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Remarks on certain complemented subspaces on groups

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1. Introduction

Let G be a locally compact group, and let m_G denote the left invariant Haar measure on G. (By a locally compact group we shall mean a locally compact Hausdorff group.) Let $L^{\infty}(G)$ denote the Banach algebra of essentially bounded Haar-measurable complex-valued functions on G with pointwise operations and essential sup norm. For a locally compact abelian (LCA) group G, J. E. Gilbert ([4]) characterized weak*-closed translation invariant complemented subspaces of $L^{\infty}(G)$ by their spectra. After that the author ([13]) determined the form of weak*-closed left and right translation invariant complemented subalgebras of $L^{\infty}(G)$ for a LCA group G and a compact group G. (Unfortunately there exists a gap in [13]. For the correction, see Zentralblatt für Math. 483. (1982), 43002.) But we don't know when closed (but not weak*-closed) left and right translation invariant subspaces (or, in particular, subalgebras) of $L^{\infty}(G)$ are complemented in $L^{\infty}(G)$.

Let $AP(G)$ and $WAP(G)$ denote the closed subalgebras of $L^{\infty}(G)$ consisting of all continuous left almost periodic functions on G and all continuous left weakly almost periodic functions on G, respectively. Our first purpose in this paper is to examine whether $AP(G)$ and $WAP(G)$ are complemented in $L^{\infty}(G)$ or not.

Let $L^{1}(G)$ denote the Banach space of all Haar-integrable complex-valued functions on G, and let $\mathscr{L}(L^{1}(G), L^{\infty}(G))$ denote the Banach space of all bounded linear operators from $L^{1}(G)$ to $L^{\infty}(G)$. Our second purpose in this paper is to define certain closed subspaces of $\mathscr{L}(L^{1}(G), L^{\infty}(G))$ for a LCA group G and to consider when their closed subspaces are complemented in $\mathscr{L}(L^{1}(G), L^{\infty}(G))$. The result obtained here seems to contain Gilbert Theorem as its special case.

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2. Preliminaries

Throughout this paper, G denotes a locally compact group. (In section 4, we shall consider only a LCA group G.)

Let X be a Banach space with its dual X^*. If $F \in X^*$ and $x \in X$, then the value of F at x is written as $F(x)$ or (x, F). (In particular, for $f \in L^1(G)$ and $g \in L^\infty(G) = (L^1(G))^*$, we always use $(f, g) = \int_G f(x)\, g(x^{-1})\, dm_G(x)$.) If Y is another Banach space, then $\mathscr{L}(X, Y)$ denotes the Banach space of all bounded linear operators from X to Y. (We also write $\mathscr{L}(X) = \mathscr{L}(X, X)$.) When Z is a closed subspace of X, we shall say that Z is complemented in X if there exists a bounded projection (i.e., a bounded linear idempotent operator) from X onto Z.

For a compact Hausdorff space S, $C(S)$ denotes the Banach algebra of all continuous complex-valued functions on S.

$L^\infty(G)$ is a commutative B^*-algebra with the complex conjugation operator as involution. Hence by Gelfand-Naimark Theorem the Gelfand transform is an isometric isomorphism from $L^\infty(G)$ onto $C(\Delta(L^\infty(G)))$ satisfying $\hat{\bar{f}} = \bar{\hat{f}}$ $(f \in L^\infty(G))$, where $\Delta(L^\infty(G))$ is the maximal ideal space of $L^\infty(G)$, $\hat{\cdot}$ is the Gelfand transform, and $\bar{\cdot}$ is the complex conjugation operator. We shall often identify $L^\infty(G)$ and $C(\Delta(L^\infty(G)))$ through the Gelfand transform.

For $s \in G$, left and right translation of a function f on G by s are denoted by $(L_s f)(x) = f(sx)$ and $(R_s f)(x) = f(xs)$ $(x \in G)$, respectively. A subspace X of $L^\infty(G)$ is said to be left [resp. right, left and right] translation invariant if $L_s f \in X$ [resp. $R_s f \in X$, $L_s f \in X$ and $R_s f \in X$] for all $s \in G$ and $f \in X$. If G is abelian, then left (and hence left and right) translation invariant subspaces of $L^\infty(G)$ are simply said to be translation invariant.

G^a denotes the almost periodic compactification of G, that is the closure of $\{L_x;\ x \in G\}$ with respect to the strong operator topology in $\mathscr{L}(AP(G))$. Then G^a is a compact group under the composition of operators as product and the strong operator topology. The map ρ defined by $\rho(x) = L_x$ is a continuous homomorphism from G to G^a, and ρ is one-to-one if and only if $AP(G)$ separates points in G. Moreover the map $\tilde{\rho}$ induced from ρ by $\tilde{\rho}(f) = f \circ \rho$ for $f \in C(G^a)$ is an isometric isomorphism from $C(G^a)$ onto $AP(G)$. (The full exposition can be seen in [2] and [7].)

