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Corestriction and p-subgroups

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1. Introduction

Let G be a finite group, p a prime number. In his study on the modular representation, J. A. Green defined G-algebras and defect groups for G-algebras [6]. After his ideas, M. Broué and L. Puig defined the Brauer homomorphism for G-algebras and obtained the "First Fundamental Theorem" for G-algebras. The classical "First Fundamental Theorem" for blocks and the Green correspondence was shown as corollaries of this theorem. They also gave the definitions of interior G-algebras and the "corestriction" of interior G-algebras as ring theoretical version of the induction of modules and showed the extention of Higman's criterion for relative projectivity and Green's theorem on the induction of absolutely indecomposable modules. See [1], [2] and [6].

On the other hand, in his paper [3], D. W. Burry showed the relation between blocks and induced modules from p-subgroups. Our purpose in this paper is to extend the Burry's result to interior G-algebras. We shall prove Theorem 1, 2, and Corollary 3.

THEOREM 1. *Let P be a p-subgroup of G, (B, σ) a local interior P-algebra with with defect group P and $(Cor_p{}^G B, \sigma^G)$ the corestriction of (B, σ). Let Br_P be the Brauer homomorphism of $(Cor_P{}^G B, \sigma^G)$ with respect to P. If b is a block of RG whose defect group contains P up to G-conjugacy, then the element $Br_P \circ \sigma^G (b)$ is non-zero.*

THEOREM 2. *Let b be a block of RG and P a subgroup of a defect group of b. Then for a local interior P-algebra (B, σ) with defect group P there exists a local interior G-algebra (A, ρ) satisfying the following conditions:*

(1) *The element $\rho(b)$ is non-zero.*

(2) *A defect group of (A, ρ) equals P up to G-conjugacy.*

(3) *The interior P-algebra (B, σ) is a source of the interior G-algebra (A, ρ).*

COROLLARY 3. (*Burry*) *Let b be a block of RG and P a subgroup* *of a defect group of b. Then for an indecomposable RP-module W with vertex P there exists an indecomposable RG-module V lying in b satisfying the following conditions*:

(1) *A vertex V equals P up to G-conjugacy.*

(2) *The RP-module W is a source of V.*

In section 2, we shall present the definitions of interior G-algebras and the "corestriction" of interior G-algebras according to Broué's note [2]. Also other definitions on interior G-algebras will be stated and several facts will be proven in section 2. In section 3, we shall prove Theorem 1, 2, and some corollaries.

Notation. Maps are usually on the left with the corresponding convension for writing composition. Let A and B be sets and f a map of A to B. For a subset C of A we denote by $f|_C$ the restriction of f to C. If H and K are groups, then the notation $H \leq K$ means that H is a subgroup of K and the notation $H < K$ means that H is a proper subgroup of K. Let H and K be subgroups of G. The notation $H \leq_G K$ means that H is contained in K up to G-conjugacy. We denote by (G/H) a set of representatives of the left coset gH in G, containing the identity element. We denote by $(H \backslash G/K)$ a set of representatives of the (H, K)-double coset HgK in G, containing the identity element.

Let o denote the complete valuation ring with a unique maxmal ideal (π), and $k = o/(\pi)$ the residue class field of characteristic p. In this paper let R equal o or k. Let RG be the group algebra of G over R and $Z(RG)$ the center of RG. We define a block of RG by a primitive idempotent of $Z(RG)$. All RG-modules considered in this paper are finitely generated R-free left RG-modules. All R-algebras assume to have the identity element 1 and to be R-free modules of finite ranks.

2. Corestriction of interior algebras

First we note some properties of R-algebras. Let A be an R-algebra. Then by Krull-Schmidt theorem, the identity element 1 has at least a primitive decomposition in A, namely $1 = e_1 + \cdots + e_m$ where $e_1, \cdots, e_m$ are mutualy orthogonal primitive idempotent of A. If $1 = f_1 + \cdots + f_n$ is another primitive decomposition of 1, then $m = n$ and there exists an invertible element a of A such that $a^{-1} e_1 a = f_1, \cdots, a^{-1} e_m a = f_m$ after a suitable rearrangement. Moreover every idempotent of A can be expressed as the sum of a set of mutualy orthogonal primitive idempotents of A.

Following [6], a G-algebra over R is an R-algebra on which G acts as a group of R-algebra automorphisms. Let A be a G-algebra and H a subgroup of G. We denote by A^H the subalgebra consisting of fixed points of A under the action of H. Let K be a subgroup of G containing H. The trace map Tr_H^K of A^H to A^K is defined by

$$\mathrm{Tr}_H^K(a)=\sum_{u\in(K/H)}u(a)$$

for $a\in A^H$. We set $A_H^K=\mathrm{Tr}_H^K(A^H)$.

