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Nonsingular rings with a countable-dimensional annihilator base

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1. Introduction

If a nonsingular ring R happens to be a countable-dimensional algebra over some field, then, as shown in [5] and [4], the structure of the maximal right quotient ring $Q_{\max}(R)$ is surprisingly restricted. No extensions of these results seem to be known if R is countable-dimensional over just a sub-division ring or even over a non-central subfield. Here we consider an even weaker condition, namely that some nonzero left annihilator ideal J of R has a *countable-dimensional annihilator base* (CDAB). This is satisfied, for example, if for some division ring $K\subseteq R$ there is a countable-dimensional vector space ${}_KV\subseteq{}_KJ$ such that V intersects non-trivially with all left annihilator ideals $J'\subseteq J$ (§ 2 contains the precise definition of a CDAB). Our principal result (Theorem 2) states that then either $Q_{\max}(R)$ has a nonzero Type I part or R cannot be a right Utumi ring. Right Utumi rings share (by definition) an important property of right self-injective rings — their complement right ideals are right annihilator ideals.

The other main theorem, and on which the principal result hinges, is that if f is an idempotent of a regular, right self-injective ring Q, and Qf has a CDAB, then fQ must be of Type I_f (Theorem 1).

One surprising corollary of these theorems is that if R is a prime, right Utumi ring with a nonzero right ideal which is of countable dimension as a left vector space over some division ring $K\subseteq R$, then R must be right Goldie (Corollary 4). This ties in with J. Lawrence's result [6] that a countable-dimensional self-injective algebra is Artinian. It is also consistent with a more general theme that a nonsingular irreducible ring R which satisfies a countability condition either satisfies a finiteness condition or is, in some sense, a long way from being self-injective.

2. Definitions, Notation, and Background

Rings are associative with identity. The left annihilator of a set X in a ring R is denoted as $l_R(X)$ or $l(X)$ depending on the context. Similarly

$r_R(X)$ or $r(X)$ denotes the right annihilator.

For a module A (over a general ring), $E(A)$ denotes its injective hull. For a natural number n, nA denotes a direct sum of n copies of A. We write $A \lesssim B$ to indicate a module A is subisomorphic to a module B, and $A \leqslant_e B$ to indicate the submodule A is essential in the module B.

For the general background on nonsingular rings and maximal right quotient rings of such rings, as well as uniform modules and the uniform dimension of a finite-dimensional module, see Goodearl [2]. We denote the maximal right quotient ring of a right nonsingular ring R by $Q_{\max}(R)$. For the theory of regular, right self-injective rings Q, and the associated theory of types, see Goodearl [3]. We remind the reader that Q has a nonzero Type I part exactly when it contains a nonzero abelian idempotent e (all idempotents in eQe are central in eQe). Also a nonsingular injective module M_R is of Type I_f exactly when M is directly finite and each nonzero submodule of M contains a nonzero abelian submodule.

A *right Utumi* ring is a right nonsingular ring in which each complement right ideal is a right annihilator ideal, equivalently, every right ideal of R with zero left annihilator is essential in R. These rings were introduced by Utumi in [7]. In terms of $Q=Q_{\max}(R)$, a right nonsingular ring R is right Utumi if and only if Q is left intrinsic over R, that is nonzero left ideals of Q intersect nontrivially with R [7, Theorem 2. 2]. In particular, regular right self-injective rings, semi-prime right and left Goldie rings, commutative semi-prime rings, and the nonsingular right CS-rings of [1] are all right Utumi rings.

Let us say that a module ${}_KV$ has *countable uniform dimension* if V contains an essential submodule which is a countable direct sum of uniform K-modules, equivalently V contains only countable direct sums of nonzero submodules and each nonzero submodule contains a uniform submodule. For example this is true if ${}_KV$ is a countable-dimensional vector space over a division ring K, or if ${}_KV$ is a countably generated unitary module over a semisimple Artinian ring K, or even if ${}_KV$ is a countably generated nonsingular module over a left nonsingular ring K whose maximal left quotient ring has a countably generated essential left socle.

Finally we introduce a new concept:

DEFINITION. A left annihilator ideal J of a ring R has a *countable-dimensional annihilator base* (CDAB) if there is a subring K of R and a K-module ${}_KV \subseteq {}_KJ$ such that ${}_KV$ has countable uniform dimension and for all left annihilator ideals $0 \neq J' \subseteq J$,

$$J' \cap V \neq 0 .$$

Note that there is no loss in generality in assuming that ${}_KV$ is actually a (countable) direct sum of uniforms.