Finally, G is said to be a maximally almost periodic group if $AP(G)$ separates points in G. Of course, maximally almost periodic groups include LCA groups and compact groups.

3. Subalgebras $AP(G)$ and $WAP(G)$

In this section we shall prove two Theorems.

THEOREM 1. *Let G be a maximally almost periodic group. Then the following statements are equivalent.*

(i) *G is finite.*
(ii) *$WAP(G)$ is complemented in $L^\infty(G)$.*
(iii) *$AP(G)$ is complemented in $L^\infty(G)$.*
(iv) *$C(G^a)$ is complemented in $L^\infty(G^a)$.*

In order to prove Theorem 1, we need two Lemmas.

LEMMA 1. *Let G be a infinite compact metrizable group. Then $C(G)$ is uncomplemented in $L^\infty(G)$.*

PROOF. We know that the Gelfand transform is an isometric isomorphism from $L^\infty(G)$ onto $C(\Delta(L^\infty(G)))$. Moreover, as is well known, $\Delta(L^\infty(G))$ is extremely disconnected, that is the closure of every open subset of $\Delta(L^\infty(G))$ is also open. Since G is infinite and metrizable, $C(G)$ has infinite dimension and is separable. By a result of Grothendieck ([6]. p. 169) we conclude that $C(G)$ is uncomplemented in $L^\infty(G)$. Q. E. D.

LEMMA 2. *Let G be a compact group, and let H be a closed normal subgroup of G. If $C(G)$ is complemented in $L^\infty(G)$, then $C(G/H)$ is complemented in $L^\infty(G/H)$.*

PROOF. Let π be the natural homomorphism from G onto G/H. We now define two bounded linear operators I: $L^\infty(G/H)\rightarrow L^\infty(G)$ and J: $L^\infty(G)\rightarrow L^\infty(G/H)$, as follows:

$$(If)(x)=(f\circ\pi)(x) \text{ for } x\in G \text{ and } f\in L^\infty(G/H)$$

and
$$(Jg)(xH)=\int_H g(x\xi)\, dm_H(\xi) \text{ for } x\in G \text{ and } g\in L^\infty(G),$$

where m_H is the normalized Haar measure on H. Since $C(G)$ is complemented in $L^\infty(G)$, there exists a bounded projection P from $L^\infty(G)$ onto $C(G)$. Now we define $Q=JPI$. Then it is easy to verify that Q is a bounded projection from $L^\infty(G/H)$ onto $C(G/H)$. Q. E. D.

PROOF OF THEOREM 1. (i)$\Rightarrow$(ii): If G is finite, then $L^\infty(G)=WAP(G)$ and therefore clearly (i) implies (ii).

(ii)$\Rightarrow$(iii): Let m be a (unique) two-sided invariant mean on $WAP(G)$, that is a bounded linear functional m on $WAP(G)$ such that

(a) $m(1)=1$,

(b) $|m(f)|\leq\|f\|_\infty$ for every $f\in WAP(G)$,

(c) $m(L_s f)=m(R_s f)=m(f)$ for every $s\in G$ and $f\in WAP(G)$.

(See [2] about the existence of such m.) If we define

$$W_0=\{f\in WAP(G)\,;\ m(|f|)=0\},$$

then W_0 is a closed subspace of $WAP(G)$ and we have

$$WAP(G)=AP(G)\oplus W_0 \quad ([2]).$$

Hence (ii) implies (iii).

(iii)⇨(iv): Since $L^\infty(G)$ and $L^\infty(G^a)$ have the 1-extension property, there exist bounded linear operators $I\colon L^\infty(G)\to L^\infty(G^a)$ and $J\colon L^\infty(G^a)\to L^\infty(G)$ such that

