipped with the uniform norm. Then condition (3) means that the maps $\Phi_k(f)$, $k=1, 2, \cdots$, defined by

$$\Phi_k(f)(\psi) = \int_0^\infty \psi(t)\, L_k(f)(t)\, dt\,, \qquad \psi \in C_0\,, \tag{4}$$

are equibounded linear functionals on C_0, that is, they map the closed unit ball of C_0 into a bounded set not depending on k.

In this note the above characterization of Stieltjes transforms is extended to functions f on $(0, \infty)$ with values in a quasi-complete locally convex space X. Defining $L_k(f)$ and $\Phi_k(f)$ as in (1) and (4) respectively, but with values now in X, it is shown that f is the Stieltjes transform of a vector measure on $[0, \infty)$, if and only if, f has weak derivatives of all orders in $(0, \infty)$, in the sense of [3; Definition 3. 2. 3], and the maps $\Phi_k(f)$ take the closed unit ball of C_0 into a weakly compact subset of X, not depending on k.

A problem of fundamental importance in Spectral Theory consists of finding criteria for an operator to be of scalar-type in the sense of N. Dunford [2]. In the final section of this note, the result characterizing the Stieltjes transforms of vector measures is used to establish a criterion for an operator on X with spectrum in $[0, \infty)$ to be a scalar-type spectral operator. This is an extension to locally convex spaces of a result proved by S. Kantorovitz [3] in the case where X is a reflexive Banach space.

Preliminaries

Let X be a quasi-complete locally convex Hausdorff space. The space of continuous linear functionals on X is denoted by X'. Let $\boldsymbol{C}$ denote the complex number field and $\mathscr{B}$ the σ-algebra of all Borel subsets of $[0, \infty)$.

By a vector measure $m: \mathscr{B} \to X$ is meant a function on $\mathscr{B}$ which is σ-additive. For each $x' \in X'$, the $\boldsymbol{C}$-valued measure $E \mapsto \langle m(E), x' \rangle$, $E \in \mathscr{B}$, is denoted by $\langle m, x' \rangle$. The measure m is said to be regular if, for every $x' \in X'$, the complex measure $\langle m, x' \rangle$ is regular (i. e., its variation is regular).

A complex-valued, $\mathscr{B}$-measurable function f on $[0, \infty)$ is said to be m-integrable if it is integrable with respect to every measure $\langle m, x' \rangle$, $x' \in X'$, and if, for every $E \in \mathscr{B}$, there exists an element $\int_E f dm$ of X such that

$$\left\langle \int_E f dm, x' \right\rangle = \int_E f d\langle m, x' \rangle\,,$$

for each $x' \in X'$. Bounded measurable functions are always m-integrable ([6], Lemma II 3. 1). Hence, the Stieltjes transform, $\hat{m}$, of any vector measure $m: \mathscr{B} \to X$ can be defined by (2).

Let dt denote Lebesgue measure on $(0, \infty)$. A function $F: (0, \infty) \to X$ is said to be (Pettis) integrable if for every Borel subset E of $(0, \infty)$, there exists an element $\int_E F(t)\, dt$ of X such that

$$\left\langle \int_E F(t)\,dt, x' \right\rangle = \int_E \left\langle F(t), x' \right\rangle dt\,, \qquad x' \in X'\,.$$

Let $L(X)$ denote the space of all continuous linear operators on X, equipped with the topology of pointwise convergence. The identity operator on X is denoted by I.

A map $P\colon \mathscr{B} \to L(X)$ is called a spectral measure if it is σ-additive, multiplicative and $P([0, \infty)) = I$. Of course, the multiplicativity of P means that $P(E \cap F) = P(E)\,P(F)$, for every $E \in \mathscr{B}$ and $F \in \mathscr{B}$. Since $L(X)$ is itself a locally convex space it is clear that spectral measures are vector measures.

