Hokkaido Mathematical Journal Vol. 14 (1985), p. 175-248

Cauchy Problem for Fuchsian Hyperbolic Operators

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(Received November 13, 1984)

0. Introduction.

In this paper we will be concerned with the Cauchy problem:

$$(0.1)\qquad \begin{cases} Pu=f \\ D_t^j u|_{t=0}=g_j, \qquad j=0, 1, \ldots, m-k-1, \end{cases}$$

for a class of differential operators $P(t, x, D_t, D_x)$ with ≪regular singularities≫ at $t=0$. Precisely, for two positive integers $k, m, k\leq m, P$ has the form:

$$(0.2)\qquad P=t^k P_m+t^{k-1}P_{m-1}+\ldots+P_{m-k},$$

where the P_{m-j} are of order $m-j, j=0, \ldots, k,$ and have C^{∞} coefficients defined on some cylinder $\boldsymbol{R}_t\times M(M=\boldsymbol{R}^n$ or, more generally, M is a C^{∞} n-dimensional manifold without boundary).

We suppose that P_m is *strictly hyperbolic* with respect to dt and call P a *Fuchsian hyperbolic operator of weight* $m-k$.

As in the case of an ordinary differential equation of Fuchs type (see e.g. [2], [24]) we are led to consider the (reduced) *indicial polynomial* of P, i.e., denoting by a_{m-j} the coefficient of D_t^{m-j} in P_{m-j} and supposing $a_m\equiv 1$, the k-th order polynomial:

$$(0.3)\qquad I_P(x, \zeta)=\sum_{j=0}^{k}\left(\frac{1}{\sqrt{-1}}\right)^{m-j} a_{m-j}(0, x)\zeta(\zeta-1)\ldots(\zeta-(k-j-1)),$$

$x\in M,\ \zeta\in\boldsymbol{C}$.

To make (0.1) meaningful, at least at a formal power series level, one has to require that all the traces of u at $t=0$ can be recovered from the Cauchy data g_j and the corresponding traces of f.

Implementing this fact is equivalent to impose the following *Fuchs condition* on P:

$$(0.4)\qquad I_P(x;\zeta)\neq 0,\ \forall x\in M,\ \forall\zeta\in\boldsymbol{Z}_+=\{0, 1, \ldots\}.$$

Problem (0.1), even for more general classes of degenerate operators P, has been studied by several authors starting with the pioneering work of Baouendi and Goulaouic [4] (see also [3]). In [4], under condition (0.4), Cauchy-Kowalewski and Holmgren type theorems (among others) were proved supposing the coefficients of P to be analytic in the space variables and sufficiently regular with respect to t. More recently, a deep analysis of Cauchy problem (0.1) for Fuchsian hyperbolic operators with analytic coefficients has been performed by Tahara [19] in the hyperfunction framework. In this important work existence and uniqueness results are proved as well as structure theorems concerning the singularities of hyperfunction solutions of the equation $Pu=f$.

Operators with regular singularities have also been considered by Kashiwara and Oshima [15] in the analytic setting.

It is worth noting that Fuchsian hyperbolic operators of the form (0.2) are microlocal models of operators with multiple characteristics of variable multiplicity and non-involutive intersection. From this point of view we mention here the work of Ôaku [17] in the analytic category and B. L. P. [6] in the C^∞ setting (see also the references quoted therein).

The above mentioned works give an almost complete picture of problem (0.1) in the analytic-hyperfunction situation. We think, however, that in the C^∞ setting the available results concerning the Cauchy problem (0.1) are far from being as complete as in the analytic case.

The main contributions have been given by Tahara in a long series of papers (see [20], [21]), where the author considers the Cauchy problem (0.1) for a wide class of operators including (0.2) and gives various existence and uniqueness results when the data f and g_j are smooth functions or belong to suitable Sobolev spaces which take into account the t-degeneracy. The technique of proof relies essentially on energy estimates for abstract singular equations and on a tricky reduction of (0.1) to an equivalent $m\times m$ singular system.

Contributions to the study of hyperbolic singular systems have been given by Alinhac [1].

We mention also the important contribution given by Roberts [18] in proving a Calderon type local uniqueness result for smooth flat solutions of Cauchy problem (0.1).

However, as far as we know, results concerning C^∞-singularities of distribution solutions of problem (0.1) are still lacking. The aim of this paper is to fill in part the gap existing between the analytic and C^∞ analysis of problem (0.1).

To be definite, we will suppose that the Cauchy data g_j are distributions on M while f is a *regular distribution*, i.e. $f\in C^\infty(\boldsymbol{R}_t;\mathscr{D}'(M))$ and:

(0.5) $$WF(f)\cap\{t, x, \tau, \xi)\in T^*(\boldsymbol{R}\times M)\backslash 0 \mid t\neq 0, \xi=0\}=\phi.$$

For regular distributions an adapted notion of wave front set will be used; precisely, we say that a point $(x, \xi)\in T^*M\backslash 0$ *does not* belong to $\partial WF(f)$ iff for some classical pdo $B(x, D_x)$, elliptic near (x, ξ), we have $Bf\in C^\infty(]-\varepsilon, \varepsilon[\times M)$ for some $\varepsilon>0$ (for this notion of *boundary wave front set* we refer the reader to Chazarain [8] and Melrose-Sjöstrand [16]). In the statement of our main result we denote by $\Phi_j^t, j=1,\dots, m$, the Hamiltonian flow on $T^*M\backslash 0$ of the *hyperbolic roots* $\tau=\lambda_j(t, x, \xi)$ of the equation $\sigma_m(P_m)$ $(t, x, \tau, \zeta)=0$. For simplicity we state our theorem in the case where M is a compact manifold.

THEOREM. *Let P a Fuchsian hyperbolic operator of weight $m-k$, defined in $\boldsymbol{R}\times M$ and satisfying condition* (0. 4).