DEFINITION 2.1. *A G-algebra A is called a local G-algebra if A^G is a local ring.*

LEMMA 2.2 (Green) *Let A be a G-algebra and let H and K be subgroups of G. Then the following assertions hold:*

(1) *If $H\leq K$, then $\mathrm{Tr}_H^G(a)=\mathrm{Tr}_K^G\mathrm{Tr}_H^K(a)$ for $a\in A^H$.*

(2) $\mathrm{Tr}_H^G(a)=\sum_{u\in(K\backslash G/H)}\mathrm{Tr}_{uHu^{-1}\cap K}^K(u(a))$ *for $a\in A^H$.*

(3) $\mathrm{Tr}_H^G(a)\,\mathrm{Tr}_K^G(a')=\sum_{u\in(K\backslash G/H)}\mathrm{Tr}_{uHu^{-1}\cap K}^G(u(a)\,a')$ *for $a\in A^H$ and $a'\in A^K$.*

(4) *A_H^G is a two-sided ideal of A^G.*

DEFINITION 2.3 *Let P be a p-subgroup of G. We denote by $I^P(A)$ the ideal of A^P defined by*

$$I^P(A)=\sum_{Q<P}A_Q^P+\pi A^P\,.$$

We set $A(P)=A^P/I^P(A)$. The Brauer homomorphism $\mathrm{Br}_P(A)$ of G-algebra A with respect to P is the canonical homomorphism of A^P onto $A(P)$.

By Lemma 2.2 and Definition 2.3 it is easy to see the following lemma, which is a characterization of defect groups.

LEMMA 2.4 (Green, Broué-Puig) *If e is a primitive idempotent of A^G, then there exists a unique p-subgroup D of G up to G-conjugacy satisfying the following conditions:*

(1) $e\in A_D^G$.

(2) *If $H\leq G$ and $e\in A_H^G$, then $D\leq_G H$.*

(3) $\mathrm{Br}_D(A)(e)$ *is non-zero.*

(4) *If $\mathrm{Br}_P(A)(e)$ is non-zero for a p-subgroup for P, then $P\leq_G D$.*

Any such p-subgroup D is called a defect group of a primitive idempotent e of A^G in the G-algebra A. If A is a local G-algebra, then a defect group of A is a defect group of a unique primitive idempotent 1 of A^G in the G-algebra A.

Now we shall describe interior G-algebras and the "corestriction" of interior G-algebras.

Definition 2.5 *An interior G-algebra* (A, ρ) *over* R *is an* R*-algebra* A *with an* R*-algebra homomorphism of the group algebra* RG *to* A.

Let (A, ρ) be an interior G-algebra. Define the G-action on A by $g(a)=\rho(g)a\rho(g^{-1})$ where $a\in A$ and $g\in G$. Then the interior G-algebra (A, ρ) is a G-algebra.

Definition 2.6 *Let* (A, ρ) *and* (A', ρ') *be interior G-algebras. Then a morphism* $\phi: (A, \rho)\rightarrow(A', \rho')$ *is a R-algebra homomorphism* ϕ *of* A *to* A' *satisfying* $\phi\circ\rho=\rho'$. *A morphism* ϕ *of* (A, ρ) *to* (A', ρ') *is an isomorphism if the homomorphism* ϕ *of* A *to* A' *is an isomorphism. We say* (A, ρ) *is isomorphic to* (A', ρ') *if there exists an isomorphism of* (A, ρ) *to* (A', ρ').

Definition 2.7 *Let* (A, ρ) *and* (A', ρ') *be interior G-algebras and* e' *an idempotent of* A'^G. *Define an R-algebra homomorphism* $\rho'_{e'}$ *of* RG *to* $e'A'e'$ *by* $\rho'_{e'}(x)=e'\rho'(x)$ *for* $x\in RG$. *Then we have an interior G-algebra* $(e'A'e', \rho'_{e'})$. *A direct embedding of* (A, ρ) *into* (A', ρ') *is an isomorphism of* (A, ρ) *to* $(e'A'e', \rho'_{e'})$ *for an idempotent* e' *of* A'^G. *We call* (A, ρ) *is directly embedded into* (A', ρ') *if there exists a direct embedding of* (A, ρ) *into* (A', ρ').

An interior G-algebra (A, ρ) is naturally an interior H-algebra for a subgroup H of G. This interior H-algebra is called the restriction of (A, ρ) and denoted by $(\mathrm{Res}^G{}_H A, \rho_H)$.