A spectral measure $P\colon \mathscr{B} \to L(X)$ is said to be equicontinuous if its range, $\{P(E)\,;\, E \in \mathscr{B}\}$, is an equicontinuous part of $L(X)$. For such spectral measures every P-essentially bounded, measurable function is P-integrable (see § 1 of [7] for example). If the space X barrelled, then P is necessarily equicontinuous.

Let $T \in L(X)$. If $\lambda \in \boldsymbol{C}$ is such that $R(\lambda\,;\, T) = (\lambda I - T)^{-1}$ exists in $L(X)$, then $R(\lambda\,;\, T)$ is called the resolvent operator of T at λ. Define $R(\infty\,;\, T)$ to be the zero operator. If it is clear which operator T is being considered, then $R(\lambda\,;\, T)$ is denoted simply by $R(\lambda)$. The resolvent set of T, which is denoted by $\rho(T)$, consists of those points λ in the extended complex plane, $\boldsymbol{C}^*$, for which the resolvent map $R(\,\cdot\,) = R(\cdot\,;\, T)$ is defined and holomorphic in a neighbourhood of λ. The complement of $\rho(T)$ in $\boldsymbol{C}^*$ is denoted by $\sigma(T)$ and is called the spectrum of T.

The resolvent equations,

$$R(\lambda) - R(\mu) = (\mu - \lambda)\, R(\lambda)\, R(\mu) = (\mu - \lambda)\, R(\mu)\, R(\lambda)\,,$$

are valid for all points $\lambda \in \rho(T)$ and $\mu \in \rho(T)$. Also, for each $x \in X$, the X-valued function $R(\cdot\,;\, T)(x)$ has weak derivatives of all orders in $\rho(T)$.

A characterization of Stieltjes transforms of vector measures

In the following two lemmas, f is a complex-valued function with derivatives of all orders in $(0, \infty)$.

1. Lemma. *Let f satisfy* (3). *Then the limit*

$$A = \lim_{t \to 0+} t f(t)$$

exists, and

$$f(t) = \lim_{k \to \infty} \int_0^\infty (s+t)^{-1} L_k(f)(s)\, ds + A t^{-1}\,, \qquad t \in (0, \infty)\,. \qquad (5)$$

PROOF: See [8], VIII 15.

2. LEMMA. *If there exists a constant* K *such that* (3) *holds, then* $\lim_{k\to\infty} \Phi_k(f)(\psi)$ *exists, for every* $\psi \in C_0$.

PROOF: By Lemma 1, it follows that (5) holds. Therefore, if $\mathscr{A}$ denotes the linear subspace of C_0 consisting of all functions of the form

$$\psi:\ s \mapsto \sum_{i=1}^{n} \alpha_i (s+t_i)^{-1}, \qquad s \geq 0,$$

where $\alpha_i \in \boldsymbol{C}$ and $t_i > 0$ are arbitrary, then it is clear that $\lim_{k\to\infty} \Phi_k(f)(\psi)$ exists, for all $\psi \in \mathscr{A}$.

Let $\varepsilon > 0$ and $\psi \in C_0$ be given. Since $\mathscr{A}$ is dense in C_0, there exists an element φ in $\mathscr{A}$ such that $\|\varphi-\psi\|_\infty < \varepsilon/3K$. Then

$$\left|\Phi_k(f)(\psi)-\Phi_j(f)(\psi)\right| \leq \left|\Phi_k(f)(\psi-\varphi)\right| + \left|\Phi_k(f)(\varphi)-\Phi_j(f)(\varphi)\right| +$$
$$+\left|\Phi_j(f)(\varphi-\psi)\right| < \|\varphi-\psi\|_\infty K + \varepsilon/3 + \|\psi-\varphi\|_\infty K < \varepsilon,$$

for k, j sufficiently large.

Let f: $(0, \infty) \to X$ have weak derivatives of all orders. Then it is clear that the Widder differential operators (1) can be applied to f giving a family of X-valued functions

$$t \mapsto L_k(f)(t),\ t \in (0, \infty), \qquad k = 1, 2, \cdots. \tag{6}$$

The collection of linear maps (4) is said to be weakly equicompact if the subset,

$$\left\{\Phi_k(f)(\psi);\ \psi \in C_0,\ \|\psi\|_\infty \leq 1, \quad k = 1, 2, \cdots\right\}, \tag{7}$$

of X, is relatively weakly compact; (see [4]).

