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On the curvature of Riemannian submanifolds of codimension 2

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Introduction

Let (M, g) be an n-dimensional Riemannian manifold which is isometrically immersed into the $n+k$-dimensional Euclidean space $\boldsymbol{R}^{n+k}$. Then the curvature transformation R of (M, g) satisfies the condition

$$(*) \qquad \text{rank } R(X, Y) \leqq 2k$$

for any tangent vectors $X, Y \in T_xM$, where we consider $R(X, Y)$ as a linear endomorphism of T_xM. Using this condition, Agaoka and Kaneda gave in [4] some estimates on the dimension of the Euclidean space into which Riemannian symmetric spaces can be locally isometrically immersed. For example they proved that the complex projective space $P^n(\boldsymbol{C})$ cannot be locally isometrically immersed in codimension $n-1$. But if $k \geqq (n-1)/2$, the condition $(*)$ does not impose any restrictions on the curvature of n-dimensional Riemannian submanifolds of $\boldsymbol{R}^{n+k}$.

Our first purpose of this paper is, using the representation theory of $GL(n, \boldsymbol{R})$, to determine the polynomial relations of the curvature tensor of $M^n \subset \boldsymbol{R}^{n+k}$, up to degree 3 explicitly (Theorem 1.4) and to find a new condition on the curvature tensor which serves as the obstruction in the cases $M^4 \subset \boldsymbol{R}^6$ and $M^5 \subset \boldsymbol{R}^7$. (See §1 and §2. Note that in these cases, the inequality $(*)$ reduces to a trivial condition.) Our second purpose is to express this new relation appeared in degree 3 in a simple form which is easy to calculate (Proposition 3.3, Theorem 3.4). As applications of this curvature relation, we prove that Riemannian symmetric spaces $P^2(\boldsymbol{C})$, $SU(3)/SO(3)$ and their non-compact dual spaces cannot be isometrically immersed in codimension 2 even locally (Corollary 3.5). As for $P^2(\boldsymbol{C})$ and its dual space, this result can be proved, using the theorems in Ôtsuki [18] and Weinstein [23] (see Remark (1) after Corollary 3.5). But, as for $SU(3)/SO(3)$ and its dual space, this is a new result, which cannot be obtained by a previously known method.

Now we explain our method briefly. Let V be an n-dimensional real vector space and let K be the space of curvature like tensors on V (see §1). We define a quadratic map $\gamma_k : S^2 V^* \otimes \boldsymbol{R}^k \longrightarrow K$ by

$$\gamma_k(\alpha)(X,Y,Z,W)=\langle\alpha(X,Z),\alpha(Y,W)\rangle-\langle\alpha(X,W),\alpha(Y,Z)\rangle$$

for $\alpha\in S^2V^*\otimes\boldsymbol{R}^k$ and $X, Y, Z, W\in V$, where we consider $\alpha\in S^2V^*\otimes\boldsymbol{R}^k$ as an $\boldsymbol{R}^k$-valued symmetric bilinear map on V and $\langle\,,\rangle$ is a positive definite inner product of $\boldsymbol{R}^k$. If an n-dimensional Riemannian manifold (M, g) is isometrically immersed into $\boldsymbol{R}^{n+k}$, then the Gauss equation of this immersion at $x\in M$ is expressed in the form $\gamma_k(\alpha)=R$ where R and α are the curvature tensor and the second fundamental form at x, respectively, i. e., the curvature R must be contained in the image of γ_k. (We identify T_xM with V and the normal space $T_x^\perp M$ with $\boldsymbol{R}^k$.) Therefore, we call that the Gauss equation for R has a solution if and only if $R\in\mathrm{Im}\,\gamma_k$. Our main purpose is to look for non-trivial homogeneous polynomials on K which vanish identically on $\mathrm{Im}\,\gamma_k$. If a polynomial ϕ satisfies such a condition and if $R\in K$ satisfies $\phi(R)\neq 0$, then it follows that $R\notin\mathrm{Im}\,\gamma_k$ and hence any Riemannian manifolds (M, g) whose curvature at one point of M is R cannot be isometrically immersed into $\boldsymbol{R}^{n+k}$. Thus the polynomials on K which vanish on $\mathrm{Im}\,\gamma_k$ serve as the obstructions to the existence of local isometric immersions of n-dimensional Riemannian manifolds into $\boldsymbol{R}^{n+k}$.

In order to state our first result in detail we reformulate this problem in the following form. Let

$$\gamma_k^{p*}: S^pK^*\longrightarrow S^{2p}(S^2V^*\otimes\boldsymbol{R}^k)^*$$

be the dual map of γ_k, i. e., γ_k^{p*} is a linear map defined by $\gamma_k^{p*}(\phi)(\alpha)=\phi(\gamma_k(\alpha))$ for $\phi\in S^pK^*$ and $\alpha\in S^2V^*\otimes\boldsymbol{R}^k$, where we regard S^pK^* as a space of homogeneous polynomials on K with degree p. Then our problem is to determine the kernel of γ_k^{p*} for each k and p. The group $GL(V)$ acts on K and $S^2V^*\otimes\boldsymbol{R}^k$ by $(g\cdot R)(X, Y, Z, W)=R(g^{-1}X, g^{-1}Y, g^{-1}Z, g^{-1}W)$ and $(g\cdot\alpha)(X, Y)=\alpha(g^{-1}X, g^{-1}Y)$, respectively, for $g\in GL(V)$, $R\in K$ and $\alpha\in S^2V^*\otimes\boldsymbol{R}^k$. Since γ_k is $GL(V)$-equivariant with respect to these actions, the linear map γ_k^{p*} is also $GL(V)$-equivariant. In particular $\mathrm{Ker}\,\gamma_k^{p*}$ is a $GL(V)$-invariant subspace of S^pK^*. Our first purpose is to determine the character and the generator of each $GL(V)$-irreducible component of S^pK^* for $p=1, 2, 3$ completely and decide whether these generators belong to $\mathrm{Ker}\,\gamma_k^{p*}$ or not (§1 and §2). The results may be stated as follows (Proposition 1. 2, Theorem 1. 4) : The spaces K^*, S^2K^* and S^3K^* are sum of 1, 4 and 17 $GL(V)$-irreducible components (for sufficiently large n), the subspaces $\mathrm{Ker}\,\gamma_1^{2*}$, $\mathrm{Ker}\,\gamma_1^{3*}$ and $\mathrm{Ker}\,\gamma_2^{3*}$ consist of 1, 10 and 2 irreducible components, respectively, and other subspaces $\mathrm{Ker}\,\gamma_k^{p*}$ for $p\leqq 3$ are all trivial. The polynomials belonging to $\mathrm{Ker}\,\gamma_1^{2*}$ and the one component of $\mathrm{Ker}\,\gamma_2^{3*}$ correspond to the condition $(*)$ stated before and another component of $\mathrm{Ker}\,\gamma_2^{3*}$ is a new type of condition, which serves as the obstructions in the case $M^n\subset\boldsymbol{R}^{n+2}$ for $n\geqq 4$. The subspace $\mathrm{Ker}\,\gamma_1^{3*}$ is not a new condition because it is contained in the ideal generated by Ker γ_1^{2*}. In summary the essential polynomial relations of the curvature tensor up to degree 3 are exhausted by Ker γ_1^{2*} and Ker γ_2^{3*}. To obtain these results, many computations will be required and in some cases it is almost impossible to calculate by hand because the generators of the irreducible components of S^pK^* are lengthy in general. Hence we use the algebraic programming system REDUCE 2 at Kyoto University in many places of this paper.

Next we state the second main result in detail. Using the fact that the new relation contained in Ker γ_2^{3*} is an invariant of $GL(V)$ in the case $n=4$, we prove that this obstruction is essentially equivalent to the equality Tr $(*\circ R)^3=0$, where we consider R as a symmetric endomorphism of $\wedge^2 V$ $(V=\boldsymbol{R}^4)$ and $*:\wedge^2 V\longrightarrow\wedge^2 V$ is the star operator with respect to some orientation of V (see §3). Moreover we prove that the condition Tr $(*\circ R)^5=0$ also holds if R is contained in the image of γ_2 (Theorem 3. 4). In the case $n\geqq 5$, we restrict the curvature operator to a 4-dimensional subspace of V. Then the
same conclusions hold if $R\in\mathrm{Im}\ \gamma_2$ (see Remark (1) after Theorem 3. 4).

Theoretically our method is well applied to higher degree and codimensional case. But in practice it is hard to carry out even if we use the system REDUCE 2 because the number of irreducible components and the length of the generators of S^pK^* increase rapidly as p becomes large.

§ 1. The Gauss equation and the representations of GL(n, $\boldsymbol{R}$).

In this section we fix our notations and state the first main results of this paper (Theorem 1. 4). For this purpose we review some known results on the character of $GL(n, \boldsymbol{R})$. For details, see [9] and [6].

Let V be an n-dimensional real vector space and K be the vector space of curvature like tensors on V, i. e.,

$$K=\{R\in\wedge^2 V^*\otimes\wedge^2 V^*\,|\,\underset{X,Y,Z}{\mathfrak{S}} R(X,Y,Z,W)=0,\ X,Y,Z,W\in V\},$$

where V^* is the dual space of V and $\underset{X,Y,Z}{\mathfrak{S}}$ implies the cyclic sum with respect to X, Y and Z. Let $\boldsymbol{R}^k$ by the k-dimensional Euclidean vector space with the inner product $\langle\,,\rangle$ and we put $E_k=S^2V^*\otimes\boldsymbol{R}^k$ for each positive integer k. Then the general linear group $GL(V)$ acts on both spaces K and E_k naturally. We first define a $GL(V)$-equivariant quadratic map $\gamma_k: E_k\longrightarrow K$ by

$$\gamma_k(\alpha)(X, Y, Z, W)=\langle\alpha(X, Z),\ \alpha(Y, W)\rangle-\langle\alpha(X, W),\ \alpha(Y, Z)\rangle$$

for $\alpha\in E_k$ and $X, Y, Z, W\in V$. Then our problem is, as stated in Introduction, to determine the kernel of the dual linear map $\gamma_k^{p*}: S^pK^*\longrightarrow S^{2p}E_k^*$ $(p=1, 2, 3)$ defined by $\gamma_k^{p*}(\phi)(\alpha)=\phi(\gamma_k(\alpha))$ for $\phi\in S^pK^*$ and $\alpha\in E_k$.

Before stating the results, we review the general theory of the character of an irreducible representation of $GL(V)$, according to Iwahori [9]. (In [9] all theorems are stated over the field of complex numbers $\boldsymbol{C}$, but the same results hold if we use the field of real numbers instead of $\boldsymbol{C}$. See the exercise 9, *p.* 121 in [9].)

Let $l=(l_1, \cdots, l_n)\in\mathbf{Z}^n$. We define a rational function $\xi(l)\in\mathbf{Z}[\varepsilon_1, \cdots, \varepsilon_n, \varepsilon_1^{-1}, \cdots, \varepsilon_n^{-1}]$ by

$$\xi(l)=\begin{vmatrix} \varepsilon_1^{l_1} & \varepsilon_1^{l_2}\cdots\varepsilon_1^{l_n} \\ \varepsilon_2^{l_1} & \varepsilon_2^{l_2}\cdots\varepsilon_2^{l_n} \\ & \cdots\cdots\cdots \\ \varepsilon_n^{l_1} & \varepsilon_n^{l_2}\cdots\varepsilon_n^{l_n} \end{vmatrix}$$

For example if $\delta=(n-1, n-2, \cdots, 2,1,0)$, then we have $\xi(\delta)=\prod_{i<j}(\varepsilon_i-\varepsilon_j)$ (Vandermonde's determinant). For an element $\lambda=(\lambda_1, \cdots, \lambda_n)\in\mathbf{Z}^n$ we define a rational function $S_\lambda(\varepsilon)=S_{\lambda_1,\dots\dots,\lambda_n}(\varepsilon_1, \cdots, \varepsilon_n)\in\mathbf{Z}[\varepsilon_1, \cdots, \varepsilon_n, \varepsilon_1^{-1}, \cdots, \varepsilon_n^{-1}]$ by

$$S_\lambda(\varepsilon)=\xi(\lambda+\delta)/\xi(\delta),$$

where $\lambda+\delta=(\lambda_1+n-1, \lambda_2+n-2, \cdots, \lambda_{n-1}+1, \lambda_n)$. $S_\lambda(\varepsilon)$ is called a *Schur function* corresponding to $\lambda=(\lambda_1, \cdots, \lambda_n)$. It is known that if $\lambda\in\mathbf{Z}^n$ satisfies the condition $\lambda_1\geqq\lambda_2\geqq\cdots\geqq\lambda_n\geqq 0$, $S_\lambda(\varepsilon)$ is a homogeneous polynomial of $\{\varepsilon_1, \cdots, \varepsilon_n\}$. There is a one-to-one correspondence between the set of real irreducible polynomial representations of $GL(V)$ and the set of Schur functions $S_\lambda(\varepsilon)$ satisfying $\lambda_1\geqq\lambda_2\geqq\cdots\geqq\lambda_n\geqq 0$. The correspondence is given as follows: Let $\rho: GL(V)\longrightarrow GL(m, \boldsymbol{R})$ be an irreducible polynomial representation of $GL(V)$ and let $a_1, \cdots, a_n$ be the eigenvalues of an element $g\in GL(V)$. Then there exists uniquely a Schur function $S_\lambda(\varepsilon)$ $(\lambda_1\geqq\lambda_2\geqq\cdots\geqq\lambda_n\geqq 0)$ such that the character $\mathrm{Tr}\,\rho(g)$ is equal to $S_\lambda(a_1, \cdots, a_n)$. In the following we identify $S_\lambda(\varepsilon)$ with the character of the irreducible representation or the representation space itself. We often abbreviate $S_\lambda(\varepsilon)$ as S_λ and omit zeroes appeared in $\lambda=(\lambda_1, \cdots, \lambda_n)$. For example we write $S_{2,1,0,\dots\dots,0}(\varepsilon)$ as $S_{2,1}$. The polynomial $S_\lambda+S_\mu$ implies a sum of irreducible representations (or spaces) with characters S_λ and S_μ. The degree of S_λ is given by the dimension formula $D(l_1, \cdots, l_n)/D(n-1, n-2, \cdots, 2,1,0)$ where $l=\lambda+\delta$ and $D(\alpha_1, \cdots, \alpha_n)=\prod_{i>j}(\alpha_i-\alpha_j)$ ([9, *p.* 115]). In particular S_λ is an invariant of $GL(V)$ (i. e., 1-dimensional representation of $GL(V)$) if and only if $\lambda_1=\cdots=\lambda_n$. We say that $S_\lambda(\varepsilon)$ is of depth i if $\lambda=(\lambda_1, \cdots, \lambda_n)$ satisfies $\lambda_i\neq 0$ and $\lambda_{i+1}=0$. We consider $S_{\lambda_1,\dots\dots,\lambda_n}$ to be zero in the case $m>n$ and $\lambda_m\neq 0$.