Definition 2.8 *Let* H *be a subgroup of* G *and* (B, σ) *an interior H-algebra. Then* B *is naturally considered to be* (RH, RH)*-bimodule. So we can define an R-module*

$$RG\otimes_{RH}B\otimes_{RH}RG\,.$$

This R-module is denote by $\mathrm{Cor}_H{}^G B$. *Define the product in* $\mathrm{Cor}_H{}^G B$ *and the homomorphism* σ_G *of* RG *to* $\mathrm{Cor}_H{}^G B$ *as the followings*:

$$(\textit{product})\quad (g\otimes z\otimes h^{-1})(g'\otimes z'\otimes h'^{-1}) = \begin{cases} 0 & \text{if } hH\neq g'H \\ g\otimes z\sigma(h^{-1}g')z'\otimes h'^{-1} & \text{if } hH=g'H \end{cases}$$

$$(\textit{homomorphism})\quad \sigma^G(x)=\sum_{u\in(G/H)} xu\otimes l\otimes u^{-1}$$

where g, h, g', *and* $h'\in G$, z *and* $z'\in B$, *and* $x\in RG$. *Then we have an interior G-algebra* $(\mathrm{Cor}_H{}^G B, \sigma^G)$. *This interior G-algebra is called the corestriction of the interior H-algebra* (B, σ).

Example 2.9 Let V be an RG-module and $\mathrm{End}_R(V)$ an R-endomorphism ring of V. Let ρ be the representation of RG to $\mathrm{End}_R(V)$ afforded by the RG-module V. Then we have an interior$G$-algebra $(\mathrm{End}_R(V), \rho)$. This interior G-algebra is called the interior G-algebra induced by an RG-module V. Let $(\mathrm{End}_R(V'), \rho')$ be the interior G-algebra induced by an RG-module V'. Then V is isomorphic to a direct summand of V' if and only if $(\mathrm{End}_R(V), \rho)$ is directly embedded into $(\mathrm{End}_R(V'), \rho')$. Moreover the RG-module V is indecomposable if and only if $(\mathrm{End}_R(V), \rho)$ is a local interior G-algebra. Then a vertex of V is equals to a defect group of $(\mathrm{End}_R(V), \rho)$.

Let H be a subgroup of G and U an RH-module, $(\mathrm{End}_R(U), \sigma)$ the interior H-algebra induced by U. Then the interior G-algebra $(\mathrm{Cor}_H{}^G\mathrm{End}_R(U), \sigma^G)$ is isomorphic to the interior G-algebra induced by the RG-module U^G. See [2].

Let (B, σ) be an interior H-algebra for a subgroup H of G. In this section let $e_{gg'}$ equal the element $g\otimes l\otimes g'^{-1}$ in the R-algebra $\mathrm{Cor}_H{}^G B$ where g and $g' \in G$. The following lemmas are due to Broué.

LEMMA 2.10 *If g, g', h, and h' are elements of G, then the followings hold*:

(1) $g e_{11} g'^{-1} = e_{gg'}$.

(2) $e_{gg'} e_{hh'} = \delta_{g'h} e_{gh'}$

where δ is the Kronecker's delta. In particular the R-algebra $\mathrm{Cor}_H{}^G B$ is to the matrix ring of degree $|G:H|$ over the R-algebra B with the natural basis $\{e_{gg'} : g, g' \in (G/H)\}$.

LEMMA 2.11 (Higman) *If (A, ρ) is an interior G-algebra and H is a subgroup of G, then the following conditions are equivalent*:

(1) $A_H{}^G = A^G$.

(2) *(A, ρ) is directly embedded into $(\mathrm{Cor}_H{}^G(\mathrm{Res}^G{}_H A), (\rho_H)^G)$.*

(3) *There exists an interior H-algebra (B, σ) such that (A, ρ) is directly embedded into $(\mathrm{Cor}_H{}^G B, \sigma^G)$.*

Let μ be an automorphism of G, H a subgroup of G, and (B, σ) an interior H-algebra. Then we have the interior $\mu^{-1}(H)$-algebra (B^μ, σ^μ) whose underlying R-algebra B^μ equals B and the homomorphism σ^μ is defined by $\sigma^\mu(x) = \sigma(\mu^{-1}(x))$ for $x \in RH$. If the automorphism μ is the interior automorphism by the element g of G, namely $\mu: h \mapsto ghg^{-1}$ for $h \in G$, then we denote by (B^g, σ^g) the interior $g^{-1}Hg$-algebra (B^μ, σ^μ). The following lemma is easily verified by Lemma 2.1, Lemma 2.8, and [4] p. 112 Lemma 4.2.

LEMMA 2.12 *Let (A, ρ) be a local interior G-algebra. Assume that the followings*:

(1) *A defect group of (A, ρ) equals P up to G-conjugacy.*

(2) *(A, ρ) is directly embedded into $(\mathrm{Cor}_P{}^G B, \rho^G)$ for an interior P-algebra (B, σ).*

(3) *H is a subgroup of G satisfying $A_H{}^G = A^G$.*

Then there exists a local interior H-algebra (C, τ) such that (C, τ) is directly embedded into $(\mathrm{Res}^G{}_H A, \rho_H)$ and (A, ρ) is directly embedded into $(\mathrm{Cor}_H{}^G C, \tau^G)$. Moreover for any such interior H-algebra (C, τ) there exists an element g of G such that $gPg^{-1} \leq H$ and (C, τ) is directly embedded into $(\mathrm{Cor}_{gPg^{-1}}{}^H(B^g), (\sigma^g)^H)$.