Let $C_{00}((0, \infty))$ denote the space of continuous functions on $(0, \infty)$ having compact support, equipped with the uniform norm. The symbol $B(\cdot, \cdot)$ denotes the Beta function.

1. THEOREM. *A function* f: $(0, \infty) \to X$ *is the Stieltjes transform of a (unique) regular,* X*-valued measure on* $\mathscr{B}$*, if and only if, it has weak derivatives of all orders, each of the functions* (6) *is integrable and the collection of maps* (4) *is weakly equicompact.*

PROOF: Let m: $\mathscr{B} \to X$ be a regular vector measure. It is a consequence of the Dominated Convergence Theorem ([6], Theorem II. 2) that $\hat{m}$ has derivatives of all orders (in the given topology of X) and that

$$D^k(\hat{m})(t) = (-1)^k k! \int_0^\infty (s+t)^{-k-1} dm(s), \qquad t \in (0, \infty), \tag{8}$$

for every $k=1, 2, \cdots$. The Leibnitz formula and (8) imply that

$$L_k(\hat{m})(t) = c'_k \int_0^\infty t^{k-1} s^k (s+t)^{-2k} dm(s), \qquad t \in (0, \infty), \tag{9}$$

for each $k=1, 2, \cdots$, where $c'_1=1$ and $c'_k=B(k-1, k+1)^{-1}$ for $k \geq 2$. It is clear from (9) that the functions $t \mapsto L_k(\hat{m})(t)$, $t \in (0, \infty)$, are continuous for each $k=1, 2, \cdots$.

Fix $k \geq 1$. Let $\psi \in C_{00}((0, \infty))$. For every $\varphi \in C_{00}((0, \infty))$ there exists $x_\varphi \in X$ such that

$$\langle x_\varphi, x' \rangle = \int_0^\infty \varphi(t)\, \psi(t) \left\langle L_k(\hat{m})(t), x' \right\rangle dt, \qquad x' \in X',$$

(see [1] III Proposition 3. 2); moreover if $||\varphi||_\infty \leq 1$, then x_φ belongs to the closed convex hull of the range of $\psi L_k(\hat{m})$ which is a compact set. Accordingly $\psi L_k(\hat{m})$ is integrable ([5], Lemma 3).

It follows from (9) and the Fubini theorem that

$$\int_0^\infty \psi(t)\, L_k(\hat{m})(t)\, dt = \int_0^\infty \left(c'_k \int_0^\infty \psi(t)\, t^{k-1} s^k (s+t)^{-2k} dt \right) dm(s). \tag{10}$$

Since

$$\left| \int_0^\infty \psi(t)\, t^{k-1} s^k (s+t)^{-2k} dt \right| \leq ||\psi||_\infty \int_0^\infty s^k t^{k-1} (s+t)^{-2k} dt = ||\psi||_\infty B(k, k),$$

for each $s \geq 0$, and $c'_k B(k, k) \leq 1$ for $k \geq 2$, it follows from (10) that

$$\int_0^\infty \psi(t)\, L_k(\hat{m})(t)\, dt \in ||\psi||_\infty \overline{coR}(m), \qquad \psi \in C_{00}((0, \infty)),$$

where $\overline{coR}(m)$ denotes the closed balanced convex hull of the range of m. Hence, the set

$$\left\{ \int_0^\infty \psi(t)\, L_k(\hat{m})(t)\, dt\, ;\ \psi \in C_{00}((0, \infty)), ||\psi||_\infty \leq 1 \right\},$$

being a subset of $\overline{coR}(m)$, is relatively weakly compact ([5], Lemma 1). Lemma 3 of [5] implies that $L_k(\hat{m})$ is integrable,

If $\psi \in C_0$, then the restriction of $\psi L_k(\hat{m})$ to $(0, \infty)$ is continuous. A similar argument as that used for $L_k(\hat{m})$, applied to the function $\psi L_k(\hat{m})$, $k=1, 2, \cdots$, shows that $\psi L_k(\hat{m})$ is integrable and that (7) is contained in $\overline{coR}(m)$. Hence, the maps (4) are weakly equicompact.