Now we define subspaces K_S, $K_A \subset \wedge^2 V^* \otimes \wedge^2 V^*$ by

$$K_S = \{R \in \wedge^2 V^* \otimes \wedge^2 V^* \mid R(X, Y, Z, W) = -R(Z, W, X, Y)\}$$

and $$K_A = \wedge^4 V^*.$$

Each of these spaces is $GL(V)$-invariant by the natural action.

LEMMA 1.1 (*cf.* [7, *p.* 882]). *$GL(V)$-irreducible decomposition of $\wedge^2 V^* \otimes \wedge^2 V^*$ is given by*

$$\wedge^2 V^* \otimes \wedge^2 V^* = K \oplus K_S \oplus K_A.$$

In particular K is a $GL(V)$-irreducible subspace of $\wedge^2 V^* \otimes \wedge^2 V^*$.

PROOF. We consider the dual space $\wedge^2 V \otimes \wedge^2 V$. The character of $\wedge^2 V$ is given by $S_{1,1}$ and using the Littlewood-Richardson rule (cf. [7, *p.* 879], [11], [15]), we have $S_{1,1} \cdot S_{1,1} = S_{2,2} + S_{2,1,1} + S_{1,1,1,1}$. Hence $\wedge^2 V^* \otimes \wedge^2 V^*$ splits into 3 irreducible components. We can easily verify that $\wedge^2 V^* \otimes \wedge^2 V^*$ is a direct sum of 3 spaces K, K_S and K_A and each space is non-trivial. Therefore this gives the irreducible decomposition of $\wedge^2 V^* \otimes \wedge^2 V^*$. q. e. d.

REMARK. Using the dimension formula, we have $\dim S_{2,2} = \frac{1}{12} n^2(n^2-1)$, $\dim S_{2,1,1} = \frac{1}{8} n(n-2)(n-1)(n+1)$ and $\dim S_{1,1,1,1} = \frac{1}{24} n(n-1)(n-2)(n-3)$. On the other hand it is already known that $\dim K = \frac{1}{12} n^2(n^2-1)$ (cf. [13, *p.* 63]) and hence the character of K^* is given by $S_{2,2}$.

Next we calculate the characters of $S^p K^*$ for $p=2$ and 3. In general let $\chi(g)$ be the character of an irreducible representation ρ of $GL(V)$ and let $\chi_s(g)$ be the character of the symmetric s-product of ρ. Then $\chi_s(g)$ is given by the following (see [9, *p.* 121]):

$$\chi_s(g) = \frac{1}{s!} \begin{vmatrix} \chi(g) & -1 & & 0 & \\ \chi(g^2) & \chi(g) & -2 & & \\ \chi(g^3) & \chi(g^2) & \chi(g) & \cdots\cdots\cdots & \\ & \cdots\cdots\cdots\cdots\cdots & & & -(s-1) \\ \chi(g^s) & \chi(g^{s-1}) & \chi(g^{s-2}) \cdots & \chi(g) & \end{vmatrix}.$$

For example we have

$$\chi_2(g) = \frac{1}{2}\chi(g)^2 + \frac{1}{2}\chi(g^2) \tag{1.1}$$

and

$$\chi_3(g) = \frac{1}{6}\chi(g)^3 + \frac{1}{2}\chi(g)\cdot\chi(g^2) + \frac{1}{3}\chi(g^3). \tag{1.2}$$

Next if we put $p_m = S_m$ and $\sigma_m = \varepsilon_1^m + \cdots + \varepsilon_n^m$, we have the following formulas ([9, *p.* 137, *p.* 120]):

$$S_\lambda = \det(p_{\lambda_{i+j-i}})_{1 \leq i,\, j \leq n} \tag{1.3}$$

and

(1. 4) $$\sigma_m+\sigma_{m-1}p_1+\cdots+\sigma_1 p_{m-1}=mp_m.$$

In particular p_m (resp. σ_m) can be expressed as a polynomial of $\{\sigma_1, \cdots, \sigma_m\}$ (resp. $\{p_1, \cdots, p_m\}$). Using these formulas, we determine the character of S^2K^* in the following way. First, if we put $\chi(g)=S_{2,2}$, we have from (1. 3) and (1. 4)

$$\chi(g)=p_2^2-p_3p_1 =\frac{1}{12}(\sigma_1^4+3\sigma_2^2-4\sigma_1\sigma_3).$$

It is easy to see that $\chi(g^2)$ is equal to

$$\frac{1}{12}(\sigma_2^4+3\sigma_4^2-4\sigma_2\sigma_6)$$

and we substitute these equalities into (1. 1). Then $\chi_2(g)$ is expressed as a polynomial of $\{\sigma_1, \cdots, \sigma_6\}$. Next using the formula (1. 4) once again, we rewrite $\chi_2(g)$ as a polynomial of $\{p_1, \cdots, p_6\}$ and finally using (1. 3) repeatedly, we express this polynomial as a sum of Schur functions. But in practice this procedure requires many calculations and hence we use the algebraic programming system REDUCE 2 to obtain the final expression. For the character of S^3K^*, it can be calculated in the same way by using the formula (1. 2) in this case. Thus we obtain the following proposition.

PROPOSITION 1. 2. *The characters of S^2K^* and S^3K^* are given by the following*:

$$S^2K^*: \quad S_{4,4}+S_{4,2,2}+S_{3,3,1,1}+S_{2,2,2,2},$$

$$S^3K^*: \quad S_{6,6}+S_{6,4,2}+S_{6,2,2,2}+S_{5,5,1,1}+S_{5,4,2,1}+S_{5,3,3,1} + S_{5,3,2,1,1}+S_{4,4,4}+2S_{4,4,2,2}+S_{4,4,1,1,1,1}+S_{4,3,3,1,1} + S_{4,3,2,2,1}+S_{4,2,2,2,2}+S_{3,3,3,3}+S_{3,3,2,2,1,1}+S_{2,2,2,2,2,2}.$$

REMARK. In the case $n=4$, S^3K^* is a sum of 10 irreducible components because $S_{5,3,2,1,1}$, $S_{4,4,1,1,1,1}$, etc., reduce to trivial spaces. The character of S^2K^* is already determined in [21, *p.* 383] by a different method.

Next we calculate the characters of $S^{2p}E_k^*$ for $p\leqq 3$ and $k=1, 2$. The character and the generator of each irreducible component of $S^pE_1^*$ ($E_1=S^2V^*$) is already known and in the case $n\geqq p$, the number of irreducible components of $S^pE_1^*$ is equal to the partition number of p (cf. [1], [21, *p.* 378]. See also [3] and [8]). As a representation space E_k^* is isomorphic to $\overbrace{E_1^*\oplus\cdots\oplus E_1^*}^{k}$ and hence we have

$$S^pE_k^*\cong\sum_{p_1+..+p_k=p}S^{p_1}E_1^*\otimes\cdots\otimes S^{p_k}E_1^*.$$

Thus the character of $S^pE_k^*$ can be calculated by using the Littlewood-Richardson rule. As a result, we havé the following lemma.

LEMMA 1. 3. *The characters of $S^{2p}E_k^*$ ($p=1$, 2, 3 and $k=1$, 2) are given* *by the following*:

$S^2E_1^*$: $S_4+S_{2,2}$,

$S^4E_1^*$: $S_8+S_{6,2}+S_{4,4}+S_{4,2,2}+S_{2,2,2,2}$,

$S^6E_1^*$: $S_{12}+S_{10,2}+S_{8,4}+S_{6,6}+S_{8,2,2}+S_{6,4,2}+S_{4,4,4}+S_{6,2,2,2}$
$+S_{4,4,2,2}+S_{4,2,2,2,2}+S_{2,2,2,2,2,2}$,

$S^2E_2^*$: $3S_4+3S_{2,2}+S_{3,1}$,

$S^4E_2^*$: $5S_8+9S_{6,2}+6S_{4,4}+9S_{4,2,2}+5S_{2,2,2,2}+3S_{7,1}+3S_{5,3}$
$+4S_{5,2,1}+3S_{4,3,1}+3S_{3,2,2,1}+S_{3,3,1,1}$,

$S^6E_2^*$: $7S_{12}+5S_{11,1}+15S_{10,2}+9S_{9,3}+8S_{9,2,1}+18S_{8,4}+9S_{8,3,1}$
$+19S_{8,2,2}+6S_{7,5}+12S_{7,4,1}+12S_{7,3,2}+3S_{7,3,1,1}+9S_{7,2,2,1}$
$+10S_{6,6}+8S_{6,5,1}+27S_{6,4,2}+S_{6,4,1,1}+S_{6,3,3}+12S_{6,3,2,1}$
$+19S_{6,2,2,2}+S_{5,5,2}+3S_{5,5,1,1}+8S_{5,4,3}+12S_{5,4,2,1}$
$+3S_{5,3,3,1}+9S_{5,3,2,2}+4S_{5,3,2,1,1}+8S_{5,2,2,2,1}+10S_{4,4,4}$
$+6S_{4,4,3,1}+18S_{4,4,2,2}+3S_{4,3,3,1,1}+9S_{4,3,2,2,1}+15S_{4,2,2,2,2}$
$+S_{3,3,3,1,1,1}+3S_{3,3,2,2,1,1}+5S_{3,2,2,2,2,1}+7S_{2,2,2,2,2,2}$.

Now we state the first main results of this paper. Its proof will be done in §2.

THEOREM 1.4. *The characters of the invariant subspaces* Ker γ_k^{p*} $(p=1, 2, 3)$ *are given by the following*:

(1) *The case of* $k=1$.

Ker $\gamma_1^{1*}=\{0\}$, Ker $\gamma_1^{2*}=S_{3,3,1,1}$,

Ker $\gamma_1^{3*}=S_{5,5,1,1}+S_{5,4,2,1}+S_{5,3,3,1}+S_{5,3,2,1,1}+S_{4,4,2,2}$
$+S_{4,4,1,1,1,1}+S_{4,3,3,1,1}+S_{4,3,2,2,1}+S_{3,3,3,3}+S_{3,3,2,2,1,1}$.

(2) *The case of* $k=2$.

Ker $\gamma_2^{1*}=$ Ker $\gamma_2^{2*}=\{0\}$, Ker $\gamma_2^{3*}=S_{4,4,1,1,1,1}+S_{3,3,3,3}$.

(3) *The case of* $k\geqq 3$.

Ker $\gamma_k^{1*}=$ Ker $\gamma_k^{2*}=$ Ker $\gamma_k^{3*}=\{0\}$.

REMARK. (1) As stated in Introduction, the spaces $S_{3,3,1,1}\subset S^2K^*$ and $S_{4,4,1,1,1,1}\subset S^3K^*$ correspond to the condition rank $R(X, Y)\leqq 2k$ for $X, Y\in T_xM$ $(k=1, 2$, respectively) and the space $S_{3,3,3,3}\subset$ Ker γ_2^{3*} is a new condition. For details, see §2 and §3. We remark that the inequality rank $R(X, Y)\leqq 4$ is useful only in the range $n\geqq 6$ while the condition $S_{3,3,3,3}$ serves as the actual obstruction in the case $M^n\subset \boldsymbol{R}^{n+2}$ for $n\geqq 4$.

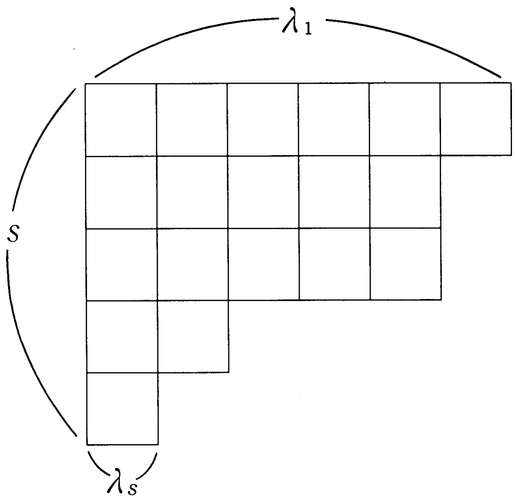
(2) The inclusions Ker $\gamma_1^{2*}\supset S_{3,3,1,1}$, Ker $\gamma_1^{3*}\supset S_{5,5,1,1}+\cdots+S_{3,3,2,2,1,1}$ and Ker $\gamma_2^{3*}\supset S_{4,4,1,1,1,1}+S_{3,3,3,3}$ can be directly checked from Proposition 1.2 and Lemma 1.3. In fact, since $\gamma_k^{p*}: S^pK^*\longrightarrow S^{2p}E_k^*$ is a $GL(V)$-equivariant map, the irreducible component of S^pK^* is mapped by γ_k^{p*} to $\{0\}$ or the non-trivial irreducible space with the same character. The space S^2K^* contains the irreducible component $S_{3,3,1,1}$, but the space $S^4E_1^*$ does not contain $S_{3,3,1,1}$ and hence we have $\gamma_1^{2*}(S_{3,3,1,1})=\{0\}$, i.e., $S_{3,3,1,1}\subset$ Ker γ_1^{2*}.

Other inclusions can be verified in the same way. To prove the " equality " in this theorem, we have to look for a generator of each irreducible component of S^pK^* ($p=1, 2, 3$), to substitute $\alpha \in E_k$ into these generators and to decide whether they are zero or not as polynomials on E_k. We achieve this procedure in §2.

§ 2. Irreducible components of $S^pK^*(p\leqq 3)$.

In this section we give a method to obtain the generators of the irreducible components of S^pK^* for $p=1, 2, 3$ and prove Theorem 1. 4. For this purpose we first review some facts on the Young tableaux and the $GL(V)$-irreducible subspaces of the tensor space $V\otimes\cdots\otimes V$ (cf. [6], [9]).

By a Young diagram of signature $\lambda=(\lambda_1, \lambda_2, \cdots, \lambda_s)$ $(\lambda_1\geqq\lambda_2\geqq\cdots\geqq\lambda_s>0)$ we mean an array of boxes such that the number of boxes in the i-th row is λ_i $(i=1, \cdots, s)$:



We call s the *depth* of this Young diagram. By a Young tableau we mean the Young diagram whose boxes are filled with the integers $1, \cdots, q$, where q is the number of boxes of this diagram (i. e., $q=\sum_{i=1}^{s}\lambda_i$). For example the following is the Young tableau of signature $(3, 3, 2, 1)$:

4	1	3
9	5	7
2	8	
6		

We denote the Young tableau by the letter B and the (i, j)-component of B by $B(i, j)$.