Let (A, ρ) be a local interior G-algebra with defect group P. By Lemma 2.12 we have an interior P-algebra (B, σ) satisfying (B, σ) is directly embedded into $(\mathrm{Res}^G{}_P A, \rho_P)$ and (A, ρ) is directly embedded into $(\mathrm{Cor}_P{}^G B, \sigma^G)$. This interior P-algebra is called a source of (A, ρ). If (B, σ) is a source of (A, ρ), then the interior P-algebra (B^g, σ^g) for $g \in N_G(P)$ is also a source of (A, ρ). On the contrary, if (B', ρ') is another source of (A, ρ), then there exists an element $g \in N_G(P)$ such that (B^g, σ^g) is isomorphic to (B', σ').

LEMMA 2.13 *Let P be a normal p-subgroup of G and (R, σ_0) the interior P-algebra induced by the trivial RP-module R_P. If $(\mathrm{Cor}_P{}^G R, (\sigma_0)^G)$ is the corestriction of (R, σ_0) and b is a block of RG, then $(\sigma_0)^G(b)$ is non-zero.*

PROOF. By Example 2.9 $(\mathrm{Cor}_P{}^G R, (\sigma_0)^G)$ is isomorphic to the interior G-algebra induced by the induced module $R_P{}^G$. Let ν be the R-homomorphism of RG onto RG/P induced by the natural homomorphism of G onto G/P. Since P is normal there exists an R-algebra isomorphism ϕ of $(\mathrm{Cor}_P{}^G R)^G$ to RG/P satisfying $\phi \circ \sigma_0{}^G(x) = \nu(x)$ for $x \in Z(RG)$. Therefore the kernel of the homomorphism $\nu|_{Z(RG)}$ is contained in the Jacobson radical of $Z(RG)$. See [8] p. 144. This implies that $(\sigma_0)^G(b)$ is non-zero.

LEMMA 2.14 *Let H be a subgroup of G and (B, σ) and (B', σ') interior H-algebras. If the kernel of σ is contained in the kernel of σ', the kernel of of σ^G is contained in the kernel of σ'^G. If furthermore σ is injective, then so is σ^G.*

PROOF. Any element g of G can be written uniquely as

$$g = \bar{g}\underline{g} \quad \text{where } \bar{g} \in (G/H) \quad \text{and} \quad \underline{g} \in H.$$

Let $x = \sum_{g \in G} a_g g$ $(a_g \in R)$ be an element of the kernel of σ^G. Then we have

$$\sigma^G(x) = 0.$$

By the definition of the homomorphism σ^G, we have the followings:

$$\sigma^G(x)=\sum_{u\in(G/H)}\sum_{v\in(G/H)}\Big(\sum_{h\in H}a_{uvhu^{-1}}\sigma(\underline{uv}\,h)\Big)e_{\underline{uv}\,u}$$
$$=\sum_{u\in(G/H)}\sum_{v\in(G/H)}\Big(\sigma\big(\sum_{h\in H}a_{uvhu^{-1}}\underline{uv}\,h\big)\Big)e_{\underline{uv}\,u}$$
$$=0\,.$$

Since the element $e_{\underline{uv}\,u}$ $(u, v\in(G/H))$ are distinct, Lemma 2.10 implies

$$\sigma\big(\sum_{h\in H}a_{uvhu^{-1}}\underline{uv}\,h\big)=0$$

for all u and $v\in(G/H)$. Because the kernel of σ is contained in the kernel of σ', we obtain

$$\sigma'\big(\sum_{h\in H}a_{uvhu^{-1}}\underline{uv}\,h\big)=0$$

for all u and $v\in(G/H)$. Therefore we have

$$\sigma'^g(x)=\sigma'^G\big(\sum_{g\in G}a_g\,g\big)=0$$

by the same calculation of σ'^G. Thus x is contained in the kernel of σ'^G, we have proved the first statement. The second statement of Lemma follows from the first statement.

3. Corestriction and blocks

In this section, we shall prove Theorem 1, 2, and some corollaries. We assume that P is a p-subgroup of G, (B, σ) is a local interior P-algebra with defect group P, and (A, ρ) equals the interior G-algebra $(\mathrm{Cor}_P{}^G B, \sigma^G)$.