$$I(f)=\tilde{\rho}^{-1}(f) \qquad \text{for all } f\in AP(G)$$

and

$$J(g)=\tilde{\rho}(g) \qquad \text{for all } g\in C(G^a) \ ([8],\ \S\, 11.).$$

But we can prove directly that there exists such a bounded linear operator J without referring to the 1-extension property. Let $M(G)$ (resp. $M(G^a)$) denote the Banach space of all bounded regular complex Borel measures on G (resp. G^a). Let $\tau\colon M(G)\to M(G^a)$ be the bounded linear operator defined by $\int_{G^a} g\,d(\tau(\mu))=\int_G \tilde{\rho}(g)\,d\mu$ for $g\in C(G^a)$ and $\mu\in M(G)$. Let δ_e be the bounded linear functional on $C(G^a)$ defined by $\delta_e(g)=g(e)$ for $g\in C(G^a)$. (e is the identity element of G^a.) Choose and fix one Hahn-Banach extension F of δ_e to $L^\infty(G^a)$. For $g\in L^\infty(G^a)$, we define $Jg\in L^\infty(G)$ as a bounded linear functional on $L^1(G)$ as follows:

$$(h, Jg)=F\big(\tau(h)*g\big),$$

where $h\in L^1(G)$ and $\tau(h)*g(x)=\int_{G^a} g(y^{-1}x)\,d(\tau(h))(y)$. Then it is clear that J is a bounded linear operator from $L^\infty(G^a)$ to $L^\infty(G)$. To see that $J(g)=\tilde{\rho}(g)$ for all $g\in C(G^a)$, let $g\in C(G^a)$ and $h\in L^1(G)$. Then

$$\begin{aligned}(h, Jg)&=F\big(\tau(h)*g\big)\\&=\big(\tau(h)*g\big)(e)\\&=\int_{G^a} g(y^{-1})\,d\big(\tau(h)\big)(y)\\&=\int_G h(y)\,\tilde{\rho}(g)(y^{-1})\,dm_G(y)\\&=\big(h, \tilde{\rho}(g)\big).\end{aligned}$$

Hence we have $J(g)=\tilde{\rho}(g)$ for all $g\in C(G^a)$.

By our assumption, there exists a bounded projection P from $L^\infty(G)$ onto $AP(G)$. Let Q: $L^\infty(G^a)\rightarrow L^\infty(G^a)$ be the bounded linear operator defined by $Q=IPJ$. Since $Qg=IPJ(g)=\tilde{\rho}^{-1}(PJ(g))$ for every $g\in L^\infty(G^a)$, we have $Qg\in C(G^a)$ for every $g\in L^\infty(G^a)$. Moreover, for each $f\in C(G^a)$, we have $Qf=IPJ(f)=IP(\tilde{\rho}(f))=I(\tilde{\rho}(f))=\tilde{\rho}^{-1}(\tilde{\rho}(f))=f$. Hence Q is a bounded projection from $L^\infty(G^a)$ onto $C(G^a)$.

(iv)⇨(i): Suppose that G is infinite. Since G is a maximally almost periodic group, $AP(G)$ separates points in G, and therefore the natural homomorphism ρ from G to G^a is one-to-one. Hence G^a is a infinite compact group. After this we shall use the results and notation in [7], §§ 27 and 28. Let Σ be the dual object of G^a. Since G^a is infinite, Σ is also infinite by Lemma (28. 1). Hence there exists a countable subset P_0 of Σ. Then $A(G^a, P_0)$ is a closed normal subgroup of G^a. (For the definition of $A(G^a, P_0)$, see (28. 3).) Now we put $G_0=A(G^a, P_0)$. By Theorems (28. 5) and (28.9), we have $A(\Sigma, G_0)=[P_0]$. (For the definition of $A(\Sigma, G_0)$, see (28. 7).) Since P_0 is countable, it follows from the definition of the brackets $[\cdot]$ ((27. 35)) that $[P_0]$ is countable. Hence $A(\Sigma, G_0)$ is a countable subset of Σ. Thus it follows from Corollary (28. 11) that G^a/G_0 is metrizable. Since by Corollary (28. 10) the dual object of G^a/G_0 is infinite, G^a/G_0 is infinite ((27. 57)). Hence by our Lemmas 1 and 2, we conclude that $C(G^a)$ is uncomplemented in $L^\infty(G^a)$. Q. E. D.

REMARK 1. (a) Theorem 1 isn't necessarily true without the assumption that G is a maximally almost periodic group. For example, take $G=SL(2, \boldsymbol{C})$ (= the special linear group of degree 2 over the complex number field $\boldsymbol{C}$). Then G admits no nontrivial, finite dimensional, unitary representations ([7], (22. 22)). Hence $AP(G)$ consists of all constant functions on G ([7], (33. 26)), and therefore $AP(G)$ is complemented in $L^\infty(G)$. More generally, every simple noncompact connected Lie group G has this property ([12]).