Conversely, suppose that f: $(0, \infty) \to X$ is a function having weak derivatives of all orders such that the functions (6) are integrable and such that the maps (4) are weakly equicompact.

If $x' \in X'$, define the function $g_{x'} : (0, \infty) \to \boldsymbol{C}$ by

$$g_{x'}(t) = \langle f(t), x' \rangle, \qquad t \in (0, \infty).$$

Then $g_{x'}$ has derivatives of all orders and

$$L_k(g_{x'})(t) = \langle L_k(f)(t), x' \rangle, \ t \in (0, \infty), \qquad k = 1, 2, \cdots. \tag{11}$$

Fix $x' \in X'$. Since the set (7) is weakly bounded there is a constant $K_{x'}$ such that

$$\left| \langle \Phi_k(f)(\psi), x' \rangle \right| \leq K_{x'}, \ \psi \in C_0, \ \|\psi\|_\infty \leq 1, \qquad k = 1, 2, \cdots.$$

Accordingly for each $k = 1, 2, \cdots$, it follows that

$$\begin{aligned} \int_0^\infty \left| L_k(g_{x'})(t) \right| dt &\leq \sup \left\{ \left| \int_0^\infty \psi(t) L_k(g_{x'})(t)\, dt \right| ; \ \psi \in C_0, \ \|\psi\|_\infty \leq 1 \right\} \\ &= \sup \left\{ \left| \langle \Phi_k(f)(\psi), x' \rangle \right| ; \ \psi \in C_0, \ \|\psi\|_\infty \leq 1 \right\} \leq K_{x'}. \end{aligned}$$

Hence, Lemma 2 implies the existence of

$$\lim_{k \to \infty} \Phi_k(g_{x'})(\psi) = \lim_{k \to \infty} \langle \Phi_k(f)(\psi), x' \rangle, \qquad \psi \in C_0.$$

Since $x' \in X'$ was arbitrary, it follows that for fixed $\psi \in C_0$ the sequence $(\Phi_k(f)(\psi))_{k=1}^\infty$ is weakly Cauchy. The relative weak compactness of (7) implies that this sequence is weakly convergent. Thus, for each $\psi \in C_0$, there is a unique $\Phi(f)(\psi) \in X$ such that the weak limit,

$$\Phi(f)(\psi) = \lim_{k \to \infty} \Phi_k(f)(\psi), \tag{12}$$

exists. This defines a weakly compact linear map $\Phi(f) : C_0 \to X$. Accordingly, there exists a regular measure $m : \mathscr{B} \to X$ such that

$$\Phi(f)(\psi) = \int_0^\infty \psi(t)\, dm(t), \qquad \psi \in C_0; \tag{13}$$

(see [5], Proposition 1).

Fix $\alpha > 0$ and $x' \in X'$. Let $t \in (0, \alpha)$. It follows from (11) and the definition of $L_1(g_{x'})$ that

$$\left\langle \int_t^\alpha L_1(f)(s)\, ds, \ x' \right\rangle = \alpha g_{x'}(\alpha) - t g_{x'}(t).$$

This identity and Lemma 1 imply that $\lim_{t \to 0+} \langle \int_t^\alpha L_1(f)(s) ds, x' \rangle$ exists, for each $x' \in X'$. Since $\langle \int_E L_1(f)(s)\,ds\,;\ E \subseteq (0, \infty),\ E\ \mathit{Borel}\}$ is contained in the closure of the relatively weakly compact set (7), it follows that the weak limit,

$$A = \lim_{t \to 0+} tf(t)\,, \tag{14}$$

exists.