LEMMA 3.1 *If P is a normal p-subgroup of G and e is a primitive idempotent of A^G, then a defect group of e equals P. In particular we have* $\mathrm{Br}_P(e)$ *is non-zero where* Br_P *is the Brauer homomorphism of (A, ρ) with respect to P.*

PROOF. Let D be a defect group of e. By Lemma 2.4 and Lemma 2.12 we obtain $D\leq P$. Suppose that $D<P$. Then for the interior G-algebra (eAe, ρ_e) we have $(eAe)_D{}^G=(eAe)^G$. Therefore there exists a local interior D-algebra (B', σ') with defect group D such that (eAe, ρ_e) is directly embedded into $(\mathrm{Cor}_D{}^G B', \sigma'^G)$, and so $(\mathrm{Res}^G{}_P eAe, (\rho_e)_P)$ is directly embedded into $(\mathrm{Res}^G{}_P$ $Cor_D{}^G B', (\sigma'^G)_P)$. Let e'_{11} equal the element $1\otimes1\otimes1$ in $\mathrm{Cor}_D{}^G B'$. By Lemma 2.2 and Lemma 2.11 we have

$$\mathrm{Tr}_D{}^G(e'_{11})=\sum_{u\in P\backslash G/D}\mathrm{Tr}_{uDu^{-1}\cap P}{}^P\big(g(e'_{11})\big)$$
$$=1\,.$$

This implies a defect group of the primitive idempotent e'_{11} of $(\mathrm{Cor}_D{}^G B')^P$ in $(\mathrm{Res}^G{}_P \mathrm{Cor}_D{}^G B', (\sigma'^G)_P)$ is contained in D up to G-conjugacy. Therefore a defect group of a primitive idempotent of $(eAe)^P$ in $(\mathrm{Res}^G{}_P eAe, (\rho_e)_P)$ is contained in D up to G-conjugacy.

On the other hand, since (eAe, ρ_e) is directly embedded into (A, ρ), $(\mathrm{Res}^G{}_P eAe, (\rho_e)_P)$ is directly embedded into $(\mathrm{Res}^G{}_P A, \rho_P)$. Let e_{11} be the element $1\otimes 1\otimes 1$ in $\mathrm{Cor}_P{}^G B$. Because P is normal, Lemma 2. 2 implies

$$\begin{aligned}\mathrm{Tr}_P{}^G(e_{11}) &= \sum_{v\in(G/P)} v(e_{11}) \\ &= 1 .\end{aligned}$$

Since P is normal in G and $v(e_{11})$ is a primitive idempotent of $(\mathrm{Cor}_p{}^G B)^P$ a defect group of $v(e_{11})$ in $(\mathrm{Res}^G{}_P A, \rho_P)$ equals P. Therefore a defect group of a primitive idempotent of $(eAe)^P$ in $(\mathrm{Res}^G{}_P eAe, (\rho_e)_P)$ is P. This is a contradiction.

LEMMA 3. 2 *Let N equals the normalizer $N_G(P)$ and $(\tilde{A}, \tilde{\rho})$ be the G-algebra $(\mathrm{Cor}_P{}^N B, \sigma^N)$. Then the N-algebras $A(P)$ and $\tilde{A}(P)$ are isomorphic, namely there exists an R-algebra isomorphism of $A(P)$ to $\tilde{A}(P)$, which is compatible with the N-action. In particular the R-algebras $A(P)^N$ and $\tilde{A}(P)^N$ are isomorphic.*

PROOF. Let $1=e_1+\cdots+e_r+e_{r+1}+\cdots+e_n$ be a primitive decomposition of the identity element 1 in A^G satisfying ; if $i=1, \cdots, r$, then a defect group of e_i equals P up to G-conjugacy ; otherwise a defect group of e_i equals a proper subgroup of P up to G-conjugacy. Since $A^G \subset A^N$ we have a primitive decomposition of the idempotent e_i $(1\leq i\leq r)$ in A^n :

$$e_i = f_{i1} + \cdots + f_{in_i}$$

By the same argument of [5] Lemma 2, for the primitive idempotent e_i of A^G there exists a unique idempotent f_{i1} of A^N such that a defect group of f_{i1} in $(\mathrm{Res}^G{}_N A, \sigma_N)$ equals P up to G-conjufacy. Let f be an idempotent of A^N defined by

$$f = f_{11} + \cdots + f_{r1} .$$

Then we have

$$1-f = \sum_{\substack{1\leq i\leq r\\ 2\leq j}} f_{ij} + \sum_{r+1\leq i} e_i$$

For f_{ij} $(1\leq i\leq r$ and $2\leq j)$ let D_{ij} be a defect group of f_{ij} in $(\mathrm{Res}^G{}_N A, \rho_N)$ and let

$$f_{ij} = h_1 + \cdots + h_s$$

be a primitive decomposition of f_{ij} in A^P. By Lemma 2. 2 a defect group of h_t $(1 \leq t \leq s)$ is a subgroup of D_{ij} uo to N-conjugacy. Hence a defect group of h_t is a proper subgroup of P. This implies

$$h_t \in \sum_{Q<P} A_Q{}^P + \pi A^P = I^P(A).$$

Therefore we obtain

$$f_{ij} \in I^P(A).$$

Because a defect group of e_i in (A, ρ) is a proper subgroup of P up to G-conjugacy for $r+1 \leq i$, it is easy to see that

$$e_i \in I^P(A)$$

for $r+1 \leq i$. Therefore the idempotent $1-f$ is contained in the ideal $I^P(A)$. Thus the inclusion of $(f \mathrm{Res}^G{}_N Af, (\rho_N)_f)$ into $(\mathrm{Res}^G{}_N A, \rho_N)$ induces an N-algebra isomorphism of $fAf(P)$ to $A(P)$.