Then for every regular distribution f and for every Cauchy data $g_j\in\mathscr{D}'(M)$, $j=0, 1,\cdots, m-k-1$, there exists a unique regular distribution u which solves (0. 1). *Furthermore, the following description of the singularities holds*:

i) $\partial WF(u)\subset\partial WF(f)\cup\bigcup_{j=0}^{m-k-1} WF(g_j)$.

ii) $WF(u|_{\{t\neq 0\}\times M})\subset\{(t, x, \tau, \xi)\mid t\neq 0, (t, x, \tau, \xi)\in WF(f)\}\cup$
$\bigcup_{j=1}^{m}\{(t, x, \lambda_j(t, x, \xi), \xi)\mid\exists s, \frac{s}{t}\in]0, 1[, \exists(y, \eta)\in T^*M\backslash 0,$
$(s, y, \lambda_j(s, y, \eta), \eta)\in WF(f), (x, \xi)=\Phi_j^{t-s}(y, \eta)\}\cup$
$\bigcup_{j=1}^{m}\{(t, x, \lambda_j(t, x, \xi), \xi)\mid t\neq 0,$
$\exists(y, \eta)\in\partial WF(f)\cup\bigcup_{r=0}^{m-k-1} WF(g_r), (x, \xi)=\Phi_j^t(y, \eta)\}$.

iii) *Denoting by N^*M the conormal bundle of $\{0\}\times M$ in $\boldsymbol{R}\times M$, we have*:

$WF(u)\cap N^*M=WF(f)\cap N^*M$.

iv) *If for some $x_0\in M$ we have*:

a) $\pi^{-1}(0, x_0)\cap WF(f)=\phi$ (π: $T^*(\boldsymbol{R}\times M)\backslash 0\to\boldsymbol{R}\times M$ *is the canonical projection*),

b) $(x_0, \xi_0)\in\bigcup_{j=0}^{m-k-1} WF(g_j)$, *for some* $\xi_0\neq 0$,

then there exists at least one $j\in\{1,\dots, m\}$ such that either $WF(u|_{\{t>0\}\times M})$ or $WF(u|_{\{t<0\}\times M})$ contains a small arc of the bicharacteristic defined by the factor $\tau-\lambda_j(t, x, \xi)$ and issued from the point $(0, x_0, \lambda_j(0, x_0, \xi_0), \xi_0)$.

We make some comments concerning the theorem.

Existence and uniqueness will be proved by constructing (see Chapt. 4) a right and a left parametrix for P. This construction and the preparations which are needed form the core of the paper. We have been unable to construct parametrices directly for P(this has been done in a very particular case in [7]) and therefore we decided to follow a somewhat different procedure. In Chapt. 1 we show how an equation $Pu=f$ for an operator of weight 0, can be reduced to an equivalent (mod. $C^\infty(\boldsymbol{R}\times M)$) singular system of the form:

$$(0.6) \qquad \mathscr{P}u=(I_N tD_t-tA(t, x, D_x)-B(t, x, D_x))u=f,$$

where $A(t)$ and $B(t)$ are $N\times N$ matrices of classical pdo's of order 1 and 0 respectively, having the following crucial properties:

i) $\sigma_1(A)(t, x, \xi)$ is a diagonal matrix whose eigenvalues are the hyperbolic roots $\lambda_j(t, x, \xi)$, $j=1,\dots, m$.

ii) $\mathrm{Re}\ \sigma_0(B)(0, x, \xi)\leq -I_N$, for every $(x, \xi)\in T^*M\backslash 0$.

In proving ii) we exploit in an essential way the Fuchs condition (0.4). We point out that our reduction is rather different from the one performed by Tahara [20] (the operators in (0.6) are classical pdo's and, more substantially, they depend smoothly on t, contrary to what happens in [20]). In our reduction N is larger than m, precisely $N=m(m+1)/2$.

We observe that for *scalar* operators of the form (0.6) left and right parametrices can be obtained with only minor modifications following the explicit constructions performed by Hanges in [12]. Therefore, having the system (0.6), one is tempted to decouple this system as in the classical strictly hyperbolic case (see e.g. Taylor [22]), by reducing it via an intertwining elliptic operator, to a system of the same type with $B(t)$ in a diagonal (or block diagonal) form.

Unfortunately, the classical decoupling procedure breaks down in our case due to the t-degeneracy. In Chapt. 3 we show that system (0.6) can actually be decoupled for large values of $t|\xi|$. Roughly speaking, supposing A and B to be independent of (t, x) and taking partial Fourier transform with respect to x in (0.6), by putting $z=t|\xi|$ we get a system of the form:

$$(0.7) \qquad I_N zD_z-zA(\xi')-B(\xi'), \quad \xi'=\xi/|\xi|.$$

By a well known classical procedure (Cfr. [2], [24]), the hypothesis on $A(\xi')$ allows to find two power series $q(\xi', z)=\sum_{j\geq 0} q_{-j}(\xi')z^{-j}$ (with $q_0(\xi')=I_N$) and $\tilde{b}(\xi', z)=\sum_{j\geq 0} b_{-j}(\xi')\, z^{-j}$ (where the matrices $b_{-j}(\xi')$ are block-diagonal) such that the following equality holds at a formal power series level:

(0.8) $$(I_N zD_z - zA(\xi') - B(\xi'))q(\xi', z) = q(\xi', z)(I_N zD_z - zA(\xi') - \tilde{b}(\xi', z)).$$

To pass from the formal level to a correct operator level we work within a class of pdo's whose symbols have a suitable behaviour in the t-variable. These symbols and the corresponding operators are studied in Chapt. 2. Precisely, we work with symbols $a(t, x, \xi) \in S^{p,q}$ which satisfy (local) estimates of the type:

(0.9) $$|\partial_t^j \partial_x^\alpha \partial_\xi^\beta a(t, x, \xi)| \leq \text{const.}\ (1+|\xi|)^{p-|\beta|}(|t|+1/|\xi|)^{q-j}.$$

Summing up, in Chapt. 3 we prove that there exist an invertible matrix $Q \in OPS^{0,0}$ and a matrix $\tilde{B} \in OPS^{0,0}$ such that, putting $\tilde{\mathscr{P}} = I_N tD_t - tA(t, x, D_x) - \tilde{B}(t, x, D_x)$ we have:

(0.10) $$\mathscr{P} Q = Q \tilde{\mathscr{P}},$$

modulo operators which map regular distributions into smooth functions.

The matrix $\tilde{B}$ has the property that its symbol is (block) diagonal for $t|\xi|$ large. In Chapt. 4 we construct parametrices for the system $\tilde{\mathscr{P}}$ following (in spirit, if not in detail) the work of Hanges referred to above.

Combining the left and right parametrices with known existence and uniqueness results when the data in (0.1) are smooth functions, we get the first part of the Theorem.

Chapt. 5 is devoted to the analysis of the singularities. We point out that operators in the classes $S^{p,q}$ are not (in general) microlocal with respect to t at $t=0$; precisely, they do not preserve distributions whose wave front set is disjoint from the conormal bundle N^*M. To get some control on the behaviour of the solutions at $t=0$ we found it convenient to use the boundary wave front set $\partial WF(\cdot)$, a set which is preserved by the action of $OPS^{p,q}$.