Let $\mathfrak{S}_q$ be the symmetric group of degree q and we put $T_q=\overbrace{V\otimes\cdots\otimes V}^{q}$ $(V=\boldsymbol{R}^n)$. We define a group representation

$$\rho:\mathfrak{S}_q\longrightarrow GL(T_q)$$

by $\rho(\tau)(X_1\otimes\cdots\otimes X_q)=X_{\tau^{-1}(1)}\otimes\cdots\otimes X_{\tau^{-1}(q)}$. We denote by $\boldsymbol{R}[\mathfrak{S}_q]$ the group algebra of $\mathfrak{S}_q$, i. e., $\boldsymbol{R}[\mathfrak{S}_q]=\{\sum_i a_i\tau_i \mid a_i\in\boldsymbol{R}, \tau_i\in\mathfrak{S}_q\}$ equipped with the natural sum and product. The group homomorphism ρ induces the ring homomorphism from $\boldsymbol{R}[\mathfrak{S}_q]$ to $\mathrm{End}(T_q)$, which we denote by the same letter ρ.

Let B be a Young tableau of depth$\leqq n$ such that the number of boxes is equal to q. Then using this tableau, a $GL(V)$-invariant irreducible subspace of T_q can be constructed in the following way. We define subgroups $\mathfrak{H}_B$ and $\mathfrak{V}_B\subset\mathfrak{S}_q$ by $\mathfrak{H}_B=\{\sigma\in\mathfrak{S}_q\mid\sigma$ interchanges the numbers of each row$\}$ and $\mathfrak{V}_B=\{\sigma\in\mathfrak{S}_q\mid\sigma$ interchanges the numbers of each column$\}$. Note that if the signature of B is $\lambda=(\lambda_1, \cdots, \lambda_s)$, then $\mathfrak{H}_B$ is isomorphic to $\mathfrak{S}_{\lambda_1}\times\overset{s}{\cdots}\times\mathfrak{S}_{\lambda_s}$. Now we define a Young symmetrizer $\hat{c}_B\in\boldsymbol{R}[\mathfrak{S}_q]$ by

$$\hat{c}_B=\sum_{\tau\in\mathfrak{H}_B}\sum_{\sigma\in\mathfrak{V}_B}(-1)^{\sigma}\sigma\tau,$$

where $(-1)^{\sigma}$ implies the signature of σ. Then the space $\rho(\hat{c}_B)T_q$ is a $GL(V)$-invariant irreducible subspace of T_q, and in addition, the character of $\rho(\hat{c}_B)T_q$ is given by S_λ, where λ is the signature of B. We remark that if the depth of B exceeds the dimension of V, then $\rho(\hat{c}_B)T_q$ reduces automatically to a trivial space.

Using this theory, we give a method to obtain irreducible subspaces of $S^p K^*$. By definition K is a $GL(V)$-invariant subspace of $\wedge^2V^*\otimes\wedge^2V^*$ and $\wedge^2V^*\otimes\wedge^2V^*$ can be considered as a subspace of $T_4^*=V^*\otimes V^*\otimes V^*\otimes V^*$ in a natural way. Hence we have a surjective linear map $T_4\longrightarrow K^*$ and this induces a map $T_{4p}\longrightarrow\overbrace{K^*\otimes\cdots\otimes K^*}^{p}$. Combining this map with the canonical projection $\overbrace{K^*\otimes\cdots\otimes K^*}^{p}\longrightarrow S^pK^*$, we obtain $\pi: T_{4p}\longrightarrow S^pK^*$, which is a $GL(V)$-equivariant surjective map. Explicitly π is given by

$$\pi(X_1\otimes\cdots\otimes X_{4p})(R)=R(X_1, \cdots, X_4)\cdots R(X_{4p-3}, \cdots, X_{4p})$$

for $X_i\in V$ and $R\in K$. Let B be a Young tableau of signature $\lambda=(\lambda_1, \cdots, \lambda_s)(s\leqq n)$ such that $\sum\lambda_i=4p$. Then the space $\pi(\rho(\hat{c}_B)T_{4p})$ is either an irreducible subspace of S^pK^* with the character S_λ or a trivial space $\{0\}$ because π is $GL(V)$-equivariant.

PROPOSITION 2.1. *Let B be a Young tableau of signature $(\lambda_1, \cdots, \lambda_s)$ $(s\leqq n$ and $\sum\lambda_i=4p)$ and let μ_i be the number of boxes appeared in the i-th column of B. (Hence $\mu_1=s$.) For each element $X_1, \cdots, X_s\in V$ we define I_B*

$(X_1, \cdots, X_s) \in S^pK^*$ by

$$I_B(X_1 \cdots, X_s)(R) = \sum_{\substack{\sigma_k \in \mathfrak{S}_{\mu_k} \\ k=1,\dots,\lambda_1}} (-1)^{\sigma_1} \cdots (-1)^{\sigma_{\lambda_1}} R(Y_1, \cdots, Y_4) \cdots R(Y_{4p-3}, \cdots, Y_{4p}), \cdots, Y_{4p}),$$

where $Y_1, \cdots, Y_{4p} \in V$ are given by $Y_{B(l,k)} = X_{\sigma_k(l)} (1 \leqq l \leqq s,\ 1 \leqq k \leqq \lambda_1)$. Then the subspace $\pi(\rho(\hat{c}_B) T_{4p})$ is generated by the element $I_B(X_1, \cdots, X_s)$, i. e., $\pi(\rho(\hat{c}_B) T_{4p}) = \{\sum_i a_i I_B(X_{i_1}, \cdots, X_{i_s}) | a_l \in \boldsymbol{R}, X_{i_a} \in V\}$.

EXAMPLE. Consider the following tableau B of signature $(3, 3, 1, 1)$:

1	3	7
2	4	8
5		
6		

Then the space $\pi(\rho(\hat{c}_B) T_8) \subset S^2K^*$ is generated by

$$I_B(X_1, \cdots, X_4)(R) = \sum (-1)^{\sigma_1} (-1)^{\sigma_2} (-1)^{\sigma_3} R(X_{\sigma_1(1)}, X_{\sigma_1(2)}, X_{\sigma_2(1)}, X_{\sigma_2(2)}) \times R(X_{\sigma_1(3)}, X_{\sigma_1(4)}, X_{\sigma_3(1)}, X_{\sigma_3(2)}).$$

By direct calculations we have

$$(**) \quad \frac{1}{32} I_B(X_1, \cdots, X_4)(R) = R(X_1, X_2, X_1, X_2) R(X_1, X_2, X_3, X_4) - R(X_1, X_2, X_1, X_3) R(X_1, X_2, X_2, X_4) + R(X_1, X_2, X_1, X_4) R(X_1, X_2, X_2, X_3).$$

In the case $n \geqq 4$, this polynomial represents a non-trivial element of S^2K^* and hence the invariant irreducible subspace $S_{3,3,1,1}$ of S^2K^* (cf. Proposition 1. 2) is generated by the polynomial $(**)$. If we use the following tableau B:

7	1	4
2	5	6
8		
3		

,

then we have

$$I_B(X_1, \cdots, X_4)(R) = \sum (-1)^{\sigma_1} (-1)^{\sigma_2} (-1)^{\sigma_3} R(X_{\sigma_2(1)}, X_{\sigma_1(2)}, X_{\sigma_1(4)}, X_{\sigma_3(1)}) \times R(X_{\sigma_2(2)}, X_{\sigma_3(2)}, X_{\sigma_1(1)}, X_{\sigma_1(3)}) = 0,$$

and hence $\pi(\rho(\hat{c}_B) T_8) = \{0\}$ in this case.

PROOF of PROPOSITION 2. 1. For elements $Z_1, \cdots, Z_{4p} \in V$ we put $A(Z_1, \cdots, Z_{4p}) = \sum_{\tau \in \mathfrak{H}_B} \sum_{\sigma \in \mathfrak{V}_B} (-1)^{\sigma} \rho(\sigma\tau)(Z_1 \otimes \cdots \otimes Z_{4p}) \in T_{4p}$. Then we have $\rho(\hat{c}_B) T_{4p} = \{A(Z_1, \cdots, Z_{4p}) \mid Z_i \in V\}$ and $A(Z_{\tau(1)}, \cdots, Z_{\tau(4p)}) = A(Z_1, \cdots, Z_{4p})$ for each $\tau \in \mathfrak{H}_B$. Therefore by putting $Z_{B(1,1)} = \cdots = Z_{B(1,\lambda_1)} = X_1, \cdots, Z_{B(s,1)} = \cdots = Z_{B(s,\lambda_s)} = X_s$, we know that the space $\rho(\hat{c}_B) T_{4p}$ is generated by the element $\sum_{\sigma \in \mathfrak{V}_B} (-1)^{\sigma} \rho(\sigma)(Z_1 \otimes \cdots \otimes Z_{4p})$. Next we consider the first alternative sum $\sum_{\sigma_1 \in \mathfrak{S}_{\mu_1}}$ (the first column of B). Since $Z_1 \otimes \cdots \otimes Z_{4p}$ is equal to

$$\cdots \otimes \overset{B(1,1)}{X_1} \otimes \cdots \otimes \overset{B(2,1)}{X_2} \otimes \cdots \otimes \overset{B(\mu_1,1)}{X_{\mu_1}} \otimes \cdots,$$

we have

$$\begin{aligned}
&\sum_{\sigma_1 \in \mathfrak{S}_{\mu_1}} (-1)^{\sigma_1} \rho(\sigma_1)(Z_1 \otimes \cdots \otimes Z_{4p}) \\
&= \sum_{\sigma_1 \in \mathfrak{S}_{\mu_1}} (-1)^{\sigma_1} \{\cdots \otimes \overset{B(1,1)}{X_{\sigma_1^{-1}(1)}} \otimes \cdots \otimes \overset{B(2,1)}{X_{\sigma_1^{-1}(2)}} \otimes \cdots \otimes \overset{B(\mu_1,1)}{X_{\sigma_1^{-1}(\mu_1)}} \otimes \cdots\} \\
&= \sum_{\sigma_1 \in \mathfrak{S}_{\mu_1}} (-1)^{\sigma_1} \{\cdots \otimes \overset{B(1,1)}{X_{\sigma_1(1)}} \otimes \cdots \otimes \overset{B(2,1)}{X_{\sigma_1(2)}} \otimes \cdots \otimes \overset{B(\mu_1,1)}{X_{\sigma_1(\mu_1)}} \otimes \cdots\}.
\end{aligned}$$

Hence by putting $Y_{B(l,1)} = X_{\sigma_1(l)}$, *we have*

$$\sum_{\sigma_1 \in \mathfrak{S}_{\mu_1}} (-1)^{\sigma_1} \rho(\sigma_1)(Z_1 \otimes \cdots \otimes Z_{4p}) = \sum_{\sigma_1 \in \mathfrak{S}_{\mu_1}} (-1)^{\sigma_1} \{\cdots \otimes \overset{B(1,1)}{Y_{B(1,1)}} \otimes \cdots \otimes \overset{B(\mu_1,1)}{Y_{B(\mu_1,1)}} \otimes \cdots\}.$$

Repeating the same procedure to the other columns of B, we have

$$\sum_{\sigma \in \mathfrak{V}_B} (-1)^{\sigma} \rho(\sigma)(Z_1 \otimes \cdots \otimes Z_{4p}) = \sum_{\substack{\sigma_k \in \mathfrak{S}_{\mu_k} \\ k=1,\ldots,\lambda_1}} (-1)^{\sigma_1} \cdots (-1)^{\sigma_{\lambda_1}} Y_1 \otimes \cdots \otimes Y_{4p},$$

where $Y_{B(l,k)} = X_{\sigma_k(l)}$. Then mapping this element by π, we obtain the desired result. q. e. d

Next using Proposition 2. 1, we obtain the generators of the irreducible components of $S^p K^*$ for $p=1, 2, 3$. In the following we list up the characters S_λ, the tableaux B and the generators $I_B(X_1, \cdots, X_s)$ of the spaces $\pi(\rho(\hat{c}_B) T_{4p}) \subset S^p K^*$. (We divide I_B by a non-zero constant such that I_B is expressed in a simple form.) For the spaces $S_{4,3,2,2,1}$, $S_{4,2,2,2,2}$, $S_{3,3,2,2,1,1}$ and $S_{2,2,2,2,2,2} \subset S^3 K^*$, the explicit expression of these generators are too long to write down here and hence we omit them. Except the case $2S_{4,4,2,2} \subset S^3 K^*$, the generators may be obtained by using a different tableau B of the same signature unless $I_B = 0$, but we list up here only one of them. In some cases, in order to calculate the generators I_B, we use the algebraic programming system REDUCE 2. For simplicity we write $R(X_i, X_j, X_k, X_l)$ as R_{ijkl} etc.

(1°) The case $p=1$.

$S_{2,2}$, $B_{2,2} = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}$, $I_{B_{2,2}} = R_{1212}$.

(This implies that the space of curvature like tensors is generated by the element $R(X_1, X_2, X_1, X_2)$, i. e., the curvature is determined by its sectional curvature.)