Let $1 = \tilde{f}_1 + \cdots + \tilde{f}_{\tilde{s}} + \tilde{f}_{\tilde{s}+1} + \cdots + \tilde{f}_{\tilde{n}}$ be a primitive decomposition of the identity element 1 in $\tilde{A}^N$ satisfying; if $i=1, \cdots, \tilde{s}$, then a defect group of $\tilde{f}_i$ equals P up to N-conjugacy; otherwise a defect group of $\tilde{f}_i$ equals a proper subgroup of P up to N-conjugacy. Let $\tilde{f}$ be an idempotent of $\tilde{A}^N$ defined by

$$\tilde{f} = \tilde{f}_1 + \cdots + \tilde{f}_{\tilde{s}}.$$

By the same argument on (A, ρ), the inclusion of $(\tilde{f}\tilde{A}\tilde{f}, \tilde{\rho}_{\tilde{f}})$ into $(\tilde{A}, \tilde{\rho})$ induces an N-algebra isomorphism of $\tilde{f}\tilde{A}\tilde{f}(P)$ to $\tilde{A}(P)$.

Finally we shall define an isomorphism of the interior N-algebra $(f \mathrm{Res}^G{}_N Af, (\rho_N)_f)$ to the interior N-algebra $(\tilde{f}\tilde{A}\tilde{f}, \tilde{\rho}_{\tilde{f}})$. Then it is easy to see that this isomorphism induces an isomorphism of the N-algebra $A(P)$ to the N-algebra $\tilde{A}(P)$. Let e_{11} be the element $1 \otimes 1 \otimes 1$ in A and e' the element $\mathrm{Tr}_P{}^N(e_{11})$ of A^N. Let $e_{uu'}$ be the element $u \otimes 1 \otimes u'^{-1}$ of A for u and $u' \in (G/P)$ and $e_{vv'}$ the element $v \otimes 1 \otimes v'^{-1}$ of $\tilde{A}$ for v and $v' \in (N/P)$. Let β be an R-linear map of $\tilde{A}$ into A defined by

$$\beta: \tilde{e}_{vv'} \longmapsto e_{vv'}$$

for v and $v' \in (N/P)$. Then β is a direct embedding of $(\tilde{A}, \tilde{\rho})$ into $(\mathrm{Res}^G{}_N A, \rho_N)$. Furthermore

$$\beta(1) = \mathrm{Tr}_P{}^N(e_{11}) = e'.$$

Since $\tilde{f}_1, \cdots, \tilde{f}_{\tilde{s}}$ are mutualy orthogonal primitive idempotents of $\tilde{A}^N$ whose defect groups equal P up to N-conjugacy the element $\beta(\tilde{f}_1), \cdots, \beta(\tilde{f}_{\tilde{s}})$ are mutualy orthogonal primitive idempotents of $(e'Ae')^N$ whose defect groups equal P up to N-conjugacy. By Lemma 2. 2 we have

$$1-e' = \mathrm{Tr}_P{}^G(e_{11}) - \mathrm{Tr}_P{}^N(e_{11})$$
$$= \sum_{\substack{u\in(N\backslash G/P)\\ u\notin N}} \mathrm{Tr}_{uPu^{-1}\cap N}{}^N\big(u(e_{11})\big).$$

It follows that

$$f\cdot e' = (f_1+\cdots+f_r)\cdot e' = f.$$

So we obtain

$$\beta(\tilde{f}) = f.$$

Therefore the direct embeddeing β induces an isomorphism of $(\tilde{f}\tilde{A}\tilde{f}, \rho_{\tilde{f}})$ to $(f\,\mathrm{Res}^G{}_N Af, (\rho_N)_f)$, completing the proof of Lemma.

THEOREM 1. *Let* $\mathrm{Br}_P(A)$ *be the Brauer homomorphism of* (A, ρ) *with respect to* P. *If* b *is a block of* RG *whose defect group contains* P *up to* G-*conjugacy, then the element* $\mathrm{Br}_P(A)\circ\rho(b)$ *is non-zero.*

PROOF. We shall prove Theorem in case that P is a normal subgroup of G. In order to prove $\mathrm{Br}_P(A)\circ\rho(b)$, By Lemma 3. 1 it is sufficiant to see that $\rho(b)$ is non-zero. Let (R, σ_0) be the interior P-algebra defined in Lemma 2. 13. Since the kernel of σ_0 is the augmentation ideal of RP it is a unique maxmal ideal of RP. Therefore the kernel of σ is contained in the kernel of σ_0. By Lemma 2. 14 the kernel of ρ $(=\sigma^G)$ is contained in the kernel of $(\sigma_0)^G$. Therefore Lemma 2. 13 implies $(\sigma_0)^G(b)$ is non-zero, and so $\rho(b)$ is non-zero.