(b) In [13] it was proved that if G is a LCA group or a compact group, and if A is a weak*-closed left and right translation invariant subalgebra of $L^\infty(G)$, then A is complemented in $L^\infty(G)$ if and only if A is self-adjoint, that is $f\in A$ implies $\bar{f}\in A$. But by Theorem 1 we can see that if G is a infinite maximally almost periodic group, then there are always closed left and right translation invariant subalgebras of $L^\infty(G)$ which are self-adjoint but uncomplemented in $L^\infty(G)$.

Let G be a LCA group. We define closed subspaces $\mathscr{M}(L^1(G), L^\infty(G))$, $\mathscr{C}\mathscr{M}(L^1(G), L^\infty(G))$, and $\mathscr{W}\mathscr{C}\mathscr{M}(L^1(G), L^\infty(G))$ of $\mathscr{L}(L^1(G), L^\infty(G))$ as follows:

$$\mathscr{M}\left(L^1(G),\ L^\infty(G)\right)=\left\{T\in\mathscr{L}\left(L^1(G),\ L^\infty(G)\right);\ TL_s=L_sT \text{ for all } s\in G\right\}.$$

$$\mathscr{C}\mathscr{M}\left(L^1(G),\ L^\infty(G)\right)=\left\{T\in\mathscr{M}\left(L^1(G),\ L^\infty(G)\right);\ T \text{ is compact}\right\}.$$

$$\mathscr{W}\mathscr{C}\mathscr{M}\left(L^1(G),\ L^\infty(G)\right)=\left\{T\in\mathscr{M}\left(L^1(G),\ L^\infty(G)\right);\ T \text{ is weakly compact}\right\}.$$

For two closed subspaces $\mathscr{C}\mathscr{M}(L^1(G), L^\infty(G))$ and $\mathscr{W}\mathscr{C}\mathscr{M}(L^1(G), L^\infty(G))$ of $\mathscr{M}(L^1(G), L^\infty(G))$, we have the following Corollary.

COROLLARY. *Let G be a infinite LCA group. Then $\mathscr{C}\mathscr{M}(L^1(G), L^\infty(G))$ and $\mathscr{W}\mathscr{C}\mathscr{M}(L^1(G), L^\infty(G))$ are uncomplemented in $\mathscr{M}(L^1(G), L^\infty(G))$.*

PROOF. As is well known, we can define an isometric linear isomorphism from $L^\infty(G)$ onto $\mathscr{M}(L^1(G), L^\infty(G))$ by the correspondence between $f\in L^\infty(G)$ and the convolution operator C_f defined by $C_f g=f*g$ for each $g\in L^1(G)$ ([9]). By this correspondence, $AP(G)$ and $WAP(G)$ are isometrically linear isomorphic to $\mathscr{C}\mathscr{M}(L^1(G), L^\infty(G))$ and $\mathscr{W}\mathscr{C}\mathscr{M}(L^1(G), L^\infty(G))$, respectively ([3]). Hence this Corollary is clear by Theorem 1. Q. E. D.

In relation to the equivalence of (i) and (iii) in Theorem 1, we have the following Theorem.

THEOREM 2. *Let G be a locally compact group, and let B be a closed left and right translation invariant subalgebra of $AP(G)$. If B is complemented in $L^\infty(G)$, then B is finite dimensional.*

PROOF. We may suppose that $B\neq\{0\}$. Let $B_1=\tilde{\rho}^{-1}(B)$, then B_1 is a closed subalgebra of $C(G^a)$. Since $\rho(G)$ is dence in G^a, it is easy to see that B_1 is left and right translation invariant. Moreover with the same argument as that in the implication (iii)$\Rightarrow$(iv) in Theorem 1, we obtain that B_1 is complemented in $L^\infty(G^a)$. Let $H=\{x\in G^a;\ f(x)=f(e) \text{ for all } f\in B_1\}$. ($e$ is the identity element of G^a.) Then it is easy to see that H is a closed normal subgroup of G^a. Let π be the natural homomorphism from G^a onto G^a/H, and let $\tilde{\pi}:\ L^\infty(G^a/H)\to L^\infty(G^a)$ be the map induced from π by $\tilde{\pi}(f)=f\circ\pi$ for $f\in L^\infty(G^a/H)$. Let $B_2=\tilde{\pi}^{-1}(B_1)$, then B_2 is a closed left and right translation invariant subalgebra of $C(G^a/H)$. Also we can verify easily that B_2 is complemented in $L^\infty(G^a/H)$. By Glicksberg Theorem ([5]) we obtain that B_2 is self-adjoint. From the definition of H, B_2 separates points in G^a/H. Moreover since B_2 is left and right translation invariant and $B_2\neq\{0\}$, B_2 vanishes identically at no point in G^a/H. Therefore it follows from Stone-Weierstrass Theorem that $B_2=C(G^a/H)$. Consequently $C(G^a/H)$ is complemented in $L^\infty(G^a/H)$. By Theorem 1 G^a/H is finite, and so B_2 $(=C(G^a/H))$ is finite dimensional. Hence we conclude that B is finite dimensional. Q. E. D.