3. The Main Theorems

THEOREM 1. *Let Q be a regular, right self-injective ring and suppose f is an idempotent such that Qf has a countable-dimensional annihilator base. Then fQ is of Type I_f.*

PROOF. Let ${}_KV$ be a CDAB for Qf, for some subring K of Q. Let

$$V_1 \subset V_2 \subset \cdots \subset V_n \subset \cdots$$

be a chain of K-submodules of V such that each V_n has uniform dimension n and $\bigcup_1^\infty V_i$ is an essential submodule of ${}_KV$.

Firstly we observe that fQ must be directly finite, otherwise by [3, Theorem 10.19 and Proposition 10.21] Qf contains an uncountable direct sum $\bigoplus_\alpha Qa_\alpha$ of nonzero left ideals and then $\sum_\alpha Qa_\alpha \cap V$ is an uncountable direct sum of nonzero K-submodules of V, contrary to the assumption that ${}_KV$ has countable uniform dimension.

Let f_1Q and f_2Q be respectively the Type I_f and Type II_f parts of fQ, where f_1, f_2 are idempotents in fQf. Then $fQ=f_1Q\oplus f_2Q$, and Qf_2 inherits a CDAB from Qf, namely $V\cap Qf_2$. Thus in order to show fQ is of Type I_f it will suffice to assume fQ is of Type II_f and then deduce that $fQ=0$.

So we assume fQ is of Type II_f and has ${}_KV$ as a CDAB. Since the Type I part of fQ is zero, by [3, Proposition 10.28] each submodule of fQ can be written as a direct sum of 3 pairwise isomorphic submodules. (The choice of a 3-part splitting is inspired by the observation $\sum_1^\infty 1/3^n=1/2$.) We use this property to inductively construct independent summands $B_1, \cdots, B_n, \cdots$ of fQ such that

(1) $3B_n \cong B_{n-1}$ $\quad(\forall n>1)$

(2) $l(B_1+\cdots+B_n)\cap V_n=0$

(3) $fQ=(B_1\oplus\cdots\oplus B_n)\oplus C_n$ for some $C_n\cong(B_1\oplus\cdots\oplus B_n)\oplus B_n$.

To begin the induction, we write $fQ=A_1\oplus A_2\oplus A_3$ where $A_1\cong A_2\cong A_3$. Since $\bigcap_1^3 l(A_i)\cap Qf=0$, we have $\bigcap_1^3 (l(A_i)\cap V_1)=0$. But each $l(A_i)\cap V_1$ is a K-submodule of the uniform module V_1, so we can choose $B_1\in\{A_1, A_2, A_3\}$ such that

$$l(B_1)\cap V_1=0\,.$$

For C_1 we take the sum of the two A_i not equal to B_1. Clearly (2) and

(3) hold.

Now suppose for some $n \geqslant 1$ we have constructed $B_1, \cdots, B_n$ with the desired properties (1), (2), (3). Noting that $B_i \cong 3^{n-i} B_n$ for $i=1, \cdots, n$ by (1), and that $C_n \cong (B_1 \oplus \cdots \oplus B_n) \oplus B_n$ by (3), we can obtain a decomposition of C_n as a direct sum of $(3^{n-1}+3^{n-2}+\cdots+3+1)+1=(3^n+1)/2$ copies of B_n. Splitting each of these summands into a direct sum of 3 pairwise isomorphic modules then produces a decomposition

$$C_n = D_1 \oplus \cdots \oplus D_k$$

where $k=3(3^n+1)/2$, $D_1 \cong D_2 \cong \cdots \cong D_k$, and $3D_i \cong B_n$ for each i. Let $Y = l(B_1+\cdots+B_n) \cap V_{n+1}$, and note that Y is a K-submodule of V. From (2) we have

$$V_n \oplus Y \subseteq V_{n+1}$$

and so since V_n and V_{n+1} have uniform dimensions of n and $n+1$ respectively, either $Y=0$ or Y is a uniform submodule of ${}_K V$. From (3), $B_1+\cdots+B_n+D_1+\cdots+D_k=fQ$ implies $\bigcap_{i=1}^{k} l(B_1+\cdots+B_n+D_i) \cap Qf=0$. Hence

$$\bigcap_{i=1}^{k} \big(l(B_1+\cdots+B_n+D_i) \cap V_{n+1}\big) = 0 .$$

But each $l(B_1+\cdots+B_n+D_i) \cap V_{n+1}$ is a K-submodule of Y, and $Y=0$ or is uniform. Thus for some j

$$l(B_1+\cdots+B_n+D_j) \cap V_{n+1} = 0 .$$

Set $B_{n+1}=D_j$ and $C_{n+1}=$the sum of all the $k-1$ other D_i.