Now we shall prove in general case. Let (RG, id_{RG}) be a interior G-algebra where id_{RG} is the identity map of RG and C equal the centralizer $C_G(P)$. Since $\rho(I^P(RG))$ is contained in $I^P(A)$ we have an R-algebra homomorphism $\rho(P)$ of RC^N to $A(P)^N$ satisfying the following diagram is commutative:

$$\begin{array}{ccc} RG^G & \xrightarrow{\ \rho\ } & A^G \\ {\scriptstyle \mathrm{Br}_P(RG)}\downarrow & & \downarrow{\scriptstyle \mathrm{Br}_P(A)} \\ RC^N & \xrightarrow{\ \rho(P)\ } & A(P)^N. \end{array}$$

Similarly for $(\tilde{A}, \tilde{\rho})$ we have R-algebra homomorphism $\tilde{\rho}(P)$ of RC^N to $\tilde{A}(P)^N$ satisfying the following diagram is commutative:

$$\begin{array}{ccccc} & RN^N & \xrightarrow{\tilde{\rho}} & \tilde{A}^N & \\ \mathrm{Br}_P(RG) & \downarrow & & \downarrow & \mathrm{Br}_P(\tilde{A}) \\ & RC^N & \xrightarrow{\tilde{\rho}(P)} & \tilde{A}(P)^N\,. & \end{array}$$

Let α be the R-algebra isomorphism of $\tilde{A}(P)^N$ to $A(P)^N$ induced by the isomorphism β of $(\tilde{f}\tilde{A}\tilde{f}, \tilde{\rho}_{\tilde{f}})$ to $(f\,\mathrm{Res}^G{}_N Af, (\rho_N)_f)$ in Lemma 3. 2. Then the following diagram is commutative:

$$\begin{array}{ccccc} & RG^G & \xrightarrow{\rho} & A^G & \\ \mathrm{Br}_P(RG) & \downarrow & & \downarrow & \mathrm{Br}_P(A) \\ & RC^N & \xrightarrow{\rho(P)} & A(P)^N & \\ & \| & & \uparrow & \alpha \\ & RC^N & \xrightarrow{\tilde{\rho}(P)} & \tilde{A}(P)^N & \\ \mathrm{Br}_P(RN) & \uparrow & & \uparrow & \mathrm{Br}_P(\tilde{A}) \\ & RN^N & \xrightarrow{\tilde{\rho}} & \tilde{A}^N\,. & \end{array} \qquad (*)$$

Let ι be the inclusion of RC^N into RN^N. Then for the Brauer homomorphism $\mathrm{Br}_P(RN)$ of RC^N to RN^N with respect to P we have

$$\mathrm{Br}_P(RN)\circ\iota = \mathrm{id}_{RC^N}$$

where id_{RC^N} is the identity map of RC^N. let b' be the primitive idempotent $\iota\circ\mathrm{Br}_P(A)(b)$ of RN^N. By the first part of Theorem the element $\mathrm{Br}_P(\tilde{A})\circ\tilde{\rho}(b')$ is non-zero. The commutativity of the diagram $(*)$ implies

$$\begin{aligned} \tilde{\rho}(P)\circ\mathrm{Br}_P(RG)(b) &= \tilde{\rho}(P)\circ\mathrm{Br}_P(RN)\circ\iota\circ\mathrm{Br}_P(RG)(b) \\ &= \tilde{\rho}(P)\circ\mathrm{Br}_P(RN)(b') \\ &= \mathrm{Br}_P(\tilde{A})\circ\tilde{\rho}(b')\,. \end{aligned}$$

Therefore the element $\tilde{\rho}(P)\circ\mathrm{Br}_P(RG)(b)$ is non-zero. Similarly by the commutativity of the diagram $(*)$, we have

$$\begin{aligned} \mathrm{Br}_P(A)\circ\rho(b) &= \rho(P)\circ\mathrm{Br}_P(RG)(b) \\ &= \alpha\circ\tilde{\rho}(P)\circ\mathrm{Br}_P(RG)(b)\,. \end{aligned}$$

Since α is an isomorphism the element $\mathrm{Br}_P(A)\circ\rho(b)$ is non-zero. Thus complete the proof of Theorem.