Properties i) and ii) in the Theorem are proved by a direct inspection of the left parametrix. To prove properties iii) and iv) we rely heavily on the arguments contained in B. L. P. [6]. A consequence of (iv) is that, supposing $f \in C^\infty$, singularities of the Cauchy data g_j give rise to singularities of u that propagate into $t>0$ or $t<0$ on some (hyperbolic) bicharacteristic, a property which does not seem to be a priori obvious.

An important related question is to decide whether a branching of the singularities phenomenon may occur for a solution of Cauchy problem (0.1), i.e. decide when a singularity of u on some (hyperbolic) bicharacteristic give rise to the appearence of a singularity of u on a different bicharacteristic.

In Chapt. 5 (see Theorem 5.5) we give sufficient conditions which guarantee that such a phenomenon occur. A typical example is given by the

Euler-Poisson-Darboux equation:

$$Pu=t(D_t^2-\sum_j D_{x_j}^2)u+\alpha(t,\ x)D_t u+\sum_j\beta_j(x,\ t)D_{x_j}u+\gamma(t,\ x)u=0,$$

with $u|_{t=0}=g$, u being a regular distribution. Under the condition:

$$(0.11)\qquad 2i\zeta-\alpha(0,x)\pm\sum_j\beta_j(0,x)\xi_j/|\xi|\neq 0, \forall\zeta\in\boldsymbol{Z}_+, \forall(x,\ \xi)\in WF(g),$$

we conclude that $WF(u)$ contains both the bicharacteristics issued from the points $(0,\ x,\pm|\xi|,\ \xi)$, $(x,\ \xi)\in WF(g)$.

We do not know if (0.11) is also a necessary condition, but, as simple examples show, if no condition is imposed on P it can happen that u has singularities only along one of the two bicharacteristics.

To conclude this introduction we point out that for the notation used in the paper we refer the reader to Hörmander [13], with one exception: if S denotes a class of symbols, by $S(p\times q)$ we denote the $p\times q$ matrices with entries in S and by OPS(resp. $OPS(p\times q)$) we denote the related classes of pdo's.

1. Fuchsian operators.

We define a class of Fuchsian (hyperbolic) operators which is the main object of study in the present paper; in the differential case this class is contained in the general class of operators of Fuchsian type introduced by Baouendi and Goulaouic [4].

DEFINITION 1.1. *Let M be an n-dimensional C^∞ manifold without boundary. By $F^m_{m-k}(\boldsymbol{R}\times M)(m,\ k\in\boldsymbol{Z}_+,\ 1\leq k\leq m)$ we denote the class of all operators P, defined on the cylinder $\boldsymbol{R}\times M$, of the following type:*

$$(1.1)\qquad \begin{cases} P=\sum_{j=0}^{k} t^{k-j}P_{m-j}(t,\ x,\ \partial_t,\ D_x) \\ P_{m-j}=\sum_{h=0}^{m-j} B_{m-j-h,j}(t,\ x,\ D_x)\partial_t^h,\ j=0,\cdots,k, \end{cases}\qquad t\in\boldsymbol{R},\ x\in M$$

where $B_{m-j-h,j}\in OPS^{m-j-h}_{cl}(M)$ are classical pdo's of order $m-j-h$, depending smoothly on t.

Furthermore, the following condtions are satisfied:

1) *For $j=0$, $h=m$, $B_{0,0}(t,\ x,\ D_x)=1$.*

2) *For every $t\in\boldsymbol{R}$, $(x,\ \xi)\in T^*M\backslash 0$, the polynomial*

$$\boldsymbol{C}\ni\tau\longrightarrow\sum_{h=0}^{m}\sigma_{m-h}(B_{m-h,0})(t,\ x,\ \xi)\tau^h$$

has m simple roots $\tau=\sqrt{-1}\ \lambda_j(t,x,\xi)$, $j=1,\ldots,m$, where the $\lambda_j\in C^\infty(\boldsymbol{R}_t\times T^*M\backslash 0)$ are real functions, which will be called the hyperbolic roots of P.

To every $P\in F^m_{m-k}(\boldsymbol{R}\times M)$ we associate the (reduced) *indicial polynomial*:

$$
(1.2) \qquad \begin{cases} I_P(x, \xi; \zeta) = \sum_{j=0}^{k} \sigma_0(B_{0,j})(0, x, \xi)\zeta(\zeta-1)\dots(\zeta-(k-j-1)), \\ (x, \xi) \in S^*M, \ \zeta \in \boldsymbol{C}. \end{cases}
$$

We say that $P \in F^m_{m-k}(\boldsymbol{R} \times M)$ satisfies the *Fuchs condition* (*F. c.*) iff:

$$
(1.3) \qquad \forall (x, \xi) \in S^*M, \ \forall \zeta \in \boldsymbol{Z}_+ : I_P(x, \xi; \zeta) \neq 0.
$$

When $k=m$ it may be convenient to rewrite an operator $P \in F^m_0(\boldsymbol{R} \times M)$ in the following form:

$$
(1.4) \qquad \begin{cases} P = P(t, x, t\partial_t, D_x) = \sum_{j=0}^{m} Q_{m-j}(t, x, t\partial_t, D_x) \\ Q_{m-j} = \sum_{h=0}^{m-j} t^{m-j-h} A_{m-j-h,j}(t, x, D_x)(t\partial_t)^h, \ j=0, \dots, m, \end{cases}
$$

where $A_{m-j-h,j} \in OPS^{m-j-h}_{cl}(M)$ (depending smoothly on t). This is a trivial consequence of the identity

$$
(1.5) \qquad t^r \partial^r_t = t\partial_t(t\partial_t - 1)\dots(t\partial_t - (r-1)), \ r \in \boldsymbol{Z}_+ r \geq 1.
$$

Let us remark that both the hyperbolic roots and the indicial polynomial are unchanged and observe that with the new notation (1. 4) we have:

$$
(1.6) \qquad I_P(x, \xi; \zeta) = \sum_{j=0}^{m} \sigma_0(A_{0,j})(0, x, \xi)\zeta^{m-j}.
$$

We now list some general properties of the classes $F^m_{m-k}(\boldsymbol{R} \times M)$ which are related with the algebraic structure of these operators.