From $3D_j \cong B_n$ we have $3B_{n+1} \cong B_n$, giving (1). Clearly (2) holds for $n+1$. Also $C_n = B_{n+1} \oplus C_{n+1}$ so

$$fQ = B_1 \oplus \cdots \oplus B_n \oplus C_n = (B_1 \oplus \cdots \oplus B_{n+1}) \oplus C_{n+1}$$

and

$$\begin{aligned} C_{n+1} &\cong (k-1)\, B_{n+1} \qquad (\text{since each } D_i \cong B_{n+1}) \\ &= \big((3^n+3^{n-1}+\cdots+3+1)+1\big)\, B_{n+1} \\ &\cong (B_1 \oplus \cdots \oplus B_n \oplus B_{n+1}) \oplus B_{n+1} \end{aligned}$$

(since $B_i \cong 3^{n-i} B_n \cong 3^{n+1-i} B_{n+1}$)
which establishes (3). This completes the induction.

Let $B=E(B_1 \oplus B_2 \oplus \cdots) \subseteq fQ$. By property (2), $l(B) \cap V=0$. In consequence, $l(B) \cap Qf=0$ because V is a CDAB for Qf. Hence $B=fQ$. On the other hand by property (3), $2(B_1 \oplus \cdots \oplus B_n) \lesssim fQ$ for all n, whence $2B \lesssim$

fQ by [3, Proposition 9. 22] since fQ is directly finite. Thus $2(fQ) \lesssim fQ$. The direct finiteness of fQ now forces $fQ=0$, as required.

The above proof is based on an outline given to us by K. R. Goodearl, after he saw our original (much longer) proof.

THEOREM 2. *Let R be a right nonsingular ring which which has a nonzero left annihilator ideal with a countable-dimensional annihilator base. Then either the Type I part of $Q_{\max}(R)$ is nonzero or R is not a right Utumi ring.*

PROOF. Let $Q=Q_{\max}(R)$ and suppose R is right Utumi. Let J be a nonzero left annihilator ideal of R with a CDAB, say ${}_KV$ for some subring $K \subseteq R$. Then $J=Qf \cap R$ for some $f=f^2 \in Q$. Now since R is right Utumi, Q is left intrinsic over R by [7, Theorem 2. 2] and it follows that ${}_KV$ is also a CDAB for Qf in the ring Q. By Theorem 1 we conclude that fQ is of Type I_f and so Q has a nonzero Type I part.

4. Corollaries

Although Corollaries 1, 3 and 4 (below) are stated in a form which relies on Theorems 1 and 2 only in the case where a CDAB is a countable dimensional vector space over some division ring $K \subseteq R$ (this case seems the most interesting), these corollaries remain valid when the ideals in question have countable uniform dimension over an arbitrary subring K of R.

COROLLARY 1. *Suppose R is a right Utumi ring with a nonzero left ideal of countable dimension as a left vector space over some division ring $K \subseteq R$. Then $Q_{\max}(R)$ has a nonzero Type I part.*

REMARK. For R meet-irreducible (two-sided ideals intersect nontrivially), this means R has uniform right ideals.

PROOF. Let $V=Ra \neq 0$ be a principal left ideal of R with $\dim_K V$ countable. Let $J=l(r(V))$.

Claim: ${}_KV$ is a CDAB for the left annihilator ideal J of R.