THEOREM 2. *Let b be a block of RG and P a subgroup of a defect group of b. Then for a local interior P-algebra (B, σ) with defect group P, there exists a local interior G-algebra (A_1, ρ_1) satisfying the following properties:*

(1) *$\rho_1(b)$ is non-zero.*

(2) *A defect group of (A_1, ρ_1) equals P up to G-conjugacy.*

(3) *(B, σ) is a source of (A_1, ρ_1).*

PROOF. By Theorem 1 the idempotent $Br_P \circ \rho(b)$ is non-zero, where Br_P is the Brauer homomorphism of (A, ρ) with respect to P. Let $\rho(b) = f_1 + \cdots + f_s$ be a primitive decomposition of $\rho(b)$ in A^G. Then there exists an idempotent f_i such that $Br_P(b)$ is non-zero. We define a local interior G-algebra (A_1, ρ_1) by the interior G-algebra $(f_i A f_i, \rho_{f_i})$. Then (A_1, ρ_1) is directly embedded into (A, ρ). The statement of (1) is immediately from the definition of (A_1, ρ_1).

We shall prove the statement of (2). Because $Br_P(f_i)$ is non-zero, Lemma 2.4 implies that a defect group of f_i in (A, ρ) contains P up to G-conjugacy. Therefore by Lemma 2.1 a defect group of f_i equals P up to G-conjugacy. So a defect group of f_i equals a defect group of (A_1, ρ_1), and the statement of (2) holds.

Let (B_1, σ_1) be a source of (A_1, ρ_1). Then (B_1, σ_1) is a local interior P-algebra with defect group P. By the definition of sources, (B_1, σ_1) is directly embedded into $(\mathrm{Res}^G{}_P A_1, (\rho_1)_P)$. So (B_1, σ_1) is directly embedded into $(\mathrm{Res}^G{}_P \mathrm{Cor}_P{}^G B, (\sigma^G)^P)$. Let ϕ be this direct embedding. Let e_{11} be the element $1 \otimes 1 \otimes 1$ in A. Then by Lemma 2.2 we have an orthogonal decomposition of the identity element 1 in A^P:

$$1 = \mathrm{Tr}_P{}^G(e_{11}) = \sum_{u \in (P \backslash G/P)} \mathrm{Tr}_{uPu^{-1} \cap P}{}^P\big(u(e_{11})\big).$$

Let N be the normalizer $N_G(P)$. For $u \in N$ the idempotent $u(e_{11}) = \mathrm{Tr}_{uPu^{-1} \cap P}{}^P(e_{11})$ is a primitive idempotent of A^P. Since ϕ is a direct embedding there exists an element u of G such that the following hold:

$$\phi(1) \circ \mathrm{Tr}_{uPu^{-1} \cap P}{}^P\big(u(e_{11})\big) = \phi(1)$$

where 1 is the identity element of B_1. This implies that a defect group of (B_1, σ_1) is contained in uPu^{-1} up to G-conjugacy. Therefore we have $u \in N$. This implies $\phi(1) = u(e_{11})$. Hence by Lemma 2.12, (B, σ) is a source of (A_1, ρ_1), completing the proof of Theorem.

COROLLARY 3. *Let b be a block of RG and P a subgroup of a defect group of b. Then for an indecomposable RP-module W with vertex P there exists an indecomposable RG-module V lying in b satisfying the following properties:*

(1) *A vertex of V equals P up to G-conjugacy.*

(2) *W is a source of V.*

PROOF. By Example 2. 9, this follows immediately from Theorem 2. We recall that $k=o/(\pi)$ is a residue class field of characteristic p.

COROLLARY 4. *Suppose that k is an algebraically closed field. Let b be a block of RG and P a subgroup of a defect group of b. Then for a local interior P-algebra (B, σ) with defect group Q $(\leq P)$ and a source (C, τ) there exists a local interior G-algebra (A_1, ρ_1) satisfying the following conditions:*

(1) *$\rho_1(b)$ is non-zero.*

(2) *A defect group of (A_1, ρ_1) equals Q up to G-conjugacy.*

(3) *(C, τ) is a source of (A_1, ρ_1).*

PROOF. Since the field k is algebraicaly closed the k-algebra $C^Q/\mathrm{J}(C^Q)$ is isomorphic to k, where $\mathrm{J}(C^Q)$ is the Jacobson radical of C^Q. By the Puig's extention of the Green's theorem on absolutely indecomposable modules to the version of interior algebras, the interior P-algebra $(\mathrm{Cor}_Q{}^P C, \tau^P)$ is a local interior P-algebra. See [1] and [2]. So Lemma 2. 11 implies that the P-algebra (B, σ) is isomorphic to $(\mathrm{Cor}_Q{}^P C, \tau^P)$. Therefore the interior G-algebra (A, ρ) is isomorphic to the G-algebra $(\mathrm{Cor}_Q{}^G C, \tau^G)$, and we can applies Theorem 2 to Corollary.

COROLLARY 5. *Suppose that k is an algebraically closed field. Let b be a block of RG and P a subgroup of a defect group of b. Then for an indecomposable RP-module with vertex Q $(\leq P)$ and a source W there exists an indecomposable RG-module V lying in b satisfying the following properties:*

(1) *A vertex of V equals Q up to G-conjugacy.*

(2) *W is a source of V.*

PROOF. Immediately from Corollary 4.

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