(2°) The case $p=2$.

$$S_{4,4}\,,\ B_{4,4}=\begin{array}{|c|c|c|c|}\hline 1&3&5&7\\\hline 2&4&6&8\\\hline\end{array}\,,\quad I_{B_{4,4}}=R_{1212}{}^2.$$

$$S_{4,2,2}\,,\ B_{4,2,2}=\begin{array}{cccc} 1&3&6&8\\ 2&4&&\\ 5&7&&\end{array}\,,\quad I_{B_{4,2,2}}=\begin{vmatrix}R_{1212}&R_{1213}\\R_{1213}&R_{1313}\end{vmatrix}.$$

$$S_{3,3,1,1}\,,\ B_{3,3,1,1}=\begin{array}{ccc} 1&3&7\\ 2&4&8\\ 5&&\\ 6&&\end{array}\,,$$

$$I_{B_{3,3,1,1}}=R_{1212}R_{1234}-R_{1213}R_{1224}+R_{1214}R_{1223}\,.$$

$$S_{2,2,2,2}\,,\ B_{2,2,2,2}=\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline 5&7\\\hline 6&8\\\hline\end{array}\,,$$

$$\begin{aligned}I_{B_{2,2,2,2,}}=\;&R_{1212}R_{3434}+R_{1313}R_{2424}+R_{1414}R_{2323}-2R_{1314}R_{2324}+2R_{1214}R_{2334}\\&-2R_{1213}R_{2434}-2R_{1224}R_{1334}+2R_{1223}R_{1434}-2R_{1323}R_{1424}+R_{1234}{}^2\\&+R_{1324}{}^2+R_{1423}{}^2\,.\end{aligned}$$

(3°) The case $p=3$.

$$S_{6,6}\,,\ B_{6,6}=\begin{array}{|c|c|c|c|c|c|}\hline 1&3&5&7&9&11\\\hline 2&4&6&8&10&12\\\hline\end{array}\,,\quad I_{B_{6,6}}=R_{1212}{}^3.$$

$$S_{6,4,2}\,,\ B_{6,4,2}=\begin{array}{cccccc} 1&3&4&11&8&10\\ 2&6&9&12&&\\ 5&7&&&&\end{array}\,,$$

$$I_{B_{6,4,2}}=R_{1212}\begin{vmatrix}R_{1212}&R_{1213}\\R_{1213}&R_{1313}\end{vmatrix}.$$

$$S_{6,2,2,2}\,,\ B_{6,2,2,2}=\begin{array}{cccccc} 1&3&6&8&10&12\\ 2&4&&&&\\ 5&7&&&&\\ 9&11&&&&\end{array}\,,$$

$$I_{B_{6,2,2,2}}=\begin{vmatrix}R_{1212}&R_{1213}&R_{1214}\\R_{1213}&R_{1313}&R_{1314}\\R_{1214}&R_{1314}&R_{1414}\end{vmatrix}.$$

$$S_{5,5,1,1}\,,\ B_{5,5,1,1}=\begin{array}{ccccc} 1&3&6&7&11\\ 2&4&10&8&12\\ 5&&&&\\ 9&&&&\end{array}\,,$$

$$I_{B_{5,5,1,1}}=R_{1212}\cdot I_{B_{3,3,1,1}}\,.$$

$$S_{5,4,2,1}\,,\ B_{5,4,2,1}=\begin{array}{ccccc} 1&3&4&11&10\\ 2&6&7&12&\\ 5&8&&&\\ 9&&&&\end{array}\,,$$

$$I_{B_{5,4,2,1}} = R_{1212}{}^2R_{1334} - R_{1212}R_{1213}R_{1324} + R_{1212}R_{1214}R_{1323} - R_{1212}R_{1313}R_{1224} + R_{1212}R_{1314}R_{1223} - R_{1212}R_{1213}R_{1234} + 2R_{1213}{}^2R_{1224} - 2R_{1213}R_{1214}R_{1223} \, .$$

$$S_{5,3,3,1}\, , \quad B_{5,3,3,1} = \begin{array}{lllll} 1 & 3 & 7 & 6 & 12 \\ 2 & 4 & 8 & & \\ 5 & 10 & 11 & & \\ 9 & & & & \end{array} \, ,$$

$$I_{B_{5,3,3,1}} = \begin{vmatrix} R_{1212} & R_{1213} & R_{1223} \\ R_{1213} & R_{1313} & R_{1323} \\ R_{1214} & R_{1314} & R_{1423} \end{vmatrix} \, .$$

$$S_{5,3,2,1,1}\, , \quad B_{5,3,2,1,1} = \begin{array}{lllll} 1 & 3 & 7 & 10 & 12 \\ 2 & 4 & 8 & & \\ 5 & 11 & & & \\ 6 & & & & \\ 9 & & & & \end{array} \, ,$$

$$I_{B_{5,3,2,1,1}} = R_{1212}R_{1313}R_{1245} - R_{1214}R_{1225}R_{1313} + R_{1215}R_{1224}R_{1313} - R_{1213}{}^2R_{1245} + R_{1213}R_{1214}R_{1235} - R_{1213}R_{1215}R_{1234} + R_{1212}R_{1315}R_{1234} - R_{1212}R_{1314}R_{1235} + R_{1213}R_{1314}R_{1225} - R_{1213}R_{1315}R_{1224} + R_{1214}R_{1315}R_{1223} - R_{1215}R_{1314}R_{1223}.$$

$$S_{4,4,4}\, , \quad B_{4,4,4} = \begin{array}{llll} 1 & 3 & 5 & 8 \\ 2 & 4 & 6 & 9 \\ 7 & 11 & 12 & 10 \end{array} \, ,$$

$$I_{B_{4,4,4}} = \begin{vmatrix} R_{1212} & R_{1213} & R_{1223} \\ R_{1213} & R_{1313} & R_{1323} \\ R_{1223} & R_{1323} & R_{2323} \end{vmatrix} \, .$$

$$S_{4,4,2,2}\, , \quad B^{(1)}_{4,4,2,2} = \begin{array}{llll} 1 & 3 & 9 & 11 \\ 2 & 4 & 10 & 12 \\ 5 & 7 & & \\ 6 & 8 & & \end{array} \, , \quad B^{(2)}_{4,4,2,2} = \begin{array}{llll} 1 & 3 & 7 & 11 \\ 2 & 4 & 8 & 12 \\ 5 & 9 & & \\ 6 & 10 & & \end{array} \, ,$$

$$I_{B^{(1)}_{4,4,2,2}} = R_{1212}\cdot I_{B_{2,2,2,2}} \, ,$$

$$I_{B^{(2)}_{4,4,2,2}} = R_{1212}{}^2R_{3434} + 3R_{1212}R_{1234}{}^2 + R_{1213}{}^2R_{2424} + R_{1214}{}^2R_{2323} + R_{1223}{}^2R_{1414} + R_{1224}{}^2R_{1313} + 2R_{1212}R_{1214}R_{2334} - 2R_{1212}R_{1213}R_{2434} + 2R_{1212}R_{1223}R_{1434} - 2R_{1212}R_{1224}R_{1334} - 2R_{1213}R_{1214}R_{2324} - 2R_{1213}R_{1223}R_{1424} + 2R_{1213}R_{1224}R_{1423} - 2R_{1213}R_{1224}R_{1234} + 2R_{1214}R_{1223}R_{1234} + 2R_{1214}R_{1223}R_{1324} - 2R_{1214}R_{1224}R_{1323} - 2R_{1223}R_{1224}R_{1314} \, .$$

REMARK. If we use the tableau

$$B^{(3)}_{4,4,2,2} = \begin{array}{llll} 1 & 3 & 4 & 11 \\ 2 & 7 & 8 & 12 \\ 5 & 9 & & \\ 6 & 10 & & \end{array} \, ,$$

then it can be directly verified that the polynomial $I_{B^{(3)}_{4,4,2,2}}$ is expressed as a linear combination of $I_{B^{(1)}_{4,4,2,2}}$ and $I_{B^{(2)}_{4,4,2,2}}$, and by Proposition 2. 1 this element also generates the irreducible subspace with the character $S_{4,4,2,2}$. Note that the irreducible decomposition of the subspace $2S_{4,4,2,2} \subset S^3K^*$ is not uniquely determined because the multiplicity is 2 in this case.

$$S_{4,4,1,1,1,1}\,,\ B_{4,4,1,1,1,1} = \begin{array}{|c|c|c|c|}\hline 3&1&5&9\\\hline 4&2&6&10\\\hline 7\\\cline{1-1} 8\\\cline{1-1} 11\\\cline{1-1} 12\\\cline{1-1}\end{array}\,,$$

$$\begin{aligned} I_{B_{4,4,1,1,1,1}} = {} & R_{1212}R_{1234}R_{1256} - R_{1212}R_{1235}R_{1246} + R_{1212}R_{1236}R_{1245} - R_{1213}R_{1224}R_{1256} \\ & + R_{1213}R_{1225}R_{1246} - R_{1213}R_{1226}R_{1245} + R_{1214}R_{1223}R_{1256} - R_{1214}R_{1225}R_{1236} \\ & + R_{1214}R_{1226}R_{1235} - R_{1215}R_{1223}R_{1246} + R_{1215}R_{1224}R_{1236} - R_{1215}R_{1226}R_{1234} \\ & + R_{1216}R_{1223}R_{1245} - R_{1216}R_{1224}R_{1235} + R_{1216}R_{1225}R_{1234}\,. \end{aligned}$$

$$S_{4,3,3,1,1}\,,\ B_{4,3,3,1,1} = \begin{array}{|c|c|c|c|}\hline 1&3&7&12\\\hline 2&4&8\\\cline{1-3} 5&10&11\\\cline{1-3} 6\\\cline{1-1} 9\\\cline{1-1}\end{array}\,,$$

$$\begin{aligned} I_{B_{4,3,3,1,1}} = {} & 3R_{1213}R_{1225}R_{1334} - 3R_{1213}R_{1224}R_{1335} + 3R_{1224}R_{1235}R_{1313} + 2R_{1212}R_{1234}R_{1335} \\ & - 2R_{1212}R_{1235}R_{1334} + 2R_{1214}R_{1223}R_{1335} - 2R_{1214}R_{1235}R_{1323} - 2R_{1215}R_{1223}R_{1334} \\ & + 2R_{1215}R_{1234}R_{1323} + 2R_{1223}R_{1315}R_{1324} - 2R_{1223}R_{1314}R_{1325} + 2R_{1224}R_{1313}R_{1523} \\ & - 2R_{1224}R_{1315}R_{1323} - 2R_{1225}R_{1313}R_{1324} + 2R_{1225}R_{1314}R_{1323} + R_{1212}R_{1324}R_{1335} \\ & - R_{1212}R_{1325}R_{1334} + R_{1212}R_{1313}R_{2345} - R_{1212}R_{1314}R_{2335} + R_{1212}R_{1315}R_{2334} \\ & - R_{1212}R_{1323}R_{1345} - R_{1213}{}^2R_{2345} + R_{1213}R_{1214}R_{2335} - R_{1213}R_{1215}R_{2334} \\ & + R_{1213}R_{1223}R_{1345} - R_{1213}R_{1234}R_{1523} + R_{1213}R_{1235}R_{1423} + R_{1213}R_{1245}R_{1323} \\ & + R_{1213}R_{1314}R_{2325} - R_{1213}R_{1315}R_{2324} - R_{1214}R_{1313}R_{2325} + R_{1214}R_{1315}R_{2323} \\ & + R_{1215}R_{1313}R_{2324} - R_{1215}R_{1314}R_{2323} - R_{1223}R_{1245}R_{1313} - R_{1225}R_{1234}R_{1313}. \end{aligned}$$

$$S_{4,3,2,2,1}\,,\ B_{4,3,2,2,1} = \begin{array}{|c|c|c|c|}\hline 1&3&7&10\\\hline 2&4&8\\\cline{1-3} 5&11\\\cline{1-2} 6&12\\\cline{1-2} 9\\\cline{1-1}\end{array}\,.$$

(The polynomial $I_{B_{4,3,2,2,1}}$ can be expressed as a sum of 62 monomials.)

$$S_{4,2,2,2,2}\,,\ B_{4,2,2,2,2} = \begin{array}{|c|c|c|c|}\hline 1&3&10&12\\\hline 2&4\\\cline{1-2} 5&7\\\cline{1-2} 6&8\\\cline{1-2} 9&11\\\cline{1-2}\end{array}\,.$$

(The polynomial $I_{B_{4,2,2,2,2}}$ can be expressed as a sum of about 120 different monomials.)

$$S_{3,3,3,3}\,,\ B_{3,3,3,3} = \begin{array}{|c|c|c|}\hline 1&3&7\\\hline 2&4&8\\\hline 5&9&11\\\hline 6&10&12\\\hline\end{array}\,,$$

$$
\begin{aligned}
I_{B_{3,3,3,3}} = {} & R_{1212}R_{3434}R_{1234} - R_{1313}R_{2424}R_{1324} + R_{1414}R_{2323}R_{1423} + R_{1212}R_{1434}R_{2334} \\
& - R_{1212}R_{1334}R_{2434} + R_{1313}R_{1224}R_{2434} + R_{1313}R_{1424}R_{2324} + R_{2424}R_{1213}R_{1334} \\
& + R_{2424}R_{1314}R_{1323} + R_{1414}R_{1223}R_{2334} - R_{1414}R_{1323}R_{2324} + R_{2323}R_{1214}R_{1434} \\
& - R_{2323}R_{1314}R_{1424} + R_{3434}R_{1214}R_{1223} - R_{3434}R_{1213}R_{1224} + R_{1213}R_{1423}R_{2434} \\
& - R_{1214}R_{1323}R_{2434} - R_{1314}R_{1223}R_{2434} - R_{1213}R_{1424}R_{2334} + R_{1214}R_{1324}R_{2334} \\
& - R_{1224}R_{1314}R_{2334} - R_{1213}R_{1434}R_{2324} + R_{1234}R_{1314}R_{2324} - R_{1214}R_{1334}R_{2324} \\
& + R_{1223}R_{1324}R_{1434} - R_{1224}R_{1323}R_{1434} + R_{1224}R_{1334}R_{1423} - R_{1223}R_{1334}R_{1424} \\
& + R_{1234}R_{1323}R_{1424} - R_{1234}R_{1324}R_{1423} .
\end{aligned}
$$

$$
S_{3,3,2,2,1,1}\ ,\ B_{3,3,2,2,1,1} = \begin{array}{|c|c|c|} \hline 1 & 3 & 11 \\ \hline 2 & 4 & 12 \\ \hline 5 & 7 & \\ \cline{1-2} 6 & 8 & \\ \cline{1-2} 9 & & \\ \cline{1-1} 10 & & \\ \cline{1-1} \end{array} .
$$

(The polynomial $I_{B_{3,3,2,2,1,1}}$ can be expressed as a sum of about 220 different monomials.)

$$
S_{2,2,2,2,2,2}\ ,\ B_{2,2,2,2,2,2} = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & 7 \\ \hline 6 & 8 \\ \hline 9 & 11 \\ \hline 10 & 12 \\ \hline \end{array} .
$$

(The polynomial $I_{B_{2,2,2,2,2,2}}$ can be expressed as a sum of about 700 different monomials.)

Now using these generators, we give the proof of Theorem 1. 4. For this purpose, we have only to substitute an element $\alpha \in E_k$ into the generators of the irreducible components of $S^p K^*$ for $k=1, 2, \cdots$, i. e., we put $R=\gamma_k(\alpha)$ and determine whether they are trivial or not as polynomials on E_k .