4. Certain complemented subspaces of $\mathscr{L}(L^1(G), L^\infty(G))$.

Throughout this section G will be a LCA group. Let $\hat{G}$ denote the dual group of G. By the coset-ring $\Omega(\hat{G})$ of $\hat{G}$, we mean the ring generated by all the cosets of $\hat{G}$. Let X be a weak*-closed translation invariant subspace of $L^\infty(G)$. Then the spectrum of X, written $\sigma(X)$, is the set of all elements of $\hat{G}$ which belong to X. Let H be a subgroup of G, and let X be a weak*-closed translation invariant subspace of $L^\infty(G)$. Then we define $\mathscr{M}_H(L^1(G), X)$ as the set of all $T\in\mathscr{L}(L^1(G), L^\infty(G))$ such that $T(L^1(G))\subset X$ and $TL_s=L_sT$ for all $s\in H$. Clearly $\mathscr{M}_H(L^1(G), X)$ is a closed subspace of $\mathscr{L}(L^1(G), L^\infty(G))$.

The purpose of this section is to prove the following Theorem.

THEOREM 3. *Let G be a LCA group, and let H be a subgroup of G. Let X be a weak*-closed translation invariant subspace of $L^\infty(G)$. Then $\mathscr{M}_H(L^1(G), X)$ is complemented in $\mathscr{L}(L^1(G), L^\infty(G))$ if and only if $\sigma(X)$ belongs to $\Omega(\hat{G})$.*

REMARK 2. J. E. Gilbert ([4]) proved that if X is a weak*-closed translation invariant subspace of $L^\infty(G)$, then X is complemented in $L^\infty(G)$ if and only if $\sigma(X)$ belongs to $\Omega(\hat{G})$. (Indeed, the "only if" part is due to H. P. Rosenthal ([10]).) As we noted in the proof of Corollary to Theorem 1, $L^\infty(G)$ is isometrically linear isomorphic to $\mathscr{M}(L^1(G), L^\infty(G))$. Therefore we can view $L^\infty(G)$ as a closed subspace of $\mathscr{L}(L^1(G), L^\infty(G))$. Taking $H=G$ and $X=L^\infty(G)$ in Theorem 3, we obtain that $\mathscr{M}(L^1(G), L^\infty(G))$ is complemented in $\mathscr{L}(L^1(G), L^\infty(G))$. By this fact Gilbert Theorem can be reformulated as the statement in $\mathscr{L}(L^1(G), L^\infty(G))$ as follows: If X is a weak*-closed translation invariant subspace of $L^\infty(G)$, then X is complemented in $\mathscr{L}(L^1(G), L^\infty(G))$ if and only if $\sigma(X)$ belongs to $\Omega(\hat{G})$. With such reformulation and the identification between $L^\infty(G)$ and $\mathscr{M}(L^1(G), L^\infty(G))$, we can see that Theorem 3 for $H=G$ corresponds to Gilbert Theorem.

In order to prove Theorem 3, we need a Lemma. Let $L^1(G)\otimes_p L^1(G)$ denote the projective tensor product of two $L^1(G)$'s. Then $(L^1(G)\otimes_p L^1(G))^*$ is isometrically linear isomorphic to $\mathscr{L}(L^1(G), L^\infty(G))$ by the map Φ defined by

$$\big(g, \Phi(F)(f)\big)=(f\otimes g, F),$$

where f and $g\in L^1(G)$ and $F\in(L^1(G)\otimes_p L^1(G))^*$. Therefore through this Φ we can define the weak* topology in $\mathscr{L}(L^1(G), L^\infty(G))$. When $\mathscr{X}$ is a subset of $\mathscr{L}(L^1(G), L^\infty(G))$, we shall say that $\mathscr{X}$ is weak*-closed if $\Phi^{-1}(\mathscr{X})$ is weak*-closed in $(L^1(G)\otimes_p L^1(G))^*$. It is easy to see that $\mathscr{M}_H(L^1(G), X)$ defined above are weak*-closed subspaces of $\mathscr{L}(L^1(G), L^\infty(G))$.