i) Let $P \in F^m_{m-k}(\boldsymbol{R} \times M)$ with $k<m$; then

$$
P(t\cdot) = \tilde{P}(\cdot),
$$

for a $\tilde{P} \in F^m_{m-(k+1)}(\boldsymbol{R} \times M)$ with the same hyperbolic roots of P. Moreover:

$$
(1.7) \qquad I_{\tilde{P}}(x, \xi; \zeta) = (\zeta - (k-m)) I_P(x, \xi; \zeta).
$$

ii) Let $P \in F^m_0(\boldsymbol{R} \times M)$ be written in the form (1.4); then:

$$
(1.8) \qquad P(t, x, t\partial_t, D_x)(t\cdot) = t(P(t, x, t\partial_t + 1, D_x)\cdot).
$$

iii) Let $P \in F^m_0(\boldsymbol{R} \times M)$ be a differential operator on $\boldsymbol{R} \times M$, satisfying F.c. There exist differential operators on M, $L^P_{h,j}(x, D_x)$ of order h, j, $h=0, 1, \dots$, $j \geq h$, such that for every $u \in C^\infty(\boldsymbol{R}_t; \mathscr{D}'(\mathrm{M}))$ and for every $j=0, 1, \dots$, the following relations hold:

$$
(1.9) \qquad \partial^j_t u|_{t=0} = \sum_{r=0}^{j} L^P_{j-r,j}(x, D_x)(\partial^r_t(Pu)|_{t=0}).
$$

Moreover $L^P_{0,j}(x, D_x) = 1/I_P(x; j)$.

The proof of the above facts are left to the reader, adding only as a remark to iii) that in the differential case the indicial polynomial of an operator $P\in F^m_{m-k}(\boldsymbol{R}\times M)$ does not depend on ξ since the $B_{0,j}$ in (1.1) are in fact C^∞ functions on M.

Properties i)-iii) will be used freely in the sequel without reference.

The main purpose of this paper is the study of the Cauchy problem:

$$\text{(P. C.)}\qquad \begin{cases} Pu=f & \text{in } \boldsymbol{R}_t\times M \\ \partial_t^j u|_{t=0}=g_j & \text{in } M,\ j=0,\ 1,\dots,\ m-k-1, \end{cases}$$

for a differential operator $P\in F^m_{m-k}(\boldsymbol{R}\times M)$ satisfying F.c. (note that for $k=m$ no Cauchy data is given at $t=0$). We will be concerned with the case where $f\in C^\infty(\boldsymbol{R}_t;\mathscr{D}'(M))$ and the $g_j\in\mathscr{D}'(M)$ and prove existence and uniqueness results in the class $u\in C^\infty(\boldsymbol{R}_t;\mathscr{D}'(M))$; moreover we will relate C^∞ singularities of u to those of f and the g_j. For convenience the Cauchy problem will be studied supposing M a compact manifold; however local results in $\boldsymbol{R}_t\times\boldsymbol{R}^n_x$ will be proved and global results in $\boldsymbol{R}_t\times\boldsymbol{R}^n_x$ could be obtained as well provided suitable assumptions on the behaviour of P at infinity are made.

When f and the Cauchy data g_j are smooth functions we have the following result.

THEOREM 1.1. *Let $P\in F^m_{m-k}(\boldsymbol{R}\times M)$ (M compact) be a differential operator satisfying F. c.. Then for every $f\in C^\infty(\boldsymbol{R}\times M)$ and for every $g_j\in C^\infty(M)$, $j=0,\dots,m-k-1$, there exists a unique solution $u\in C^\infty(\boldsymbol{R}\times M)$ of the Cauchy problem (P. C.).*

The above result is essentially contained in Tahara [20]. We shall also need local existence and uniqueness results in $\boldsymbol{R}_t\times\boldsymbol{R}^n$.

THEOREM 1.2. *Let $P\in F^m_{m-k}(\boldsymbol{R}\times\boldsymbol{R}^n)$ be a differential operator satisfying F. c. Let $U\subset\subset\boldsymbol{R}^n$ and denote by $I\subset\boldsymbol{R}$ an interval containing the origin. Then:*

a) For every $f\in C_0^\infty(I\times U)$ and for every $g_j\in C_0^\infty(U)$, $j=0,1,\dots,m-k-1$, there exists $u\in C^\infty(I\times U)$ such that $Pu=f$ in $I\times U$, $\partial_t^j u|_{t=0}=g_j$, in U, $j=0,\dots,m-k-1$.

b) Let us suppose that

$$\text{(1.10)}\qquad \sup_{\substack{(t,x)\in\boldsymbol{R}\times\boldsymbol{R}^n\\ |\xi|=1\\ j=1,\dots,m}}|\lambda_j(t,x,\xi)|=\lambda<+\infty,$$

where the λ_j are the hyperbolic roots of P. Define

$$\text{(1.11)}\qquad C(t_0,x_0)=\begin{cases}\{(t,x)\in\boldsymbol{R}\times\boldsymbol{R}^n \mid t\geq 0,|x-x_0|\leq\lambda(t_0-t)\}, & t_0>0\\ \{(t,x)\in\boldsymbol{R}\times\boldsymbol{R}^n \mid t\leq 0,|x-x_0|\leq\lambda(t-t_0)\}, & t_0<0.\end{cases}$$

Then, if $u\in C^\infty(I\times U)$ and for some cone $C(t_0,x_0)\subset I\times U$ we have $Pu=0$ in $C(t_0,x_0)$, $\partial_t^j u|_{t=0}=0$ in $C(t_0,x_0)\cap\{t=0\}$, $0\leq j\leq m-k-1$, it follows *that* $u=0$ *in* $C(t_0, x_0)$.

PROOF. a) Let $C_r \subset \boldsymbol{R}^n$ be an open cube of size $r>0$ containing $\bar{U}$. Take $\theta(x) \in C_0^\infty(\boldsymbol{R}^n)$ with $\theta \equiv 1$ on $\bar{C}_r$ and $\theta(x)=0$ on $\boldsymbol{R}^n \backslash C_{2r}$. Defining $\hat{P}(t, x, \partial_t, D_x)=P(t, \theta(x)\ x, \partial_t, D_x)$, $x \in \bar{C}_{2r}$, we see that $\hat{P}=P$ in $\boldsymbol{R}_t \times U$ and $\hat{P}$ defines in a natural way a differential operator, still denoted by $\hat{P}$, on $\boldsymbol{R} \times T^n$, $T=\boldsymbol{R}/2r\boldsymbol{Z}$. Obviously $\hat{P} \in F_{m-k}^m(\boldsymbol{R} \times T^n)$ and the F.c. is preserved. Since f and the g_j can be lifted to $\boldsymbol{R} \times T^n$ and T^n respectively as smooth functions, the conclusion follows from Th. 1.1.