We first consider the case $k=1$ (codimension$=1$). In this case $\alpha \in E_1$ is a symmetric bilinear form on V. We substitute α into the generators of $S^p K^*$ listed up above. Then it is directly verified that the polynomials $I_{B_{3,3,1,1}}$, $I_{B_{5,5,1,1}}$, $I_{B_{5,4,2,1}}$, $I_{B_{5,3,3,1}}$, $I_{B_{5,3,2,1,1}}$, $I_{B_{4,4,1,1,1,1}}$, $I_{B_{4,3,3,1,1}}$, $I_{B_{4,3,2,2,1}}$, $I_{B_{3,3,3,3}}$ and $I_{B_{3,3,2,2,1,1}}$ reduce to zero and the rest of the generators are not trivial polynomials on E_1 . (In some cases, we use the system REDUCE 2 for complicated calculations once again.) The following is the list of non-trivial polynomials I_B after putting $R=\gamma_1(\alpha)$. For simplicity we write the determinant of the symmetric matrix

$$
\begin{pmatrix}
\alpha(X_1, X_1) & a(X_1, X_2) \cdots \alpha(X_1, X_r) \\
\alpha(X_2, X_1) & \alpha(X_2, X_2) \cdots \alpha(X_2, X_r) \\
& \cdots\cdots \\
\alpha(X_r, X_1) & \alpha(X_r, X_2) \cdots \alpha(X_r, X_r)
\end{pmatrix}
$$

as $\alpha_{12\cdots r, 12\cdots r}$. For example, $\alpha_{1,1}=\alpha(X_1, X_1)$.

$$I_{B_{2,2}} = \alpha_{12,12} .$$
$$I_{B_{4,4}} = \alpha_{12,12}{}^2 .$$
$$I_{B_{4,2,2}} = \alpha_{1,1}\cdot\alpha_{123,123} .$$
$$I_{B_{2,2,2,2}} = 3\alpha_{1234,1234} .$$
$$I_{B_{6,6}} = \alpha_{12,12}{}^3 .$$
$$I_{B_{6,4,2}} = \alpha_{1,1}\cdot\alpha_{12,12}\cdot\alpha_{123,123} .$$
$$I_{B_{6,2,2,2}} = \alpha_{1,1}{}^2\cdot\alpha_{1234,1234} .$$
$$I_{B_{4,4,4}} = \alpha_{123,123}{}^2 .$$
$$I_{B^{(1)}_{4,4,2,2}} = 3\alpha_{12,12}\cdot\alpha_{1234,1234} .$$
$$I_{B^{(2)}_{4,4,2,2}} = \alpha_{12,12}\cdot\alpha_{1234,1234} .$$
$$I_{B_{4,2,2,2,2}} = 6\alpha_{1,1}\cdot\alpha_{12345,12345} .$$
$$I_{B_{2,2,2,2,2,2}} = 15\alpha_{123456,123456} .$$

(It is already known that the generators of the invariant irreducible subspaces of $S^pE_1^*$ are expressed as products of the determinant of the form $\alpha_{12\cdots r,12\cdots r}$. See [1].) The intersection of the subspace $2S_{4,4,2,2}\subset S^3K^*$ and Ker γ_1^{3*} is a non-trivial irreducible subspace or $2S_{4,4,2,2}$ itself because one component of $2S_{4,4,2,2}$ is contained in Ker γ_1^{3*}. (See Remark (2) after Theorem 1. 4.) But, as we have already seen, $I_{B^{(1)}_{4,4,2,2}}$ is not an element of Ker γ_1^{3*} and hence we have $2S_{4,4,2,2}\cap$ Ker $\gamma_1^{3*}=S_{4,4,2,2}$, which is generated by the element $I_{B^{(1)}_{4,4,2,2}}-3I_{B^{(2)}_{4,4,2,2}}$. Summarizing these results, we obtain (1) of Theorem 1. 4.

Next we consider the case $k=2$ (codimension $=2$). In this case, using the system REDUCE 2, we substitute $R=\gamma_2(\alpha)$ $(\alpha\in E_2)$ into 11 polynomials $I_{B_{3,3,1,1}}, \cdots, I_{B_{3,3,2,2,1,1}}$ and $I_{B^{(1)}_{4,4,2,2}}-3I_{B^{(2)}_{4,4,2,2}}$, which are the generators of Ker γ_1^{p*}. Then we know that the polynomials $I_{B_{4,4,1,1,1,1}}$ and $I_{B_{3,3,3,3}}$ reduce to zero and the rest are non-trivial polynomials on E_2. (We omit these explicit expressions because they are lengthy.) Hence we have Ker $\gamma_2^{1*}=$ Ker $\gamma_2^{2*}=\{0\}$ and Ker $\gamma_2^{3*}=S_{4,4,1,1,1,1}+S_{3,3,3,3}$.

Finally we substitute $R=\gamma_3(\alpha)$ $(\alpha\in E_3)$ into the polynomials $I_{B_{4,4,1,1,1,1}}$ and $I_{B_{3,3,3,3}}$. Then in this case, these 2 polynomials are non-trivial on E_3 and therefore we have Ker $\gamma_k^{p*}=\{0\}$ for $k\geqq 3$ and $p\leqq 3$. Thus we complete the proof of Theorem 1. 4.

REMARK. (1) Using the metric (,) on V, we consider $R(X, Y)$ $(X, Y\in V)$ as a skew symmetric linear endomorphism of V, i. e., $R(X, Y)Z\in V$ is defined by $(R(X, Y)Z, W)=-R(X, Y, Z, W)$. Let $\{X_1, \cdots, X_n\}$ be an orthonormal base of V and we write $R(X_i, X_j, X_k, X_l)=R_{ijkl}$, as before. Then the linear endomorphism $R(X_1, X_2)$ is expressed in a matrix form:

$$\begin{pmatrix} 0 & R_{1212} & R_{1213} \cdots R_{121n} \\ -R_{1212} & 0 & R_{1223} \cdots R_{122n} \\ & \cdots\cdots\cdots & \\ -R_{121n} & -R_{122n} & -R_{123n} \cdots \quad 0 \end{pmatrix}.$$

If rank $R(X_1, X_2) \leqq 2$, then the first $(4, 4)$ principal minor of this matrix is zero, which is equal to the square of the Pfaffian $(R_{1212}R_{1234} - R_{1213}R_{1224} + R_{1214}R_{1223})^2$, and hence we have $I_{B_{3,3,1,1}}(X_1, X_2, X_3, X_4)(R)=0$. (Note that the rank of the skew symmetric matrix is always even.) Conversely if $I_{B_{3,3,1,1}}(X_1, \cdots, X_4)(R)=0$ for any $X_3, X_4 \in V$, then we have rank $R(X_1, X_2) \leqq 2$. Therefore the obstruction $S_{3,3,1,1} \subset \operatorname{Ker} \gamma_1^{2*}$ is essentially equivalent to the condition rank $R(X, Y) \leqq 2$ stated in Introduction. In the same way, in the case of codimension 2, we can check that the condition $S_{4,4,1,1,1,1} \subset \operatorname{Ker} \gamma_2^{3*}$ is equivalent to rank $R(X, Y) \leqq 4$ for $X, Y \in V$.

(2) Considering the results in Vilms [22], it is probable that in the case of codimension 1 the ideal $\sum_p \operatorname{Ker} \gamma_1^{p*}$ of the polynomial ring $\sum_p S^pK^*$ is generated by the elements of $S_{3,3,1,1} \subset S^2K^*$. By direct calculations, we can prove that the subspace $\operatorname{Ker} \gamma_1^{3*}$ of S^3K^* is contained in the ideal generated by $S_{3,3,1,1}$, but for the spaces $\operatorname{Ker} \gamma_1^{p*}$ $(p \geqq 4)$ we do not know whether it is true or not at present.

§ 3 The case of 4-dimensional Riemannian manifolds.

In this section we rewrite the remaining obstruction $S_{3,3,3,3} \subset \operatorname{Ker} \gamma_2^{3*}$ in a simple form, which is easy to calculate for a given Riemannian manifold. In the case $n=4$ the polynomial $I_{B_{3,3,3,3}}$ is an invariant of K with respect to the canonical action of $GL(4, \boldsymbol{R})$, i. e., $\rho(g)\cdot I_{B_{3,3,3,3}} = (\det g)^3 \cdot I_{B_{3,3,3,3}}$ for any $g \in GL(4, \boldsymbol{R})$ and hence we first investigate the structure and the generators of the ring of $GL(4, \boldsymbol{R})$-invariants of K for later use.

First, we prepare several lemmas. In the following we denote by F^G the ring of G-invariants of F, where F is the representation space of the group G.

LEMMA 3.1. *Let* $\operatorname{Sym}(n, \boldsymbol{C})$ *be the complex vector space of symmetric linear endomorphisms of* $\boldsymbol{C}^n$, *i. e.,* $\operatorname{Sym}(n, \boldsymbol{C}) = \{A : \boldsymbol{C}^n \longrightarrow \boldsymbol{C}^n \mid {}^tA = A\}$. *Then the ring of* $SO(n, \boldsymbol{C})$*-invariants of* $\operatorname{Sym}(n, \boldsymbol{C})$ *with respect to the natural action is a polynomial ring generated by* $\{\operatorname{Tr} A, \operatorname{Tr} A^2, \cdots, \operatorname{Tr} A^n\}$ $(A \in \operatorname{Sym}(n, \boldsymbol{C}))$.

PROOF. Let W be a subspace of $\operatorname{Sym}(n, \boldsymbol{C})$ consisting of diagonal matrices and we put

$H=\{g\in SO(n, \boldsymbol{C}) \mid g(W)\subset W\}$.

Then a generic element of $\mathrm{Sym}(n, \boldsymbol{C})$ is transformed into the element of W by the action of $SO(n, \boldsymbol{C})$. (More precisely, if the eigenvalues of $A\in\mathrm{Sym}(n, \boldsymbol{C})$ are all distinct, gAg^{-1} is a diagonal matrix for suitable $g\in SO(n, \boldsymbol{C})$.) Using this fact, it is easy to see that the natural map

$$j:\mathrm{Sym}(n, \boldsymbol{C})^{SO(n, \boldsymbol{C})}\longrightarrow W^{H}$$

is injective. We investigate the structure of H-invariants of W. We put $\tilde{H}=\{h\in H \mid h\cdot w=w \text{ for all } w\in W\}$. Then the quotient group $H/\tilde{H}$ acts on W effectively and W^{H} is isomorphic to $W^{H/\tilde{H}}$. It is directly checked that the orders of H and $\tilde{H}$ are $2^{n-1}\cdot n!$ and 2^{n-1}, respectively, and $H/\tilde{H}$ acts on W as a permutation of the diagonal elements. Since $H/\tilde{H}$ contains the permutation of i-th and j-th diagonal components, any $H/\tilde{H}$-invariant of W is a symmetric polynomial of n diagonal elements and hence $W^{H/\tilde{H}}$ is isomorphic to a polynomial ring $\boldsymbol{C}[\mathrm{Tr}\ B, \mathrm{Tr}\ B^{2}, \cdots, \mathrm{Tr}\ B^{n}]$ $(B\in W)$. Since $\mathrm{Sym}(n, \boldsymbol{C})^{SO(n, \boldsymbol{C})}$ contains elements of the form $\mathrm{Tr}\ A^{i}$ $(A\in\mathrm{Sym}(n, \boldsymbol{C}))$ and $j(\mathrm{Tr}\ A^{i})=\mathrm{Tr}\ B^{i}$, it follows that j is surjective and hence $\mathrm{Sym}(n, \boldsymbol{C})^{SO(n, \boldsymbol{C})}$ is a polynomial ring generated by $\{\mathrm{Tr}\ A, \cdots, \mathrm{Tr}\ A^{n}\}$. q. e. d.

Next we consider the $SO^{0}(p, q)$-invariants of $\mathrm{Sym}(p+q, \boldsymbol{R})$, where $SO^{0}(p, q)$ is the identity component of the group $\{g\in GL(p+q, \boldsymbol{R}) \mid {}^{t}gJg=J\}$ $(J=\mathrm{diag}(\overbrace{1, 1, \cdots, 1}^{p}, \overbrace{-1, -1, \cdots, -1}^{q}))$ and $\mathrm{Sym}(p+q, \boldsymbol{R})=\{A:\boldsymbol{R}^{p+q}\longrightarrow \boldsymbol{R}^{p+q} \mid {}^{t}AJ=JA\}$.

Lemma 3.2. *The ring of $SO^{0}(p, q)$-invariants of* $\mathrm{Sym}(p+q, \boldsymbol{R})$ *is isomorphic to the polynomial ring generated by* $\{\mathrm{Tr}\ A, \mathrm{Tr}\ A^{2}, \cdots, \mathrm{Tr}\ A^{p+q}\}$ $(A\in\mathrm{Sym}(p+q, \boldsymbol{R}))$.

Proof. We put $Q=\mathrm{diag}(\overbrace{1, 1, \cdots 1}^{p}, \overbrace{\sqrt{-1}, \sqrt{-1}, \cdots, \sqrt{-1}}^{q})$ and define a real Lie group homomorphism $h: SO^{0}(p, q)\longrightarrow SO(p+q, \boldsymbol{C})$ by $h(g)=QgQ^{-1}$ and a real linear map $c:\mathrm{Sym}(p+q, \boldsymbol{R})\longrightarrow\mathrm{Sym}(p+q, \boldsymbol{C})$ by $c(A)=QAQ^{-1}$. We construct a real homogeneous linear map $k:\mathrm{Sym}(p+q, \boldsymbol{R})^{SO^{0}(p, q)}\longrightarrow\mathrm{Sym}(p+q, \boldsymbol{C})^{SO(p+q, \boldsymbol{C})}$ in the following way. First, since the map c defined above is conjugate to a complexification of the real vector space $\mathrm{Sym}(p+q, \boldsymbol{R})$, c induces an injective real linear map $\tilde{c}$: {polynomial on $\mathrm{Sym}(p+q, \boldsymbol{R})$}$\longrightarrow${polynomial on $\mathrm{Sym}(p+q, \boldsymbol{C})$} naturally. Next we consider the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Sym}(p+q, \boldsymbol{R}) & \longrightarrow & \mathrm{Sym}(p+q, \boldsymbol{C}) \\ g\downarrow & & \downarrow h(g) \\ \mathrm{Sym}(p+q, \boldsymbol{R}) & \longrightarrow & \mathrm{Sym}(p+q, \boldsymbol{C}). \end{array} \qquad g\in SO^{0}(p, q)$$