LEMMA 4. *Let $\mathscr{X}$ be a weak*-closed subspace of $\mathscr{L}(L^1(G), L^\infty(G))$ satisfying $L_{-s}\mathscr{X}L_s$ $(=\{L_{-s}TL_s;\ T\in\mathscr{X}\})\subset\mathscr{X}$ for each $s\in G$. If $\mathscr{X}$ is complemented in $\mathscr{L}(L^1(G), L^\infty(G))$, then $\mathscr{X}\cap\mathscr{M}(L^1(G), L^\infty(G))$ is complemented in $\mathscr{M}(L^1(G), L^\infty(G))$.*

PROOF. We shall use the argument based on ideas due to K. Deleeuw. (See [5]). Let m be a invariant mean on $L^\infty(G_d)$, where G_d denotes the group G under the discrete topology. For $x\in G$ we define $\mathscr{U}_x$: $\mathscr{L}(L^1(G), L^\infty(G))\to\mathscr{L}(L^1(G), L^\infty(G))$ by $\mathscr{U}_x(T)=L_{-x}TL_x$ $(T\in\mathscr{L}(L^1(G), L^\infty(G)))$. Let $\mathscr{P}$ be a bounded projection from $\mathscr{L}(L^1(G), L^\infty(G))$ onto $\mathscr{X}$. For each $T\in\mathscr{L}(L^1(G), L^\infty(G))$ we define $\mathscr{R}(T)\in\mathscr{L}(L^1(G), L^\infty(G))$ as follows: First, define $\mathscr{Q}(T)\in(L^1(G)\otimes_p L^1(G))^*$ by the equation

$$\Big(\phi, \mathscr{Q}(T)\Big)=m_x\Big(\Big(\phi, \Phi^{-1}\Big(\mathscr{U}_{-x}\mathscr{P}\mathscr{U}_x(T)\Big)\Big)\Big)\quad \Big(\phi\in L^1(G)\otimes_p L^1(G)\Big).$$

(By $m_x((\phi, \Phi^{-1}(\mathscr{U}_{-x}\mathscr{P}\mathscr{U}_x(T))))$ we shall mean the value of the function $x\to(\phi, \Phi^{-1}(\mathscr{U}_{-x}\mathscr{P}\mathscr{U}_x(T)))$ on G by m.) Next, put $\mathscr{R}(T)=\Phi(\mathscr{Q}(T))$. Then we note that for each f and $g\in L^1(G)$,

$$\begin{aligned}\Big(g, \mathscr{R}(T)f\Big)&=\Big(g, \Phi\Big(\mathscr{Q}(T)\Big)f\Big)\\ &=\Big(f\otimes g, \mathscr{Q}(T)\Big)\\ &=m_x\Big(\Big(f\otimes g,\ \Phi^{-1}\Big(\mathscr{U}_{-x}\mathscr{P}\mathscr{U}_x(T)\Big)\Big)\Big)\\ &=m_x\Big(\Big(g,\ \mathscr{U}_{-x}\mathscr{P}\mathscr{U}_x(T)(f)\Big)\Big).\end{aligned}$$

It is easy to see that $\mathscr{R}$ is a bounded linear operator on $\mathscr{L}(L^1(G), L^\infty(G))$ with $\|\mathscr{R}\|\le\|\mathscr{P}\|$. If $T\in\mathscr{X}$, then $(g, \mathscr{R}(T)f)=m_x((g, \mathscr{U}_{-x}\mathscr{P}\mathscr{U}_x(T)(f)))=(g, Tf)$ for each f and $g\in L^1(G)$, and therefore $\mathscr{R}(T)=T$. If $T\in\mathscr{L}(L^1(G), L^\infty(G))$, then

$$\begin{aligned}\Big(\phi, \Phi^{-1}\Big(\mathscr{R}(T)\Big)\Big)&=\Big(\phi, \Phi^{-1}\Big(\Phi\Big(\mathscr{Q}(T)\Big)\Big)\Big)=\Big(\phi, \mathscr{Q}(T)\Big)\\ &=m_x\Big(\Big(\phi, \Phi^{-1}\Big(\mathscr{U}_{-x}\mathscr{P}\mathscr{U}_x(T)\Big)\Big)\Big)=0\end{aligned}$$

for each $\phi\in(\Phi^{-1}(\mathscr{X}))^\perp$, where $(\Phi^{-1}(\mathscr{X}))^\perp=\{\phi\in L^1(G)\otimes_p L^1(G);\ (\phi, F)=0$ for all $F\in\Phi^{-1}(\mathscr{X})\}$. Since $\Phi^{-1}(\mathscr{X})$ is weak*-closed, $\Phi^{-1}(\mathscr{R}(T))\in\Phi^{-1}(\mathscr{X})$ and therefore $\mathscr{R}(T)\in\mathscr{X}$. Thus we conclude that $\mathscr{R}$ is a bounded projection from $\mathscr{L}(L^1(G), L^\infty(G))$ onto $\mathscr{X}$. Moreover, we have $\mathscr{R}\mathscr{U}_a=\mathscr{U}_a\mathscr{R}$ for all $a\in G$.