To prove b) we rely on a result of Roberts [18] saying that for every neighborhood ϑ of a point $(0, x_0) \in \boldsymbol{R} \times \boldsymbol{R}^n$ there exists another neighborhood $\vartheta' \subset \vartheta$ of the same point such that if $u \in C^\infty(\vartheta)$, $\partial_t^j\ u|_{t=0}=0$ in $\vartheta \cap \{t=0\}$, $j=0, 1, \ldots, m-k-1$, and $Pu=0$ in ϑ then $u=0$ in ϑ'.

Suppose $t_0>0$; denote by $C_0=C(t_0, x_0) \cap \{t=0\}$ and for small $\varepsilon>0$ cover $C_0 \backslash \{x \in C_0 \mid d(x, \partial C_0)<\varepsilon\}=C_0^\varepsilon$ with a finite number of neighborhoods of type ϑ', $C_0^\varepsilon \subset \bigcup_j \vartheta_j'$. Take $\delta \in]0, \varepsilon[$ for which $[0, \delta] \times C_0^\varepsilon \subset \bigcup_j \vartheta_j'$ and $u=0$ on $[0, \delta] \times C_0^\varepsilon$. Put $\Gamma_\varepsilon=C(t_0-\varepsilon/\lambda\ ; x_0) \cap \{t \geq \delta\}$. Since $Pu=0$ in Γ_ε and u vanishes with all derivatives at $\Gamma_\varepsilon \cap \{t=\delta\}$, a classical argument (see e.g. [9]) yields $u=0$ on Γ_ε. Letting $\varepsilon \to 0$, the assertion is proved.

In the following chapters we will study singular hyperbolic systems of the form:

$$(1.12) \qquad (I_N t\partial_t - tA(t, x, D_x) - B(t, x, D_x))v = g,$$

where $A \in OPS_{cl}^1(\boldsymbol{R}_x^n; N \times N)$ and $B \in OPS_{cl}^0(\boldsymbol{R}_x^n; N \times N)$, are $N \times N$ matrices of classical pdo's defined on $\boldsymbol{R}^n$ (depending smoothly on t) whose structure will be specified later on.

We now show how any equation $Pu=f$, where $P \in F_0^m(\boldsymbol{R} \times \boldsymbol{R}^n)$, can be reduced to an ≪equivalent≫ system of the form (1.12). It is worth noting that the reduction proposed here is quite different from the reduction used by Tahara [20].

First a lemma is needed.

LEMMA 1.1. *Let* $P \in F_0^m(\boldsymbol{R} \times M)$ *be written in the form* (1.4). *Let* $Z(t, x, D_x) \in OPS_{cl}^1(M)$ *be a classical* 1*st order pdo on* M *(depending smoothly on* t*). Then, for every* $\gamma \in \boldsymbol{C}$, *we can rewrite* P *in the form*:

$$(1.13) \qquad P=\sum_{j=0}^{m} \sum_{h=0}^{m-j} t^{m-j-h} A_{m-j-h, j}^{(\gamma)}(t, x, D_x)(t\partial_t - tZ(t, x, D_x) - \gamma)^h$$

for some pdo's $A_{m-j-h, j}^{(\gamma)} \in OPS_{cl}^{m-j-h}(M)$ *(smoothly dependent on* t*). Moreover*:

i) *For* $j=0$, $h=m$, $A_{0,0}^{(\gamma)}(t, x, D_x)=1$.

ii) *For every* $t\in\boldsymbol{R}$, $(x, \xi)\in T^*M\backslash 0$, $\tau\in\boldsymbol{C}$:

$$\sum_{h=0}^{m}\sigma_{m-h}(A_{m-h,0}^{(\gamma)})(t, x, \xi)(\tau-\sigma_1(Z)(t, x, \xi))^h = \sum_{h=0}^{m}\sigma_{m-h}(A_{m-h,0})(t, x, \xi)\tau^h.$$

iii) *For every* $(x, \xi)\in S^*M$, $\zeta\in\boldsymbol{C}$:

$$\sum_{j=0}^{m}\sigma_0(A_{0,j}^{(\gamma)})(0, x, \xi)\zeta^{m-j}=I_P(x, \xi; \zeta+\gamma).$$

PROOF. The proof is based on the following relations which can be easily obtained by induction.

For every $j\in\boldsymbol{Z}_+$, $j\geq 1$, one can write:

$$(1.14)\qquad (t\partial_t)^j=\sum_{r=0}^{j}\binom{j}{r}(tZ(t, x, D_x)^r(t(\partial_t-Z(t, x, D_x))^{j-r} + \sum_{\substack{\beta, r, |\beta|>0\\ |\beta|+r<j}} t^{|\beta|}d_{\beta, j}(t, x, D_x)(t(\partial_t-Z(t, x, D_x))^r,$$

where the $d_{\beta, j}\in OPS_{cl}^{|\beta|}(M)$ are suitable pdo's, depending smoothly on t.

For every $j\in\boldsymbol{Z}_+$, $j\geq 1$, one can write:

$$(1.14)'\qquad t(\partial_t-Z(t, x, D_x))^j=\sum_{r=0}^{j}\binom{j}{r}\gamma^{j-r}(t(\partial_t-Z(t, x, D_x))-\gamma^r.$$

Using (1.14) and (1.14)′ in (1.4) it is a simple matter to verify that properties i)-iii) are satisfied.

Let now $P\in F_0^m(\boldsymbol{R}\times\boldsymbol{R}^n)$ (be written in the form (1.4)) and *let* $Z(t, x, D_x)$ *be equal to* $\sqrt{-1}\lambda_j(t, x, D_x)$, *where* λ_j *is one of the hyperbolic roots of* P *(the choice of* j *is inessential in what follows)*. Taking $\gamma\in\boldsymbol{C}\backslash\boldsymbol{Z}$ and defining:

$$(1.15)\qquad L=t(\partial_t-Z(t, x, D_x))-\gamma,$$

by Lemma 1.1. we can write

$$(1.16)\qquad P=\sum_{j=0}^{m}\sum_{h=0}^{m-j}t^{m-j-h}A_{m-j-h, j}^{(\gamma)}(t, x, D_x)L^h,$$

for some $A_{m-j-h, j}^{(\gamma)}\in OPS_{cl}^{m-j-h}(\boldsymbol{R}^n)$.

Denote by $\Lambda\in OPS_{cl}^1(\boldsymbol{R}^n)$ the pdo with symbol $(1+|\xi|^2)^{1/2}$.