From this diagram it follows that the $SO^{0}(p, q)$-invariants of $\mathrm{Sym}(p+q, \boldsymbol{R})$ is mapped by $\tilde{c}$ to the $SO(p+q, \boldsymbol{C})$-invariants of $\mathrm{Sym}(p+q, \boldsymbol{C})$. In fact, since $SO(p+q, \boldsymbol{C})$ is connected, we have only to check the invariance by the action of the Lie algebra $\mathfrak{o}(p+q, \boldsymbol{C})$ and this follows immediately from the fact that the differential of the group homomorphism $h: SO^0(p, q)\longrightarrow SO(p+q, \boldsymbol{C})$ is conjugate to a complexification of the real Lie algebra $\mathfrak{o}(p, q)$. Hence by restricting the map $\tilde{c}$, we obtain a real homogeneous linear map $k: \mathrm{Sym}(p+q, \boldsymbol{R})^{SO^0(p,q)}\longrightarrow \mathrm{Sym}(p+q, \boldsymbol{C})^{SO(p+q, C)}$. We have already known the generators of $\mathrm{Sym}(p+q, \boldsymbol{C})^{SO(p+q, C)}$ (Lemma 3.1) and it is easy to see that $\mathrm{Tr}\ A^i\in\mathrm{Sym}(p+q, \boldsymbol{R})^{SO^0(p,q)}$ ($A\in\mathrm{Sym}(p+q, \boldsymbol{R})$) is mapped by k to $\mathrm{Tr}\ c(A)^i$. Now let ϕ be an element of $\mathrm{Sym}(p+q, \boldsymbol{R})^{SO^0(p,q)}$. Then $k(\phi)$ is expressed as a polynomial of $\{\mathrm{Tr}\ B, \cdots, \mathrm{Tr}\ B^{p+q}\}$ with the complex coefficients. We write this polynomial in the following form:

$$k(\phi)(B)=\sum a_i f_i(\mathrm{Tr}\ B, \cdots, \mathrm{Tr}\ B^{p+q})+\sqrt{-1}\sum b_i g_i(\mathrm{Tr}\ B, \cdots, \mathrm{Tr}\ B^{p+q}),$$

where $B\in\mathrm{Sym}(p+q, \boldsymbol{C})$, a_i, $b_i\in\boldsymbol{R}$ and f_i, g_i are real polynomials. If $B=c(A)$, then $k(\phi)(B)$ is real valued, and hence we have

$$k(\phi)(c(A))=\sum a_i f_i(\mathrm{Tr}\ c(A), \cdots, \mathrm{Tr}\ c(A)^{p+q}).$$

This implies that ϕ is expressed as a real polynomial of $\{\mathrm{Tr}\ A, \cdots, \mathrm{Tr}\ A^{p+q}\}$ because k is injective. Therefore $\mathrm{Sym}(p+q, \boldsymbol{R})^{SO^0(p,q)}$ is generated by $\{\mathrm{Tr}\ A, \cdots, \mathrm{Tr}\ A^{p+q}\}$. Since these elements are independent, $\mathrm{Sym}(p+q, \boldsymbol{R})^{SO^0(p,q)}$ is a polynomial ring generated by $\{\mathrm{Tr}\ A, \cdots, \mathrm{Tr}\ A^{p+q}\}$. q. e. d.

Now we prove the following proposition. In the rest of this section V always means the real 4-dimensional vector space, unless otherwise stated.

PROPOSITION 3.3. *In the case $n=4$, the ring of invariants of K is isomorphic to the polynomial ring $\boldsymbol{R}[x_2, x_3, x_4, x_5, x_6]$ (deg $x_i=i$). The generator x_i ($i=2, \cdots, 6$) corresponds to the trace of the i-th power of the (6, 6)-matrix:*

$$\tilde{R}=\begin{pmatrix} R_{1234} & R_{1334} & R_{1434} & R_{2334} & R_{2434} & R_{3434} \\ -R_{1224} & -R_{1324} & -R_{1424} & -R_{2324} & -R_{2424} & -R_{3424} \\ R_{1223} & R_{1323} & R_{1423} & R_{2323} & R_{2423} & R_{3423} \\ R_{1214} & R_{1314} & R_{1414} & R_{2314} & R_{2414} & R_{3414} \\ -R_{1213} & -R_{1313} & -R_{1413} & -R_{2313} & -R_{2413} & -R_{3413} \\ R_{1212} & R_{1312} & R_{1412} & R_{2312} & R_{2412} & R_{3412} \end{pmatrix},$$

where $R_{ijkl}=R(X_i, X_j, X_k, X_l)$ and $\{X_1, X_2, X_3, X_4\}$ is a base of V.

PROOF. We have only to determine the structure of the ring of $SL(V)$-invariants because the $SL(V)$-invariants of the tensor space $V\otimes\cdots\otimes V$ are automatically $GL(V)$-invariants. We first construct a group homomorphism $f: SL(4, \boldsymbol{R})\longrightarrow SO^0(3, 3)$ in the following way. We fix a linear isomorphism $\psi: \wedge^2 V\longrightarrow\boldsymbol{R}^6$ once for all and define a symmetric bilinear map $\langle\!\langle\ ,\ \rangle\!\rangle$ on $\boldsymbol{R}^6$ by

$$\langle\!\langle w_1, w_2\rangle\!\rangle X_1\wedge X_2\wedge X_3\wedge X_4=\psi^{-1}(w_1)\wedge\psi^{-1}(w_2),$$

for w_1, $w_2\in\boldsymbol{R}^6$. Then $\langle\!\langle\ ,\ \rangle\!\rangle$ is a non-degenerate inner product of type (3, 3).

In fact, by putting $W_1=\psi(X_1\wedge X_2)$, $W_2=\psi(X_1\wedge X_3)$, $W_3=\psi(X_1\wedge X_4)$, $W_4=\psi(X_2\wedge X_3)$, $W_5=\psi(X_2\wedge X_4)$, $W_6=\psi(X_3\wedge X_4)$, it is directly checked that $\langle\langle W_1, W_6\rangle\rangle=-\langle\langle W_2, W_5\rangle\rangle=\langle\langle W_3, W_4\rangle\rangle=1$ and other $\langle\langle W_i, W_j\rangle\rangle=0$. Next we define a group homomorphism $f: SL(4, \boldsymbol{R})\longrightarrow GL(6, \boldsymbol{R})$ by

$$f(g)\cdot w=\psi(g\cdot\psi^{-1}(w)),$$

for $g\in SL(4, \boldsymbol{R})$ and $w\in\boldsymbol{R}^6$, where $g\cdot\psi^{-1}(w)$ implies the canonical action of $g\in SL(4, \boldsymbol{R})$ on $\psi^{-1}(w)\in\wedge^2 V$. Then it is easy to see that the equality

$$\langle\langle f(g)\cdot w_1, f(g)\cdot w_2\rangle\rangle=\langle\langle w_1, w_2\rangle\rangle$$

holds for $g\in SL(4, R)$ and $w_1, w_2\in\boldsymbol{R}^6$, i. e., $f(g)$ is an element of $O(3,3)$. Since $SL(4, \boldsymbol{R})$ is connected, we obtain a group homomorphism $f: SL(4, \boldsymbol{R})\longrightarrow SO^0(3, 3)$. (Actually, f is a double covering map onto $SO^0(3, 3)$.) From the decomposition in Lemma 1. 1, there is a surjective linear map $S^2(\wedge^2 V^*)\longrightarrow K$, which induces an $SL(V)$-equivariant inclusion $i: K^*\longrightarrow S^2(\wedge^2 V)$. Composing this inclusion with ψ, we obtain the following commutative diagram:

$$\begin{array}{ccccc} K^* & \xrightarrow{i} & S^2(\wedge^2 V) & \xrightarrow[\cong]{\psi} & S^2\boldsymbol{R}^6 \\ {\scriptstyle g}\downarrow & & \downarrow & & \downarrow{\scriptstyle f(g)} \\ K^* & \xrightarrow{i} & S^2(\wedge^2 V) & \xrightarrow[\cong]{\psi} & S^2\boldsymbol{R}^6 \end{array} \qquad g\in SL(4, \boldsymbol{R}).$$

Therefore the invariants of K is contained in the ring of $SO^0(3, 3)$-invariants of $S^2\boldsymbol{R}^6$, via the map $\psi\circ i$. (We identify $S^2\boldsymbol{R}^6$ and its dual space by the metric induced from $\langle\langle\ ,\ \rangle\rangle$.) Next using the metric $\langle\langle\ ,\ \rangle\rangle$ on $\boldsymbol{R}^6$, we consider an element $A\in S^2\boldsymbol{R}^6$ as a linear endomorphism of $\boldsymbol{R}^6$, which is symmetric with respect to $\langle\langle\ ,\ \rangle\rangle$. Then, by putting $p=q=3$ in Lemma 3. 2, the $SO^0(3, 3)$-invariants of $S^2\boldsymbol{R}^6$ is a polynomial ring generated by the elements $\{\mathrm{Tr}\, A, \mathrm{Tr}\, A^2, \cdots, \mathrm{Tr}\, A^6\}$. (Remark that $\mathrm{Tr}\, A$ is unchanged by the action $A\longmapsto PAP^{-1}$.) Now we express these invariants in terms of the components of the curvature tensor. First, we express $A\in S^2\boldsymbol{R}^6$ in a matrix form, using the base $\{W_1, \cdots, W_6\}$ of $\boldsymbol{R}^6$ defined above. If we put $A_{ij}=A(W_i, W_j)$, then $A: \boldsymbol{R}^6\longrightarrow\boldsymbol{R}^6$ is expressed in the form:

$$\begin{pmatrix} A_{16} & A_{26} & A_{36} & A_{46} & A_{56} & A_{66} \\ -A_{15} & -A_{25} & -A_{35} & -A_{45} & -A_{55} & -A_{65} \\ A_{14} & A_{24} & A_{34} & A_{44} & A_{54} & A_{64} \\ A_{13} & A_{23} & A_{33} & A_{43} & A_{53} & A_{63} \\ -A_{12} & -A_{22} & -A_{32} & -A_{42} & -A_{52} & -A_{62} \\ A_{11} & A_{21} & A_{31} & A_{41} & A_{51} & A_{61} \end{pmatrix}.$$

Since A_{16}, A_{15}, etc., correspond to R_{1234}, R_{1224}, etc., the matrix form of the element of K^* is the $(6, 6)$-matrix $\tilde{R}$ stated in this proposition. Hence the ring of invariants of K is generated by the elements $\{\mathrm{Tr}\,\tilde{R},\ \mathrm{Tr}\,\tilde{R}^2, \cdots, \mathrm{Tr}\,\tilde{R}^6\}$. But we have $\mathrm{Tr}\,\tilde{R}=0$ because $\mathrm{Tr}\,\tilde{R}=2(R_{1234}-R_{1324}+R_{1423})=0$ by Bianchi's identity. Thus to complete the proof of Proposition 3. 3, we have only to show that $\{\mathrm{Tr}\,\tilde{R}^2, \cdots, \mathrm{Tr}\,\tilde{R}^6\}$ are independent as polynomials on K. For this purpose we consider the case where $\{R_{ijkl}\}$ is expressed as a polynomial of $\{a_1, \cdots, a_5\}$ in the following way: $R_{1234}=(a_1+a_2)/2$, $R_{1423}=(a_3+a_4)/2$, $R_{1324}=(a_1+a_2+a_3+a_4)/2$, $R_{1212}=(a_1-a_2)^2/4$, $R_{1313}=\{(a_1+a_2+a_3+a_4)/2+a_5\}^2$, $R_{1414}=(a_3-a_4)^2/4$, $R_{2323}=R_{2424}=R_{3434}=1$ and other R_{ijkl} are all zero. Then the eigenvalues of the matrix $\tilde{R}$ is given by $\{a_1, a_2, a_3, a_4, a_5, -(a_1+a_2+a_3+a_4+a_5)\}$ and it can be easily proved that the 5 polynomials $\sum_{i=1}^{5} a_i^k+\{-(a_1+\cdots+a_5)\}^k$ $(k=2, \cdots, 6)$ are independent as polynomials of $\{a_1, \cdots, a_5\}$. Hence $\{\mathrm{Tr}\,\tilde{R}^2, \cdots, \mathrm{Tr}\,\tilde{R}^6\}$ are independent as polynomials on K and we complete the proof of Proposition 3. 3. q. e. d.

REMARK. (1) We complexify both the vector space K and the group $SL(V)$. Then the structure of the ring of invariants of K^c is already known. In fact the Lie algebra $\mathfrak{sl}(4, \boldsymbol{C})$ is isomorphic to $D_3=\mathfrak{o}(6, \boldsymbol{C})$ and it is known that the ring of invariants of the complex irreducible representation of D_n $(n\geqq 3)$ with highest weight $2\Lambda_1$ (with respect to the natural numbering) is isomorphic to the polynomial ring $\boldsymbol{C}[y_2, y_3, y_4, \cdots, y_{2n}]$. (See the table 3*a* in Schwarz [19, *p.* 181].)

(2) In the case $n=2$, K is a 1-dimensional vector space and the ring of invariants of K is isomorphic to the polynomial ring $\boldsymbol{R}[x_1]$. In the case $n=3$, K^* is equal to $S^2(\wedge^2 V)$ and since $\dim\wedge^2 V=\dim V=3$, the ring of invariants of K is isomorphic to the ring of invariants of the space $S^2 V$, which is isomorphic to $\boldsymbol{R}[x_3]$. But, for $n\geqq 5$, we do not know the structure of the ring of $GL(V)$-invariants of K.

By Proposition 3. 3, the generator of the space $S_{3,3,3,3}\subset\mathrm{Ker}\ \gamma_2^{3*}$ is expressed in a simple form $\mathrm{Tr}\,\tilde{R}^3$, but we can express this obstruction in a more geometrical form. First, using the metric on V, we consider $R\in K$ as a symmetric linear endomorphism of $\wedge^2 V$ and let $*: \wedge^2 V\longrightarrow\wedge^2 V$ be the star operator defined by the metric and a fixed orientation of V. If $\{X_1, \cdots, X_4\}$ is an oriented base of V, the matrix form of the endomorphism $*\circ R: \wedge^2 V\longrightarrow\wedge^2 V$ with respect to the base $\{X_1\wedge X_2, X_1\wedge X_3, X_1\wedge X_4, X_2\wedge X_3, X_2\wedge X_4, X_3\wedge X_4\}$ is just equal to $\tilde{R}$. Hence if $R\in\mathrm{Im}\ \gamma_2$, we have $\mathrm{Tr}\,(*\circ R)^3=0$. We extend this result in the following form.

THEOREM 3. 4. *The notations being as above,* $\mathrm{Tr}\,(*\circ R)^3=\mathrm{Tr}\,(*\circ R)^5=0$ *if* $R\in\mathrm{Im}\ \gamma_2$.