For let $0 \neq J' \subseteq J$ be a left annihilator ideal of R. We wish to show $J' \cap V \neq 0$. Let $Q=Q_{\max}(R)$ and write $Qa=Qf$, $aQ=eQ$, where e, f are idempotents in Q. Observe that $J=l_R(r_R(Qf))=l_R((1-f)Q \cap R)=Qf \cap R$. If we let $r_Q(J')=(1-g)Q$ for $g=g^2 \in Q$, then $J'=l_R(r_R(J'))=l_R((1-g)Q \cap R)$ $=Qg \cap R$. Also $0 \neq Qg \cap R=J' \subseteq J \subseteq Qf=Qa$, so $0 \neq ya \in Qg$ for some $y \in Q$. Write $Qy=(Q(1-e) \cap Qy) \oplus Qh$ for some $h \in Q$. Then $ya \neq 0$ implies $ye \neq 0$ and so $h \neq 0$. As R right Utumi implies Q is left intrinsic over R, there exists $q \in Q$ with $0 \neq qy \in Qh \cap R$. Now $qya \neq 0$ and so $0 \neq (qy)a \in Qg \cap Ra \subseteq$

$J' \cap V$. Thus $J' \cap V \neq 0$ as desired.

The corollary now follows from Theorem 2.

COROLLARY 2. *Suppose R is a right Utumi ring and J is a left annihilator ideal with a countable-dimensional annihilator base. Then the injective hull of any complement of $r(J)$ is of Type I_f.*

PROOF. This follows from the proof of Theorem 2. For in the notation there, $J = Qf \cap R$, $fQ \cap R$ is a complement of $r_R(J) = (1-f)Q \cap R$, and fQ is the injective hull of $fQ \cap R$. As the proof shows, fQ is of Type I_f.

COROLLARY 3. *If a right Utumi ring R has a faithful right ideal U which is countable dimensional as a left vector space over some division ring $K \subseteq R$, then $Q_{\max}(R)$ is of Type I_f.*

PROOF. Let $J = R$, $V = U$. Since U is faithful, for any nonzero left ideal $J' \subseteq J$,

$$0 \neq UJ' \subseteq V \cap J' .$$

Hence J is a left annihilator ideal with ${}_K V$ as a CDAB. From $0 = r(J)$, we infer R is a complement of $r(J)$ and that $Q_{\max}(R)$ is of Type I_f by Corollary 2.

COROLLARY 4. *Let R be a prime, right Utumi ring with a nonzero right ideal U which is countable dimensional as a left vector space over some division ring $K \subseteq R$. Then R is right Goldie.*

PROOF. Immediate from Corollary 3, since U is faithful in a prime ring and prime, regular, right self-injective rings Q of Type I_f are simple Artinian.

REMARKS.

(1) With some hesitation we point out that Corollary 4 provides another characterization of simple Artinian rings, viz. rings which are prime, regular, right Utumi, and contain a countable-dimensional *right* ideal $\neq 0$. On the other hand, the ring of linear transformations of an infinite-dimensional right vector space is prime, regular, right Utumi and can contain nonzero countable-dimensional *left* ideals but of course is neither simple or Artinian. (It is of Type I, in accordance with Corollary 1.)

(2) In particular, for a right Utumi ring R possessing a left ideal $\neq 0$ with a CDAB, $Q_{\max}(R)$ can have zero Type I_f part (as well as zero Types II and III parts). Simple examples (such as a direct product of a simple Artinian ring and a Type II regular right self-injective ring) show that $Q_{\max}(R)$ need not be Type I.

(3) It is not true that if a right Utumi ring R contains only countable

direct sums of nonzero left or right ideals, then $Q_{\max}(R)$ has a nonzero Type I part, e.g. R a simple self-injective ring of Type II_f (see [3, Proposition 5.9]). By Corollary 1, taking $V=R$ and $K=R$ in this situation will not give a CDAB ${}_KV$ for R. Thus the countability requirement for a CDAB ${}_KV$ involves more than simply having only countable direct sums of K-submodules of V.

A left ideal with a CDAB over some division ring need not itself be countable dimensional (over any division ring K):

EXAMPLE.

There exists a commutative, regular, self-injective ring Q with no countable-dimensional ideals ($\neq 0$) but each of its annihilator ideals J has a CDAB over a field. Simply let $Q=Q_{\max}(R)$ where R is a countable Boolean ring without minimal ideals. Let $J=Qf\neq 0$ where $f=f^2\in Q$. Because soc $(Q)=0$, $\bigoplus_1^\infty f_iQ\leqslant_e fQ$ for some nonzero orthogonal idempotents f_i and hence by injectivity of Q_Q

$$Qf\supseteq \prod_1^\infty Qf_i\,.$$

Consequently for any division ring $K\subseteq Q$, $\dim_K Qf$ must be uncountable. However, letting

$$V=J\cap R \quad\text{and}\quad K=\{0, 1\}$$

we observe that ${}_KV$ is a CDAB for J.

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