For each $t>0$ the function $s \mapsto (s+t)^{-1}$, $s \geq 0$, belongs to C_0. It follows from (12) and (13) that

$$\lim_{k \to \infty} \int_0^\infty (s+t)^{-1} L_k(f)(s)\,ds = \int_0^\infty (s+t)^{-1} dm(s)\,, \qquad t>0\,, \tag{15}$$

weakly in X. If $A_{x'} = \lim_{t \to 0+} t g_{x'}(t)$ for each $x' \in X'$ (cf. Lemma 1), then it is clear from (14) that $A_{x'} = \langle A, x' \rangle$, $x' \in X'$. Since Lemma 1 implies that

$$\lim_{k \to \infty} \int_0^\infty (s+t)^{-1} L_k(g_{x'})(s)\,ds = g_{x'}(t) - A_{x'} t^{-1} = \left\langle f(t) - At^{-1},\ x' \right\rangle,$$

for each $x' \in X'$, it follows from (11) and (15) that

$$f(t) - At^{-1} = \hat{m}(t)\,, \qquad t \in (0, \infty)\,.$$

Replacing m throughout by $m - m_0$, where $m_0 : \mathscr{B} \to X$ is the measure taking the value A on sets containing $\{0\}$ and zero elsewhere, we obtain (2).

Scalar-type operators with spectrum in $[0, \infty)$

It is assumed throughout this section, that in addition to being quasi-complete, the space X is barrelled.

An operator $T \in L(X)$ with spectrum in $[0, \infty)$ is said to be a scalartype spectral operator if there exists a regular spectral measure $P : \mathscr{B} \to L(X)$ such that

$$T = \int_0^\infty s\,dP(s)\,. \tag{16}$$

Let T be a continuous linear operator on X with spectrum in $[0, \infty)$. Then the resolvent operator $R(t) = (t+T)^{-1}$ of $(-T)$ is defined for each $t \in (0, \infty)$.

Let $\mathscr{M}$ denote the linear space of all complex Borel measures on $(0, \infty)$ which have finite support. If $t>0$, then ε_t will denote the Dirac point mass at t. Let $\mu = \sum_{i=1}^n \alpha_i \varepsilon_{t_i}$, $\alpha_i \in \boldsymbol{C}$, $t_i > 0$, be a member of $\mathscr{M}$. Then $\hat{\mu} : [0, \infty) \to \boldsymbol{C}$ denotes the function

$$\hat{\mu}(s) = \sum_{i=1}^n \alpha_i (s+t_i)^{-1}\,, \qquad s \geq 0\,. \tag{17}$$

The symbol $\hat{\mathscr{M}}$ denotes the subspace $\{\hat{\mu}; \mu \in \mathscr{M}\}$ of C_0.

2. THEOREM. *Let $T \in L(X)$ have spectrum in $[0, \infty)$. The operator T is a scalar-type spectral operator, if and only if, for each $x \in X$,*

(i) *the functions $t \mapsto L_k(R(\cdot)(x))(t)$, $t \in (0, \infty)$, $k=1, 2, \cdots$, are integrable and,*

(ii) *the maps $\Phi_k(R(\cdot)(x)) : C_0 \to X$, $k=1, 2, \cdots$, given by*

$$\Phi_k\big(R(\cdot)(x)\big)(\psi) = \int_0^\infty \psi(t)\, L_k\big(R(\cdot)(x)\big)(t)\, dt\,, \qquad \psi \in C_0\,, \tag{18}$$

are weakly equicompact.

PROOF: Suppose that there exists a regular spectral measure $P: \mathscr{B} \to L(X)$ such that (16) holds. Since for each $t>0$, the function $s \mapsto (s+t)^{-1}$, $s \geq 0$, is bounded and measurable, it follows from the functional calculus for P that

$$R(t) = (t+T)^{-1} = \int_0^\infty (s+t)^{-1} dP(s)\,, \qquad t \in (0, \infty)\,.$$

Hence, if $x \in X$, then the identity

$$R(t)(x) = \int_0^\infty (t+s)^{-1} dP(s)(x)\,, \qquad t \in (0, \infty)\,,$$

shows that $R(\cdot)(x)$ is the Steltjes transform of the vector measure $P(\cdot)(x)$. Theorem 1 implies that (i) and (ii) hold.