If $u\in C^\infty(\boldsymbol{R}_t; \mathscr{D}'(\boldsymbol{R}^n))$ and $Pu=f$, let us define the vector $v=(v_1^{(1)}, \dots, v_m^{(1)}, v_1^{(2)}, \dots, v_{m-1}^{(2)}, \dots, v_1^{(m)})$, where:

$$(1.17)\qquad \begin{cases} v_1^{(h)}=(t\Lambda)^{m-h}u \\ v_2^{(h)}=(t\Lambda)^{m-h-1}Lu \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, h=1,\dots, m. \\ v_{m-h}^{(h)}=t\Lambda L^{m-h-1}u \\ v_{m-h+1}^{(h)}=L^{m-h}u \end{cases}$$

The vector $v\in C^\infty(\boldsymbol{R}_t;\mathscr{D}'(\boldsymbol{R}^n))^N$, with $N=m(m+1)/2$. The following notation is convenient:

$$(1.18)\qquad \begin{cases} \beta_r(t, x, D_x)=[Z(t, x, D_x), \Lambda^r]\Lambda^{-r} \\ \theta_r(t, x, D_x)=r-\beta_r(t, x, D_x) \end{cases}, \quad r=1,\dots, m-1.$$

For $h=2,\dots, m$ we have the compatibility relations:

$$(1.19)\qquad \begin{cases} Lv_1^{(h)}= v_2^{(h-1)}+(m-h-t\beta_{m-h})v_1^{(h)} \\ Lv_2^{(h)}=v_3^{(h-1)}+(m-h-1-t\beta_{m-h-1})v_2^{(h)} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ Lv_{m-h}^{(h)}=v_{m-h+1}^{(h-1)}+(1-t\beta_1)v_{m-h}^{(h)} \\ Lv_{m-h+1}^{(h)}=v_{m-h+2}^{(h-1)} \end{cases}$$

For $h=1$, from (1.17) and (1.16) we obtain the equations:

$$(1.20)\qquad \begin{cases} Lv_1^{(1)}=t\Lambda v_2^{(1)}+(m-1-t\beta_{m-1})v_1^{(1)} \\ Lv_2^{(1)}=t\Lambda v_3^{(1)}+(m-2-t\beta_{m-2})v_2^{(1)} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ Lv_{m-1}^{(1)}=t\Lambda v_m^{(1)}+(1-t\beta_1)v_{m-1}^{(1)} \\ Lv_m^{(1)}=L^m u=f-\sum_{h=0}^{m-1}C_{h+1}^{(0)}t\Lambda v_{h+1}^{(1)}-\sum_{j=1}^{m}\sum_{h=0}^{m-j}C_{h+1}^{(j)}v_{h+1}^{(j)}, \end{cases}$$

where

$$(1.21)\qquad C_{h+1}^{(j)}(t, x, D_x)=A_{m-j-h,\, j}^{(\gamma)}(t, x, D_x)\Lambda^{-(m-j-h)}, \quad \begin{matrix} j=0,\dots, m \\ h=0,\dots, m-j. \end{matrix}$$

Note that $C_{h+1}^{(j)}\in OPS_{cl}^0(\boldsymbol{R}^n)$ (depending smoothly on t). From (1.20) and (1.19) it follows that the vector v satisfies a system of the form (1.12) where $g=(\underbrace{0,\dots,0, f,}_{m} 0,\dots,0)$ and

$$(1.22)\qquad A(t, x, D_x)= \left[\begin{array}{c|c} \overbrace{A'(t,x,Dx)}^{m} & \square_{m,N-m} \\ \hline \square_{N-m,m} & I_{N-m}Z(t,x,D_x) \end{array}\right],$$

(the first block row has height m)

with:

$$(1.23)\quad A'(t,x,D_x)=\begin{bmatrix} Z & \Lambda & 0 & \dots & 0 & 0\\ 0 & Z & \Lambda & \dots & 0 & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & \dots & Z & \Lambda\\ -C_1^{(0)}\Lambda & -C_2^{(0)}\Lambda & -C_3^{(0)}\Lambda & \dots & -C_{m-1}^{(0)}\Lambda & -C_m^{(0)}\Lambda+Z \end{bmatrix}$$

Moreover, $B(t,x,D_x)=B'(t,x,D_x)+\gamma I_N$, where B' has the form:

$$(1.24)\quad B'(t,x,D_x)=\begin{pmatrix} \overbrace{\Phi^{(m)}}^{m} & & & & \mathscr{B}'(t,x,D_x) & \\ J^{(m-1)} & \overbrace{\Phi^{(m-1)}}^{m-1} & & & & \\ & J^{(m-2)} & \overbrace{\Phi^{(m-2)}}^{m-2} & & \square & \\ & & & \ddots & & \\ \square & & & J^{(2)} & \overbrace{\Phi^{(2)}}^{3} & \\ & & & & \underbrace{J^{(1)}}_{2} & \underbrace{\Phi^{(1)}}_{1} \end{pmatrix}\begin{matrix} \}\,m \\ \}\,m-1 \\ \}\,m-2 \\ \\ \}\,2 \\ \}\,1 \end{matrix}$$

with:

$$(1.25)\quad \Phi^{(m)}(t,x,D_x)=\begin{bmatrix} \theta_{m-1} & & & & \square\\ & \theta_{m-2} & & & \\ & & \ddots & & \\ \square & & & \theta_1 & 0\\ -C_1^{(1)} & -C_2^{(1)} & \dots & -C_{m-1}^{(1)} & -C_m^{(1)} \end{bmatrix},$$

and, for $j=1,\dots,m-1$,

$$(1.26)\quad \Phi^{(m-j)}(t,x,D_x)=\begin{bmatrix} \theta_{m-1-j} & & & \square & \\ & \theta_{m-2-j} & \ddots & & \\ \square & & & \theta_1 & 0\\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

For $h=0,\dots,m-2$,

$$(1.27)\quad J^{(m-1-h)}=\left[\begin{array}{c|c} 0 & \\ 0 & -I_{m-1-h}\\ \vdots & \\ 0 & \end{array}\right]$$

and finally:

(1.28) $\mathscr{B}'(t, x, D_x)$

$$=\begin{bmatrix} & & & & & & \square_{m-1,N-m} & & & & & \\ -C_1^{(2)} & -C_2^{(2)} \dots & -C_{m-1}^{(2)} & -C_1^{(3)} & -C_2^{(3)} \dots & -C_{m-2}^{(3)} \dots & -C_1^{(m-1)} & -C_2^{(m-1)} & -C_1^{(m)} \end{bmatrix}.$$