Indeed, for $a\in G$, f and $g\in L^1(G)$, and $T\in\mathscr{L}(L^1(G), L^\infty(G))$,

$$\begin{aligned}\bigl(g, \mathscr{R}\mathscr{U}_a(T)(f)\bigr) &= m_x\Bigl(\bigl(g, \mathscr{U}_{-x}\mathscr{P}\mathscr{U}_x\bigl(\mathscr{U}_a(T)\bigr)(f)\bigr)\Bigr)\\ &= m_x\Bigl(\bigl(g, \mathscr{U}_a\mathscr{U}_{-x}\mathscr{P}\mathscr{U}_x(T)(f)\bigr)\Bigr)\\ &= m_x\Bigl(\bigl(g, L_{-a}\bigl(\mathscr{U}_{-x}\mathscr{P}\mathscr{U}_x(T)\bigr)L_a(f)\bigr)\Bigr)\\ &= m_x\Bigl(\bigl(L_{-a}g, \bigl(\mathscr{U}_{-x}\mathscr{P}\mathscr{U}_x(T)\bigr)L_a(f)\bigr)\Bigr)\\ &= \bigl(L_{-a}g, \mathscr{R}(T)L_a(f)\bigr)\\ &= \bigl(g, L_{-a}\mathscr{R}(T)L_a(f)\bigr)\\ &= \bigl(g, \mathscr{U}_a\mathscr{R}(T)(f)\bigr).\end{aligned}$$

Now if $T\in\mathscr{M}(L^1(G), L^\infty(G))$, then $L_aT=TL_a$ for all $a\in G$, and therefore for each $f\in L^1(G)$,

$$\begin{aligned}L_a\mathscr{R}(T)(f) &= L_a\mathscr{R}(T)L_{-a}(L_af)\\ &= \mathscr{U}_{-a}\mathscr{R}(T)(L_af)\\ &= \mathscr{R}\mathscr{U}_{-a}(T)(L_af)\\ &= \mathscr{R}(L_aTL_{-a})(L_af)\\ &= \mathscr{R}(T)(L_af) = \mathscr{R}(T)L_a(f).\end{aligned}$$

Hence $\mathscr{R}(\mathscr{M}(L^1(G), L^\infty(G)))\subset\mathscr{M}(L^1(G), L^\infty(G))$. Consequently we conclude that the restriction of $\mathscr{R}$ to $\mathscr{M}(L^1(G), L^\infty(G))$ is a bounded projection from $\mathscr{M}(L^1(G), L^\infty(G))$ onto $\mathscr{X}\cap\mathscr{M}(L^1(G), L^\infty(G))$. Q. E. D.

REMARK 3. (a) In the course of the proof of Lemma 4, we established the following: If $\mathscr{X}$ is a weak*-closed complemented subspace of $\mathscr{L}(L^1(G), L^\infty(G))$ satisfying $L_{-a}\mathscr{X}L_a\subset\mathscr{X}$ for each $a\in G$, then we can find a bounded projection from $\mathscr{L}(L^1(G), L^\infty(G))$ onto $\mathscr{X}$ which carries $\mathscr{M}(L^1(G), L^\infty(G))$ into $\mathscr{M}(L^1(G), L^\infty(G))$.

(b) For $1\leq s\leq\infty$, let $L^s(G)$ denote the usual Lebesgue spaces with respect to the Haar measure m_G. For $1\leq s\leq\infty$ and $1<t\leq\infty$, we can define the weak* topology in $\mathscr{L}(L^s(G), L^t(G))$ through the natural identification between $\mathscr{L}(L^s(G), L^t(G))$ and $(L^s(G)\otimes_p L^{t'}(G))^*$. ($t'$ denotes the conjugate exponent of t.) Then Lemma 4 holds with $\mathscr{L}(L^s(G), L^t(G))$ in place of $\mathscr{L}(L^1(G), L^\infty(G))$.