PROOF. Assume $R\in\mathrm{Im}\ \gamma_2$. Then we have $R=L\wedge L+M\wedge M$ for some symmetric endomorphisms L, M of V(cf. [10, *p.* 102]). We have only to prove the theorem in the case $R\in\mathrm{Im}\ \gamma_2$ is generic. Hence we may assume that L and M are non-singular endomorphisms. We denote by $(\ ,\ )$ the metric on $\wedge^2 V$ induced from the metric on V. Then we have

$$((R\circ*)(X\wedge Y),\ Z\wedge W)=(*(X\wedge Y),\ R(Z\wedge W))$$
$$=(*(X\wedge Y),\ L(Z)\wedge L(W))+(*(X\wedge Y),\ M(Z)\wedge M(W)).$$

Since $(*(X\wedge Y),\ Z\wedge W)=\det(X, Y, Z, W)$ for $X, Y, Z, W\in V$, we have

$$(*(X\wedge Y),\ L(Z)\wedge L(W))=\det(X, Y, L(Z),\ L(W))$$
$$=\det L\cdot\det(L^{-1}(X),\ L^{-1}(Y),\ Z,\ W)$$
$$=\det L\cdot(*(L^{-1}(X)\wedge L^{-1}(Y)),\ Z\wedge W).$$

Therefore we have

$$R\circ*=*\circ(\det L\cdot L^{-1}\wedge L^{-1}+\det M\cdot M^{-1}\wedge M^{-1}).$$

Using this formula, the following two equalities are directly proved.

$$(R\circ*)^3=(\det L+\det M)\cdot(R\circ*)+(\det L\cdot M\wedge M+\det M\cdot L\wedge L$$
$$+\det L\cdot ML^{-1}M\wedge ML^{-1}M+\det M\cdot LM^{-1}L\wedge LM^{-1}L)\circ*,$$

$$(R\circ*)^5=(\det L+\det M)^2\cdot(R\circ*)+2(\det L+\det M)\cdot(\det L\cdot M\wedge M+\det M\cdot L\wedge L+\det L\cdot ML^{-1}M\wedge ML^{-1}M+\det M\cdot LM^{-1}L\wedge LM^{-1}L)\circ*+(2\det LM\cdot R+(\det L)^2\cdot ML^{-1}M\wedge ML^{-1}M+(\det M)^2\cdot LM^{-1}L\wedge LM^{-1}L+(\det L)^2\cdot ML^{-1}ML^{-1}M\wedge ML^{-1}ML^{-1}M+(\det M)^2\cdot LM^{-1}LM^{-1}L\wedge LM^{-1}LM^{-1}L)\circ*.$$

From Bianchi's identity we have $\mathrm{Tr}\ (R\circ*)=\mathrm{Tr}\ (*\circ R)=\mathrm{Tr}\ \tilde{R}=0$ (see the proof of Proposition 3. 3). Since $L\wedge L,\ M\wedge M,\ ML^{-1}M\wedge ML^{-1}M$, etc., are all curvature type operators, we have $\mathrm{Tr}\ (L\wedge L)\circ*=\mathrm{Tr}\ (M\wedge M)\circ*=\mathrm{Tr}\ (ML^{-1}M\wedge ML^{-1}M)\circ*=0$, etc., from the same reason. (Remark that $ML^{-1}M$, $ML^{-1}ML^{-1}M$, etc., are symmetric endomorphisms of V.) Therefore we have $\mathrm{Tr}\ (*\circ R)^i=\mathrm{Tr}\ (R\circ*)^i=0$ for $i=3,\ 5$. q. e. d.

REMARK. (1) In the case $n\geqq 5$, we consider a 4-dimensional subspace V_4 of V and fix an orientation of V_4. Using the metric on V, we regard R as a symmetric linear endomorphism of $\wedge^2 V_4$. Then the same conclusions as in Theorem 3. 4 hold if $R\in\mathrm{Im}\ \gamma_2$. Hence the obstructions $\mathrm{Tr}\ (*\circ R)^3$, $\mathrm{Tr}\ (*\circ R)^5$ are useful in the case $n\geqq 4$.

(2) We consider the differential of the complexified quadratic map γ_2^C: $E_2^C\longrightarrow K^C$ in the case $n=4$. Then the rank of γ_{2*}^C at a generic point of E_2^C is 18. (See [5], or [7, p. 891].) Since $\dim K^C=20$ and the polynomials $\mathrm{Tr}\ (*\circ R)^3$, $\mathrm{Tr}\ (*\circ R)^5$ are independent as functions on K^C, the image $\mathrm{Im}\ \gamma_2^C$ ($\subset K^C$) is almost equal to the variety $\{R\in K^C|\mathrm{Tr}\ (*\circ R)^3=\mathrm{Tr}\ (*\circ R)^5=0\}$. But we do not know whether this variety is just equal to the closure $\overline{\mathrm{Im}\ \gamma_2^C}$ or not.

COROLLARY 3. 5. *The Riemannian symmetric spaces* $P^2(\boldsymbol{C})$, $SU(3)/SO(3)$ *and their non-compact dual spaces cannot be isometrically immersed into* *the Euclidean spaces in codimension* 2 *even locally.*

RROOF. Let R be the curvature of $P^2(\boldsymbol{C})$ or the curvature of $SU(3)/SO(3)$ restricted to some 4-dimensional subspace. Then the curvature of their non-compact dual spaces are given by $-R$. In particular Tr $(-*\circ R)^3=-$ Tr $(*\circ R)^3$. Hence we have only to prove Tr $(*\circ R)^3\neq 0$ for the spaces $P^2(\boldsymbol{C})$ and $SU(3)/SO(3)$.

(i) The case of $P^2(\boldsymbol{C})$. The curvature $R:\wedge^2 V\longrightarrow\wedge^2 V$ (V is the tangent space of $P^2(\boldsymbol{C})$ at the origin) is given by

$$\begin{aligned}
R(X_1\wedge Y_1)&=4X_1\wedge Y_1+2X_2\wedge Y_2\\
R(X_1\wedge X_2)&=X_1\wedge X_2+Y_1\wedge Y_2\\
R(X_1\wedge Y_2)&=X_1\wedge Y_2-Y_1\wedge X_2\\
R(Y_1\wedge X_2)&=Y_1\wedge X_2-X_1\wedge Y_2\\
R(Y_1\wedge Y_2)&=X_1\wedge X_2+Y_1\wedge Y_2\\
R(X_2\wedge Y_2)&=2X_1\wedge Y_1+4X_2\wedge Y_2,
\end{aligned}$$

where $\{X_1, Y_1, X_2, Y_2\}$ is a suitable oriented orthonormal base of V (see § 4). Then by direct calculations the eigenvalues of $*\circ R$ are given by $\{6,-2,-2,-2,0,0\}$ and hence we have Tr $(*\circ R)^3=192\neq 0$.

(ii) The case of $SU(3)/SO(3)$. Let $\mathfrak{su}(3)=\mathfrak{o}(3)+\mathfrak{m}$ be the canonical decomposition of the symmetric pair $(\mathfrak{su}(3), \mathfrak{o}(3))$. Then $\mathfrak{m}=\{X\in\mathfrak{su}(3)\mid {}^tX=X\}$. We use the following orthonormal base of $\mathfrak{m}$:

$$X_1=\sqrt{-1}\begin{pmatrix}1&&\\&0&\\&&-1\end{pmatrix},\ X_2=\frac{\sqrt{-1}}{\sqrt{3}}\begin{pmatrix}1&&\\&-2&\\&&1\end{pmatrix},\ X_3=\sqrt{-1}\begin{pmatrix}&1&\\1&&\\&&0\end{pmatrix},$$

$$X_4=\sqrt{-1}\begin{pmatrix}&&1\\&0&\\1&&\end{pmatrix},\ X_5=\sqrt{-1}\begin{pmatrix}0&&\\&&1\\&1&\end{pmatrix}.$$

Then the curvature $R:\wedge^2\mathfrak{m}\longrightarrow\wedge^2\mathfrak{m}$ is given by (up to a positive constant)

$$\begin{aligned}
R(X_1\wedge X_2)&=0\\
R(X_1\wedge X_3)&=X_1\wedge X_3+\sqrt{3}X_2\wedge X_3+X_4\wedge X_5\\
R(X_1\wedge X_4)&=4X_1\wedge X_4+2X_3\wedge X_5\\
R(X_1\wedge X_5)&=X_1\wedge X_5-\sqrt{3}X_2\wedge X_5+X_3\wedge X_4\\
R(X_2\wedge X_3)&=\sqrt{3}X_1\wedge X_3+3X_2\wedge X_3+\sqrt{3}X_4\wedge X_5\\
R(X_2\wedge X_4)&=0\\
R(X_2\wedge X_5)&=-\sqrt{3}X_1\wedge X_5+3X_2\wedge X_5-\sqrt{3}X_3\wedge X_4\\
R(X_3\wedge X_4)&=X_1\wedge X_5-\sqrt{3}X_2\wedge X_5+X_3\wedge X_4\\
R(X_3\wedge X_5)&=2X_1\wedge X_4+X_3\wedge X_5\\
R(X_4\wedge X_5)&=X_1\wedge X_3+\sqrt{3}X_2\wedge X_3+X_4\wedge X_5.
\end{aligned}$$

We restrict R to the 4-dimensional subspace V spanned by the oriented base $\{X_1, X_3, X_4, X_5\}$. Then the restricted curvature $R:\wedge^2 V\longrightarrow\wedge^2 V$ is given by

$$R(X_1\wedge X_3)=X_1\wedge X_3+X_4\wedge X_5$$

$$R(X_1 \wedge X_4) = 4X_1 \wedge X_4 + 2X_3 \wedge X_5$$
$$R(X_1 \wedge X_5) = X_1 \wedge X_5 + X_3 \wedge X_4$$
$$R(X_3 \wedge X_4) = X_1 \wedge X_5 + X_3 \wedge X_4$$
$$R(X_3 \wedge X_5) = 2X_1 \wedge X_4 + X_3 \wedge X_5$$
$$R(X_4 \wedge X_5) = X_1 \wedge X_3 + X_4 \wedge X_5.$$

It is easy to see that the eigenvalues of $*\circ R$ are given by $\{2, 2, 0, 0, 0, -4\}$ and hence we have Tr $(*\circ R)^3 = -48 \neq 0$.

Thus by Theorem 3.4 $P^2(\boldsymbol{C})$, $SU(3)/SO(3)$ and its non-compact dual spaces do not admit a solution of the Gauss equation in codimension 2.

q. e. d.

REMARK. (1) In the case of $P^2(\boldsymbol{C})$ and its dual space, these results can be proved by another method. First, since the dual space of $P^2(\boldsymbol{C})$ is a space of negative curvature, the corollary is obtained from the result in Ôtsuki [18, *p.* 233]. As for the space $P^2(\boldsymbol{C})$, Weinstein proved in [23] the following: Let M be an n-dimensional Riemannian manifold which is isometrically immersed into $\boldsymbol{R}^{n+2}$. Then M has positive sectional curvature if and only if the eigenvalues of $R : \wedge^2 V \longrightarrow \wedge^2 V$ are all positive. In our case, the space $P^2(\boldsymbol{C})$ has positive sectional curvature, but $R : \wedge^2 V \longrightarrow \wedge^2 V$ is not positive definite and hence it follows that $P^2(\boldsymbol{C})$ cannot be locally isometrically immersed into $\mathbf{R}^6$.

(2) It is known that $P^n(\boldsymbol{C})$ and $SU(n)/SO(n)$ can be globally isometrically imbedded into the Euclidean space of codimension n^2 and $(n^2+n+2)/2$, respectively (Kobayashi [14]) and it can be proved that $P^n(\boldsymbol{C})$ admits a solution of the Gauss equation in codimension n^2-1 (see § 4 (4)). But we do not know the least dimensional Euclidean space into which $P^n(\boldsymbol{C})$ $(n \geqq 2)$ or $SU(n)/SO(n)$ $(n \geqq 3)$ can be (locally or globally) isometrically immersed.

§ 4. Final remarks.

In this section we state some results and remarks on local isometric immersions of Riemannian manifolds.

(1) The character of S^4K^*.

By the same method as in § 1, we can determine the character of S^4K^*, using the system REDUCE 2. The result is as follows:

$S_{8,8} + S_{8,6,2} + S_{8,4,4} + S_{8,4,2,2} + S_{8,2,2,2,2} + S_{7,7,1,1} + S_{7,6,2,1} + 2S_{7,5,3,1} + S_{7,5,2,1,1}$
$+ S_{7,4,3,2} + S_{7,4,3,1,1} + S_{7,4,2,2,1} + S_{7,3,3,2,1} + S_{7,3,2,2,1,1} + S_{6,6,4} + 3S_{6,6,2,2} + S_{6,6,1,1,1,1}$
$+ S_{6,5,4,1} + S_{6,5,3,2} + 2S_{6,5,3,1,1} + 2S_{6,5,2,2,1} + S_{6,5,2,1,1,1} + 3S_{6,4,4,2} + 3S_{6,4,3,2,1} +$
$2S_{6,4,3,1,1,1} + 3S_{6,4,2,2,2}$
$+ S_{6,4,2,1,1,1,1} + S_{6,3,3,3,1} + 2S_{6,3,3,2,1,1} + S_{6,3,2,2,2,1} + S_{6,2,2,2,2,2} + 2S_{5,5,4,1,1} + 3S_{5,5,3,3}$
$+ 2S_{5,5,3,2,1} + 3S_{5,5,2,2,1,1} + S_{5,5,1,1,1,1,1,1} + 2S_{5,4,4,2,1} + S_{5,4,4,1,1,1} + 2S_{5,4,3,3,1} + 2S_{5,4,3,2,2}$

$+3S_{5,4,3,2,1,1}+S_{5,4,3,1,1,1,1}+2S_{5,4,2,2,2,1}+S_{5,4,2,2,1,1,1}+2S_{5,3,3,3,2}+2S_{5,3,3,2,2,1}$
$+S_{5,3,3,2,1,1,1}+S_{5,3,2,2,2,1,1}+2S_{4,4,4,4}+3S_{4,4,4,2,2}+3S_{4,4,3,3,1,1}+S_{4,4,3,2,2,1}$
$+S_{4,4,3,2,1,1,1}+3S_{4,4,2,2,2,2}$
$+S_{4,4,2,2,1,1,1,1}+S_{4,3,3,3,2,1}+2S_{4,3,3,2,2,1,1}+S_{4,3,2,2,2,2,1}$
$+S_{4,2,2,2,2,2,2}+S_{3,3,3,3,2,2}+S_{3,3,3,3,1,1,1,1}+S_{3,3,2,2,2,2,1,1}+S_{2,2,2,2,2,2,2,2}.$

By comparing with the character of $S^8E_3^*$, we know that $S_{5,5,1,1,1,1,1,1}$ is contained in Ker γ_3^{4*} (see Remark (2) at the end of § 1). But this relation is equivalent to the condition $(*)$ stated in Introduction (in the case $k=3$).