From Lemma 1.1, ii) it easily follows that:

$$(1.29)\quad \begin{cases} det(\zeta I_m-\sigma_1(A')(t,\ x,\ \xi)) \\ \quad =\sum_{h=0}^{m}\sigma_{m-h}(A_{m-h,\,0}^{(\gamma)})(t,\ x,\ \xi)(\zeta-\sigma_1)(Z)(t,\ x,\ \xi))^h \\ \quad =\sum_{h=0}^{m}\sigma_{m-h}(A_{m-h,0})(t,\ x,\ \xi)\zeta^h, \\ for\ every\ t\in\boldsymbol{R},(x,\ \xi)\in T^*\boldsymbol{R}^n\backslash 0,\ \zeta\in\boldsymbol{C}. \end{cases}$$

One can easily check that there exists a smooth invertible $N\times N$ matrix $U(t,\ x,\ \xi)$, $(t,\ x,\ \xi)\in\boldsymbol{R}\times T^*\boldsymbol{R}^n\backslash 0$, positively homogeneous of degree zero in ξ, such that:

$$(1.30)\qquad U^{-1}(t,\ x,\ \xi)\sigma_1(A')(t,\ x,\ \xi)U(t,\ x,\ \xi)$$

$$=\sqrt{-1}\left[\begin{array}{c|c} \overbrace{\begin{matrix}\lambda_1(t,x,\xi) & & \square \\ & \ddots & \\ \square & & \lambda_m(t,x,\xi)\end{matrix}}^{m} & \square_{m,N-m} \\ \hline \square_{N-m,m} & \frac{1}{\sqrt{-1}}\sigma_1(Z)(t,x,\xi)I_{N-m} \end{array}\right]$$

An elementary (but tedious) computation shows that:

$$(1.31)\quad \begin{cases} det\ \ (\zeta I_N-\sigma_0(B)(0,\ x,\ \xi)) \\ \quad =I_P(x,\ \xi\ ;\ \zeta)\prod_{j=1}^{m-1}\ (\zeta-\gamma-(m-j))^j, \\ for\ every\ \ (x,\ \xi)\in S^*\boldsymbol{R}^n,\ \zeta\in\boldsymbol{C}. \end{cases}$$

It is important to observe that if P satisfies the Fuchs condition (1.3) and $\gamma\notin\boldsymbol{Z}$, then $\sigma_0(B)$ $(0,\ x,\ \xi)$ has no eigenvalue in $\boldsymbol{Z}_+$ (actually a choice of γ such that $\gamma+m-j\notin\boldsymbol{Z}_+$, $j=1,\dots,\ m-1$, would suffice for this purpose).

So far we have proved that if $u\in C^\infty(\boldsymbol{R}_t\,;\mathscr{D}'(\boldsymbol{R}^n))$ solves the equation $Pu=f$ in $\boldsymbol{R}_t\times\boldsymbol{R}_x^n$, then the vector v defined by (1.17) solves the system (1. 12) with A, B having the structure specified above, and $g=(\underbrace{0,\dots,0,f}_{m},0,\dots,0)$.

Conversely, let us suppose that a vector $v \in C^\infty(\boldsymbol{R}_t; \mathscr{D}'(\boldsymbol{R}^n))^N$ satisfies the system (1.12) in $\boldsymbol{R}_t \times \boldsymbol{R}^n_x$ (with A, B and g as above).

We want to show that putting $u = v_1^{(m)}$, we have $Pu - f \in C^\infty(\boldsymbol{R}_t \times \boldsymbol{R}^n_x)$.

From (1.19) and (1.20) we obtain: $v_2^{(m-1)} = Lv_1^{(m)} = Lu$, $v_3^{(m-2)} = Lv_{(2)}^{(m-1)} = L^2 u, \dots, v_{m-1}^{(2)} = Lv_{m-2}^{(3)} = L^{m-2}u$, $v_m^{(1)} = Lv_{m-1}^{(2)} = L^{m-1}u$.

Now, from (1.20) we have

$$Lv_{m-1}^{(1)} = t\Lambda v_m^{(1)} + (1 - t\beta_1) v_{m-1}^{(1)} = t\Lambda L^{m-1}u + (1 - t\beta_1) v_{m-1}^{(1)}$$
$$= L(t\Lambda L^{m-2}u) + [t\Lambda, L]\ L^{m-2}u + (1 - t\beta_1) v_{m-1}^{(1)}.$$

In conclusion, using (1.18) we obtain the equation:

$$[t(\partial_t - Z(t, x, D_x)) - (\gamma + 1 - t\beta_1(t, x, D_x))] \cdot (v_{m-1}^{(1)} - t\Lambda L^{m-2}u) = 0. \tag{1.32}$$

Since $\gamma + 1 \notin \boldsymbol{Z}_+$, taking into account that $v_{m-1}^{(1)} - t\Lambda L^{m-2}u \in C^\infty(\boldsymbol{R}_t; \mathscr{D}'(\boldsymbol{R}^n))$, we can apply Proposition 4.4 of Hanges [12] and obtain that $v_{m-1}^{(1)} - t\Lambda L^{m-2}u \in C^\infty(\boldsymbol{R}_t \times \boldsymbol{R}^n)$.

Arguing in the same way as above, we obtain:

$$[t(\partial_t - Z(t, x, D_x)) - (\gamma + 2 - t\beta_2(t, x, D_x))] \cdot (v_{m-2}^{(1)} - (t\Lambda)^2 L^{m-3}u) \in C^\infty(\boldsymbol{R}_t \times \boldsymbol{R}^n_x). \tag{1.33}$$

Again, since $\gamma + 2 \notin \boldsymbol{Z}_+$ and $v_{m-2}^{(1)} - (t\Lambda)^2 L^{m-3}u \in C^\infty(\boldsymbol{R}_t; \mathscr{D}'(\boldsymbol{R}^n))$, the result of Hanges quoted above can be applied yielding $v_{m-2}^{(1)} - (t\Lambda)^2 L^{m-3}u \in C^\infty(\boldsymbol{R}_t \times \boldsymbol{R}^n)$. Going on in this way we obtain that relations (1.17) are satisfied mod. $C^\infty(\boldsymbol{R}_t \times \boldsymbol{R}^n_x)$. Thus, from the last equation in (1.20), we get $Pu - f \in C^\infty(\boldsymbol{R}_t \times \boldsymbol{R}^n_x)$.

To conclude, we have proved that the equation $Pu = f$, $f \in C^\infty(\boldsymbol{R}_t; \mathscr{D}'(\boldsymbol{R}^n))$, and $P \in F_0^m(\boldsymbol{R} \times \boldsymbol{R}^n)$, can be transformed into a system of the form (1.12), with the above specified structure, and that this system is equivalent mod. $C^\infty(\boldsymbol{R}_t \times \boldsymbol{R}^n_x)$ to the given scalar equation. Thus, in order to solve $Pu = f$, it will be enough to solve a system of the form (1.12) mod. C^∞.