PROOF OF THEOREM 3. Suppose that $\mathscr{M}_H(L^1(G), X)$ is complemented in $\mathscr{L}(L^1(G), L^\infty(G))$. Since $\mathscr{M}_H(L^1(G), X)$ is a weak*-closed subspace of $\mathscr{L}(L^1(G), L^\infty(G))$ and clearly $L_{-a}\mathscr{M}_H(L^1(G), X)L_a\subset\mathscr{M}_H(L^1(G), X)$ for each

$a \in G$, it follows from Lemma 4 that $\mathscr{M}_H(L^1(G), X) \cap \mathscr{M}(L^1(G), L^\infty(G))$is complemented in $\mathscr{M}(L^1(G), L^\infty(G))$. Under the correspondence identifying between $L^\infty(G)$ and $\mathscr{M}(L^1(G), L^\infty(G))$, $\mathscr{M}_H(L^1(G), X) \cap \mathscr{M}(L^1(G), L^\infty(G))$ corresponds to X. Indeed, if $f \in L^\infty(G)$ and $C_f \in \mathscr{M}_H(L^1(G), X) \cap \mathscr{M}(L^1(G), L^\infty(G))$, then $f * L^1(G) \subset X$. Since f belongs to the closure of $f * L^1(G)$ with respect to the weak* topology in $L^\infty(G)$ ([11], 7. 8. 4) and X is weak*-closed, we have $f \in X$. Conversely, if $f \in X$ then $f * L^1(G) \subset X$ ([11]. 7. 8. 4) and therefore C_f belongs to $\mathscr{M}_H(L^1(G), X) \cap \mathscr{M}(L^1(G), L^\infty(G))$. Consequently we can conclude that X is complemented in $L^\infty(G)$. By Gilbert Theorem $\sigma(X)$ belongs to $\Omega(\hat{G})$.

Conversely, suppose that $\sigma(X)$ belongs to $\Omega(\hat{G})$. By Gilbert Theorem there exists a bounded projection P from $L^\infty(G)$ onto X. We define $\mathscr{Q}$: $\mathscr{L}(L^1(G), L^\infty(G)) \to \mathscr{L}(L^1(G), L^\infty(G))$ as follows:

$$\left(g, \mathscr{Q}(T)(f)\right) = m_x\left(\left(g, L_{-h} P T L_h(f)\right)\right),$$

where f and $g \in L^1(G)$, $T \in \mathscr{L}(L^1(G), L^\infty(G))$, and m is a invariant mean on $L^\infty(H_d)$. Then clearly $\mathscr{Q}$ is a bounded linear operator on $\mathscr{L}(L^1(G), L^\infty(G))$. For $f \in L^1(G)$ and $T \in \mathscr{L}(L^1(G), L^\infty(G))$,

$$\left(g, \mathscr{Q}(T)(f)\right) = m_h\left(\left(g, L_{-h} P T L_h(f)\right)\right) = 0$$

for each $g \in X^\perp (= \{g \in L^1(G);\ (g, \phi) = 0 \text{ for all } \phi \in X\})$. Since X is weak*-closed, $\mathscr{Q}(T)(f) \in X$. Hence we have $\mathscr{Q}(T)(L^1(G)) \subset X$. For $a \in H$, f and $g \in L^1(G)$, and $T \in \mathscr{L}(L^1(G), L^\infty(G))$,

$$\begin{aligned}\left(g, L_a \mathscr{Q}(T)(f)\right) &= \left(L_a g, \mathscr{Q}(T)(f)\right) \\ &= m_h\left(\left(L_a g, L_{-h} P T L_h(f)\right)\right) \\ &= m_h\left(\left(g, L_{-(h-a)} P T L_{(h-a)}(L_a f)\right)\right) \\ &= m_h\left(\left(g, L_{-h} P T L_h(L_a f)\right)\right) \\ &= \left(g, \mathscr{Q}(T)\, L_a f\right).\end{aligned}$$

Hence $L_a \mathscr{Q}(T) = \mathscr{Q}(T) L_a$ for each $a \in H$, and we have $\mathscr{Q}(T) \in \mathscr{M}_H(L^1(G), X)$. If $T \in \mathscr{M}_H(L^1(G), X)$, then $L_h T = T L_h$ for each $h \in H$ and $Tf \in X$ for each $f \in L^1(G)$, and therefore

$$\begin{aligned}\left(g, \mathscr{Q}(T)(f)\right) &= m_h\left(\left(g, L_{-h} P T L_h(f)\right)\right) \\ &= m_h\left(\left(g, L_{-h} T L_h(f)\right)\right)\end{aligned}$$

$$= m_h\big((g, Tf)\big) = (g, Tf)$$

for each f and $g \in L^1(G)$. Hence we have $\mathscr{P}(T) = T$. Consequently we conclude that $\mathscr{P}$ is a bounded projection from $\mathscr{L}(L^1(G), L^\infty(G))$ onto $\mathscr{M}_H(L^1(G), X)$. Q. E. D.

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