Conversely, suppose that for each $x \in X$ the conditions (i) and (ii) are satisfied. It follows from Theorem 1 that for each $x \in X$, there exists a (unique) regular Borel measure $m_x: \mathscr{B} \to X$ such that

$$R(t)(x) = \int_0^\infty (s+t)^{-1} dm_x(s)\,, \qquad t \in (0, \infty)\,. \tag{19}$$

For $E \in \mathscr{B}$, define a linear operator $P(E): X \to X$ by

$$P(E)(x) = m_x(E)\,, \qquad E \in \mathscr{B}\,.$$

Firstly it is shown that $P(E)$ is continuous. For $\hat{\mu} \in \hat{\mathscr{M}}$ define a linear operator $T_{\hat{\mu}}: X \to X$ by

$$T_{\hat{\mu}}(x) = \int_0^\infty \hat{\mu}(s)\, dm_x(s)\,, \qquad x \in X\,. \tag{20}$$

If $\mu = \sum_{i=1}^n \alpha_i \varepsilon_{t_i}$, $\alpha_i \in \boldsymbol{C}$, $t_i > 0$, then it follows from (17) and (19) that

$$T_{\hat{\mu}}(x) = \sum_{i=1}^n \alpha_i R(t_i)(x)\,, \qquad x \in X\,.$$

Hence, it is clear that $T_{\hat{\mu}} \in L(X)$ for each $\hat{\mu} \in \hat{\mathscr{M}}$.

Let $\mathscr{A} = \{T_{\hat{\mu}}\,;\ \mu \in \mathscr{M},\ \|\hat{\mu}\|_\infty \leq 1\}$. If $x \in X$, then it follows from (20) that $T_{\hat{\mu}}(x) \in \overline{coR}(m_x)$, for each $\mu \in \mathscr{M}$ with $\|\hat{\mu}\|_\infty \leq 1$. Since $\overline{coR}(m_x)$ is bounded for each $x \in X$ and X is barrelled, it follows that $\mathscr{A}$ is equicontinuous.

Hence, given a continuous semi-norm p on X, there exist continuous semi-norms $q_1, \cdots, q_l$ and $\alpha > 0$ such that

$$p\Big(\int_0^\infty \hat{\mu}(s)\, dm_x(s)\Big) = p\Big(T_{\hat{\mu}}(x)\Big) \leq \alpha \|\hat{\mu}\|_\infty \max_{1 \leq i \leq l} q_i(x)\,,$$

for each $x \in X$ and $\mu \in \mathscr{M}$. Since $\hat{\mathscr{M}}$ is dense in C_0, it follows that

$$p\Big(P(E)\,(x)\Big) = p\Big(m_x(E)\Big) \leq \alpha \max_{1 \leq i \leq l} q_i(x),\quad x \in X,\ E \in \mathscr{B}\,,$$

from which the continuity of $P(E)$, $E \in \mathscr{B}$, is clear. Hence, $P\colon \mathscr{B} \to L(X)$ is a σ-additive, operator-valued measure.

Since X is barrelled, the inclusion

$$tR(t)\,(x) = \int_0^\infty t(s+t)^{-1} dm_x(s) \in \overline{coR}(m_x)\,, \qquad t \in (0, \infty)\,,$$

for each $x \in X$, shows that $\{tR(t)\,;\ t > 0\}$ is an equicontinuous part of $L(X)$.