(2) The expressions of the generators of S^pK^* for $n\leqq 4$.

The generators of the invariant irreducible subspaces of K^*, S^2K^* and S^3K^* can be expressed in a simple form if the depth of the corresponding Young diagram is at most 4. In fact, let $\tilde{R}$ be the (6, 6)-matrix stated in Proposition 3. 3 and let $(\tilde{R}^k)_{ij}$ be the (i, j)-component of $\tilde{R}^k$. Then the generators of S^pK^* are expressed in the following form:

$p=1$ $\quad I_{B_{2,2}}=\tilde{R}_{61}$.

$p=2$ $\quad I_{B_{4,4}}=(\tilde{R}_{61})^2$,

$$I_{B_{4,2,2}}=\begin{vmatrix}\tilde{R}_{51} & \tilde{R}_{52}\\ \tilde{R}_{61} & \tilde{R}_{62}\end{vmatrix},$$

$$I_{B_{3,3,1,1}}=\frac{1}{2}(\tilde{R}^2)_{61},$$

$$I_{B_{2,2,2,2}}=\frac{1}{2}\mathrm{Tr}(\tilde{R}^2).$$

$p=3$ $\quad I_{B_{6,6}}=(\tilde{R}_{61})^3$,

$$I_{B_{6,4,2}}=\tilde{R}_{61}\begin{vmatrix}\tilde{R}_{51} & \tilde{R}_{52}\\ \tilde{R}_{61} & \tilde{R}_{62}\end{vmatrix},$$

$$I_{B_{6,2,2,2}}=\begin{vmatrix}\tilde{R}_{41} & \tilde{R}_{42} & \tilde{R}_{43}\\ \tilde{R}_{51} & \tilde{R}_{52} & \tilde{R}_{53}\\ \tilde{R}_{61} & \tilde{R}_{62} & \tilde{R}_{63}\end{vmatrix},$$

$$I_{B_{5,5,1,1}}=\frac{1}{2}\tilde{R}_{61}\cdot(\tilde{R}^2)_{61},$$

$$I_{B_{5,4,2,1}}=\begin{vmatrix}\tilde{R}_{61} & (\tilde{R}^2)_{61}\\ \tilde{R}_{62} & (\tilde{R}^2)_{62}\end{vmatrix},$$

$$I_{B_{5,3,3,1}}=\begin{vmatrix}\tilde{R}_{41} & \tilde{R}_{42} & \tilde{R}_{44}\\ \tilde{R}_{51} & \tilde{R}_{52} & \tilde{R}_{54}\\ \tilde{R}_{61} & \tilde{R}_{62} & \tilde{R}_{64}\end{vmatrix},$$

$$I_{B_{4,4,4}}=\begin{vmatrix}\tilde{R}_{31} & \tilde{R}_{32} & \tilde{R}_{34}\\ \tilde{R}_{51} & \tilde{R}_{52} & \tilde{R}_{54}\\ \tilde{R}_{61} & \tilde{R}_{62} & \tilde{R}_{64}\end{vmatrix},$$

$$I_{B^{(1)}_{4,4,2,2}}=\frac{1}{2}\tilde{R}_{61}\cdot\mathrm{Tr}(\tilde{R}^2),$$
$$I_{B^{(2)}_{4,4,2,2}}=(\tilde{R}^3)_{61}\ ,$$
$$I_{B_{3,3,3,3}}=\frac{1}{6}\mathrm{Tr}(\tilde{R}^3).$$

At present we can not express the remaining generators of S^3K^* in a simple form. (Compare with the results in [1], [3], [8].)

(3) Inverse formula of the Gauss equation. (The case of codimension 1.)

In [20, *p.* 199] Thomas obtained the inverse formula of the Gauss equation in the case of codimension 1. (See also Kawaguchi [12, *p.* 43].) Namely in the case $n\geqq 3$, if $R\in K$ is a generic element of Im γ_1, then the second fundamental form $\alpha=\gamma_1^{-1}(R)$ is uniquely determined from R (up to a sign). This formula has the following representation theoretic meaning. In § 2 we have already proved that if R is contained in the image of γ_1 , then

$$I_{B_{4,2,2}}=\begin{vmatrix} R_{1212} & R_{1213} \\ R_{1213} & R_{1313}\end{vmatrix}=\alpha_{11}\begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{12} & \alpha_{22} & \alpha_{23} \\ \alpha_{13} & \alpha_{23} & \alpha_{33}\end{vmatrix}$$

and

$$I_{B_{4,4,4}}=\begin{vmatrix} R_{1212} & R_{1213} & R_{1223} \\ R_{1213} & R_{1313} & R_{1323} \\ R_{1223} & R_{1323} & R_{2323}\end{vmatrix}=\begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{12} & \alpha_{22} & \alpha_{23} \\ \alpha_{13} & \alpha_{23} & \alpha_{33}\end{vmatrix}^2 .$$

Hence if $I_{B_{4,4,4}}\neq 0$, we have

$$\alpha_{11}{}^2=\frac{\begin{vmatrix} R_{1212} & R_{1213} \\ R_{1213} & R_{1313}\end{vmatrix}^2}{\begin{vmatrix} R_{1212} & R_{1213} & R_{1223} \\ R_{1213} & R_{1313} & R_{1323} \\ R_{1223} & R_{1323} & R_{2323}\end{vmatrix}},$$

which is just equal to Thomas' inverse formula.

(4) Solutions of the Gauss equation of $P^n(\boldsymbol{C})$ $(n\geqq 2)$ and its dual space in codimension n^2-1.

Let $\mathfrak{su}(n+1)=\mathfrak{s}(\mathfrak{u}(n)+\mathfrak{u}(1))+\mathfrak{m}$ be the canonical decomposition of the symmetric pair $(\mathfrak{su}(n+1),\ \mathfrak{s}(\mathfrak{u}(n)+\mathfrak{u}(1)))$ $(n\geqq 2)$. The space

$$\mathfrak{m}=\left\{\begin{pmatrix} \overbrace{0}^{n} & \overbrace{v}^{1} \\ -{}^t\bar{v} & 0\end{pmatrix}\ \middle|\ v\in\boldsymbol{C}^n\right\}$$

is identified with the tangent space of $P^n(\boldsymbol{C})$ at the origin. We fix an orthonormal base $\{X_1,\cdots,X_n,\ Y_1,\cdots,Y_n\}$ of $\mathfrak{m}$ by

$$X_i=\left(\begin{array}{c|c} 0 & 1 \\ \hline -1 & 0 \end{array}\right)^i \quad \text{and} \quad Y_i=\sqrt{-1}\left(\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}\right)^i$$

for $i=1, \cdots, n$. Then the curvature $R: \wedge^2\mathfrak{m} \longrightarrow \wedge^2\mathfrak{m}$ of $P^n(\boldsymbol{C})$ is given by

$$R(X_i\wedge X_j)=X_i\wedge X_j+Y_i\wedge Y_j,$$
$$R(X_i\wedge Y_i)=2X_i\wedge Y_i+2\sum_{k=1}^{n}X_k\wedge Y_k,$$
$$R(X_i\wedge Y_j)=X_i\wedge Y_j+X_j\wedge Y_i,$$
$$R(Y_i\wedge Y_j)=X_i\wedge X_j+Y_i\wedge Y_j,$$

for $1\leqq i,\ j\leqq n\ (i\neq j)$. Using the metric, we consider R as an element of K as usual. First we define $\alpha\in S^2\mathfrak{m}^*\otimes \boldsymbol{R}^{n^2-1}$ as follows: Let $\{e_{ij},\ \tilde{e}_{ij}\}_{1\leqq i<j\leqq n}$ be an orthonormal base of $\boldsymbol{R}^{n^2-n}$ and let $\{v_i\}_{2\leqq i\leqq n}$ be a base of $\boldsymbol{R}^{n-1}$ such that $\|v_i\|=(v_i,\ v_j)=2$ for $i,\ j=2, \cdots, n,\ i\neq j$. Then $\{e_{ij}(i<j),\ \tilde{e}_{ij}(i<j),\ v_i\ (2\leqq i\leqq n)\}$ is a base of $\boldsymbol{R}^{n^2-1}=\boldsymbol{R}^{n^2-n}\oplus\boldsymbol{R}^{n-1}$. We set $e_{ij}=-e_{ji}$ and $\tilde{e}_{ij}=\tilde{e}_{ji}$ for $i\neq j$. We define $\alpha: \mathfrak{m}\times\mathfrak{m}\longrightarrow \boldsymbol{R}^{n^2-1}$ by

$$\alpha(X_1,\ X_1)=ka(v_2+\cdots+v_n),$$
$$\alpha(Y_1,\ Y_1)=kb(v_2+\cdots+v_n),$$
$$\alpha(X_i,\ X_i)=\alpha(Y_i,\ Y_i)=v_i,\ \alpha(X_1,\ Y_1)=\alpha(X_i,\ Y_i)=0,$$
$$\alpha(X_1,\ X_i)=a\tilde{e}_{1i},\ \alpha(Y_1,\ Y_i)=b\tilde{e}_{1i},\ \alpha(X_1,\ Y_i)=ae_{1i},$$
$$\alpha(Y_1,\ X_i)=-be_{1i},\ \alpha(X_i,\ Y_j)=e_{ij},$$

and $\alpha(X_i,\ X_j)=\alpha(Y_i,\ Y_j)=\tilde{e}_{ij}$

for $2\leqq i,\ j\leqq n,\ i\neq j$, where $k=\sqrt{2/n(n-1)}$ and $a,\ b$ are the two real solutions of the quadratic equation $x^2-2knx+1=0$. Then by direct calculations α satisfies the Gauss equation $\gamma_{n^2-1}(\alpha)=R$.

Next we construct a solution of the Gauss equation of the dual space of $P^n(\boldsymbol{C})$. Note that the curvature of the dual space is given by $-R$. Let $\{e_{ij}\ (1\leqq i<j\leqq n),\ \tilde{e}_{ij}(1\leqq i<j\leqq n),\ e_i(2\leqq i\leqq n)\}$ be an orthonormal base of $\boldsymbol{R}^{n^2-1}$ and we set $e_{ij}=e_{ji},\ \tilde{e}_{ij}=\tilde{e}_{ji}$ for $i\neq j$. We define a symmetric bilinear map $\alpha^*: \mathfrak{m}\times\mathfrak{m}\longrightarrow \boldsymbol{R}^{n^2-1}$ by

$$\alpha^*(X_i,\ X_i)=k\ \delta_{1i}(\tilde{e}_{12}+\cdots+\tilde{e}_{1n})+(1-\delta_{1i})a_ie_i,$$
$$\alpha^*(Y_i,\ Y_i)=-k\ \delta_{1i}(\tilde{e}_{12}+\cdots+\tilde{e}_{1n})+(1-\delta_{1i})b_ie_i,$$
$$\alpha^*(X_i,\ Y_i)=-k\ \delta_{1i}(e_{12}+\cdots+e_{1n}),$$
$$\alpha^*(X_i,\ X_j)=-\alpha^*(Y_i,\ Y_j)=\tilde{e}_{ij} \quad \text{and} \quad \alpha^*(X_i,\ Y_j)=e_{ij},$$

for $1\leqq i,\ j\leqq n,\ i\neq j$, where $k=\sqrt{2/(n-1)}$ and $a_i,\ b_i\ (2\leqq i\leqq n)$ are real numbers such that $a_ib_i=-4$. Then by direct calculations it is easy to see that α^* satisfies the Gauss equation $\gamma_{n^2-1}(\alpha^*)=-R$.

In the case $n=2$ (dim $P^2(\boldsymbol{C})=4$), it is directly verified that the differential of the map $\gamma_3: S^2\mathfrak{m}^*\otimes\boldsymbol{R}^3\longrightarrow K$ is of maximal rank at α constructed above. Since dim $S^2\mathfrak{m}^*\otimes\boldsymbol{R}^3=30$ and dim $K=20$, the image of γ_3 contains an open subset of K. In particular we have Ker $\gamma_3^{p*}=\{0\}$ for all p

if $n=4$, i. e., there exists no polynomial relations of the curvature tensor in the case $M^4\subset\mathbf{R}^7$.

(5) A remark on Im γ_1.

We prove that the image of the map $\gamma_1: S^2V^*\longrightarrow K$ is not closed in a usual topology if $n\geqq 3$. Let $\{X_1, \cdots, X_n\}$ be an orthonormal base of the n-dimensional Euclidean vector space V and we define a symmetric linear map $R: \wedge^2 V\longrightarrow\wedge^2 V$ by

$$R(X_1\wedge X_2)=X_1\wedge X_2,$$
$$R(X_1\wedge X_3)=X_1\wedge X_3$$

and $$R(X_i\wedge X_j)=0$$

for other $X_i\wedge X_j$. It is clear that R satisfies Bianchi's identity and hence $R\in K$. We prove that $R\notin\mathrm{Im}\ \gamma_1$. In fact if R is contained in Im γ_1, then there exists a symmetric linear map $L: V\longrightarrow V$ such that $R=L\wedge L$. Then we have $R(X_1\wedge X_2)=X_1\wedge X_2=L(X_1)\wedge L(X_2)$, which implies that $\{X_1, X_2\}$ and $\{L(X_1), L(X_2)\}$ coincide. In the same way, using $R(X_1\wedge X_3)$, it follows that $\{X_1, X_3\}=\{L(X_1), L(X_3)\}$. Hence we have

$$L(X_1)=aX_1,$$
$$L(X_2)=\frac{1}{a}X_2,$$
$$L(X_3)=\frac{1}{a}X_3$$

for some $a\in\boldsymbol{R}\setminus\{0\}$. But $R(X_2\wedge X_3)=L(X_2)\wedge L(X_3)=\frac{1}{a^2}X_2\wedge X_3\neq 0$, which is a contradiction. Therefore R is not contained in Im γ_1. We remark that R satisfies the condition rank $R(X, Y)\leqq 2$ for any $X, Y\in V$. Next for a non-zero real number t we define a symmetric linear map $L_t: V\longrightarrow V$ by

$$L_t(X_1)=tX_1,$$
$$L_t(X_2)=\frac{1}{t}X_2,$$
$$L_t(X_3)=\frac{1}{t}X_3$$

and $$L_t(X_i)=0$$

for $4\leqq i\leqq n$. Then it is easy to see that $\lim_{t\to\infty} L_t\wedge L_t=R$. Clearly $L_t\wedge L_t$ is an element of Im γ_1 and therefore Im γ_1 is not a closed set. Note that in the case $n=2$, Im γ_1 is closed because γ_1 is a surjective map.

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