Let $t > u > 0$. The resolvent equations imply that

$$tR(t)\,R(u) = t(t-u)^{-1}R(u) - \Big(tR(t)\Big)\,(t-u)^{-1}\,.$$

Since $\{tR(t)\,;\ t > 0\}$ is equicontinuous, fixing u and letting $t \to \infty$ it follows that $tR(t)\,R(u) \to R(u)$, in $L(X)$, as $t \to \infty$. Fix $x \in X$. Since $\lim_{t\to\infty} t(t+s)^{-1} = 1$ for each $s \geq 0$, it follows from the Dominated Convergence Theorem that

$$R(u)\,(x) = \lim_{t\to\infty} tR(t)\,R(u)\,(x) = \lim_{t\to\infty} \int_0^\infty t(t+s)^{-1} dm_{R(u)(x)}(s) =$$

$$= \int_0^\infty 1 dm_{R(u)(x)}(s) = m_{R(u)(x)}\Big([0, \infty)\Big)\,.$$

That is, $R(u)\,(x) = P([0, \infty))\,R(u)\,(x)$, $u > 0$, $x \in X$. Applying $(u+T)$ to both sides of this identity (on the right), we conclude that $P([0, \infty)) = I$. The multiplicitivity of P can be shown as in [4], pp. 169–170. Hence, P is a (regular) spectral measure.

It remains to verify that T is given by (16). It follows easily from $R(t) = \int_0^\infty (t+s)^{-1} dP(s)$, $t \in (0, \infty)$, that

$$\int_0^\infty s(s+t)^{-1} dP(s) = TR(t)\,, \qquad t \in (0, \infty)\,. \tag{21}$$

Furthermore for all $n>0$, $y\in X$ and $t>0$, it can be shown as in [4], pp. 170, that

$$\int_0^\infty s\chi_{[0,n]}(s)\, dP(s)\left(R(t)(y)\right)=\int_0^\infty s(t+s)^{-1}\chi_{[0,n]}(s)\, dP(s)(y)\,. \qquad (22)$$

It follows from (21), (22) and the Dominated Convergence Theorem that

$$\lim_{n\to\infty}\int_0^\infty s\chi_{[0,n]}(s)\, dP(s)\left(R(t)(y)\right)=TR(t)(y),\ t>0,\ y\in X\,.$$

If $x\in X$, then $y=(t+T)(x)$ satisfies $R(t)(y)=x$. Accordingly,

$$\lim_{n\to\infty}\int_0^\infty s\chi_{[0,n]}(s)\, dP(s)(x)=T(x)\,,$$

for every $x\in X$.

Since $[0,n]\uparrow[0,\infty)$ and $f_n(s)=s\chi_{[0,n]}(s)$, $s\geq 0$, is P-essentially bounded for each $n=1,2,\cdots$, with the limit

$$\lim_{n\to\infty}\int_0^\infty f_n(s)\, dP(s)=T$$

existing in $L(X)$, it follows from the multiplicativity and equicontinuity of P that the identity function on $[0,\infty)$ is P-integrable and that (16) is valid. This completes the proof of the theorem.

Let X be a reflexive Banach space. A subset of X is relatively weakly compact if and only if it is bounded. Define a map S: $(0,\infty)\to L(X)$ by

$$S(t)=TR(t)\left(I-R(t)\right),\qquad t\in(0,\infty)\,;$$

(see [4]). For any weakly measurable function F: $(0,\infty)\to L(X)$ define

$$|||F|||=\sup\left\{\left\|\langle F(\cdot)(x),\ x'\rangle\right\|_1;\ x\in X,\ x'\in X',\ \|x\|\leq 1,\ \|x'\|\leq 1\right\},$$

where $\|\cdot\|_1$ denotes the $L^1((0,\infty),dt/t)$-norm; (see [4]).

It follows from the Uniform Boundedness Principle that the weak equicompactness of the maps (18), for each $x\in X$, is equivalent to the existence of a constant $\alpha>0$ such that

$$|||S^k|||\leq\alpha B(k,k)\,,\qquad k=1,2,\cdots \qquad (23)$$

Furthermore, if (23) holds, then it is a consequence of the reflexivity of X that the functions (i) in the statement of Theorem 2 are integrable for each $x\in X$.

Hence, for X a reflexive Banach space, the conditions of Theorem 2 are equivalent to the existence of $\alpha>0$ such that (23) is satisfied. This result was proved by S. Kantorovitz [4].

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School of Mathematical Sciences
Flinders University of South Australia
Bedford Park, 5042, Australia