2. Classes of symbols and related operators.

This chapter is devoted to the definition of most of the relevant classes of symbols and related operators which will be used in the sequel. Since the arguments needed in the proofs are only slight modifications of the classical ones we will be rather sketchy.

DEFINITION 2.1. *Let $m, k \in \boldsymbol{R}$. By $S^{m,k}$ we denote the space of all functions $a(t, x, \xi) \in C^\infty(\boldsymbol{R}_t \times \boldsymbol{R}^n_x \times \boldsymbol{R}^n_\xi)$ such that for every $\Omega \subset\subset \boldsymbol{R}_t \times \boldsymbol{R}^n$, $j \in \boldsymbol{Z}_+$ multiindices $\alpha, \beta \in \boldsymbol{Z}^n_+$, $\delta > 0$, there exists a positive constant C for which the inequality:*

$$(2.1)\qquad |\partial_t^j\partial_x^\alpha\partial_\xi^\beta a(t,x,\xi)|\le C|\xi|^{m-|\beta|}(|t|+1/|\xi|)^{k-j},$$

holds for $(t, x)\in\Omega, |\xi|\ge\delta$.

We put

$$(2.2)\qquad S^{-\infty,k}=\bigcap_m S^{m,k},\quad S^{m,\infty}=\bigcap_k S^{m,k}.$$

The classes $S^{m,k}$ are related to the classes introduced by Boutet de Monvel in [5]. We list now some properties of the classes $S^{m,k}$ which will be of frequent use in the sequel.

First we fix a notation: unless otherwise stated, by a *cut-off function* $\chi(x, z)\in C^\infty(\mathbf{R}_x^n\times\mathbf{R}_z)$ we mean a function having the following properties:

$$(2.3)\qquad \begin{cases} \text{i)}\ \ 0\le\chi(x,z)\le 1 \\ \text{ii)}\ \ \text{For every } \omega\subset\subset\mathbf{R}_x^n,\ \chi\in C_0^\infty(\mathbf{R}_z; C^\infty(\omega)) \\ \text{iii)}\ \ \textit{For every } x\in\mathbf{R}^n,\ \chi(x,\cdot)=1 \textit{ in a neighborhood of } z=0. \end{cases}$$

LEMMA 2.1. *We have*:

1. $S^{m,k}\subset S^{m',k'}$ *iff* $m\le m'$, $m-k\le m'-k'$.
2. $S^{-\infty,k}=S^{-\infty,k'}=S_{1,0}^{-\infty}((\mathbf{R}_t\times\mathbf{R}_x^n)\times\mathbf{R}_\xi^n)$, $\forall k, k'$.
3. $S^{m,k}\subset S_{1,1}^{m+k_-}((\mathbf{R}_t\times\mathbf{R}_x^n)\times\mathbf{R}_\xi^n)$, $k_-=\max(0, -k)$.
4. $S_{1,0}^m((\mathbf{R}_t\times\mathbf{R}_x^n)\times\mathbf{R}_\xi^n)\subset S^{m,0}$.
5. *If*

$a\in S^{m,\infty}$ *and* $\chi(x, z)$ *is any cut off function, then* $\chi(x, t|\xi|)a(t, x, \xi)\in S_{1,0}^{-\infty}((\mathbf{R}_t\times\mathbf{R}_x^n)\times\mathbf{R}_\xi^n)$.

Futhermore, $(1-\chi(x, t|\xi|))a(t, x, \xi)$ *satisfies, for every* $M\in\mathbf{Z}_+$, *local estimates of the form*

$$(2.4)\qquad |\partial_t^j\partial_x^\alpha\partial_\xi^\beta[(1-\chi(x,t|\xi|))a(t,x,\xi)]|\le const.|t|^M(1+|\xi|)^{m-|\beta|}.$$

6. *Let* $a_j\in S^{m-j,k}$ *(resp.* $b_j\in S^{m,k+j}$*) for* $j=0, 1, \dots$; *then there exists a symbol* $a\in S^{m,k}$ *(resp.* $b\in S^{m,k}$*) such that*

$$(2.5)\qquad \forall M\ge 1,\ a-\sum_{j<M}a_j\in S^{m-M,k}\left(\ resp.\ b-\sum_{j<M}b_j\in S^{m,k+M}\right).$$

To express (2.5) *we shall write* $a\sim\sum_{j\ge 0}a_j$ *(resp.* $b\sim\sum_{j\ge 0}b_j$*)*.

PROOF. To prove 1. we remark that (2.1) is meaningful for $|t|+1/|\xi|$ small; moreover, for $|t||\xi|\le$const. we have $|\xi|\sim(|t|+1/|\xi|)^{-1}$, while for $|t||\xi|\ge$const. we have $|t|\sim|t|+1/|\xi|$. Using these remarks the proof of 1. is easily obtained. The equality $S^{-\infty,k}=S^{-\infty,k'}$ follows from 1. On the other hand, since $|\xi|^{-r}(|t|+1/|\xi|)^{k-j}\le$const. $|\xi|^{-r+j+k}$, $\forall j, r\in\mathbf{Z}_+$, it follows that $S^{-\infty,k}\subset S^{-\infty}_{1,0}$. The converse inclusion is proved arguing in the same way as in 1.; point 2. is proved. Point 3. is a trivial consequence of the above remark. To prove point 4. we observe that for any cut-off function χ one can write: $\partial_t^j\partial_x^\alpha\partial_\xi^\beta a=\chi(x, t|\xi|)\partial_t^j\partial_x^\alpha\partial_\xi^\beta a+(1-\chi(x, t|\xi|))\partial_t^j\partial_x^\alpha\partial_\xi^\beta a=\mathrm{I}_1+\mathrm{I}_2$, $a\in S^m_{1,0}((\boldsymbol{R}\times\boldsymbol{R}^n)\times\boldsymbol{R}^n)$. To prove that I_1 satisfies inequality (2.1) it is enough to note that locally in (t, x) and for $|\xi|\geq$ const., we have $\chi(x, t|\xi|)\leq$const. $|\xi|^j\leq$const. $(|t|+1/|\xi|)^{-j}$. On the other hand, since $|\partial_t^j\partial_x^\alpha\partial_\xi^\beta a|\leq$const. $(1+|\xi|)^{m-|\beta|}|t|^{-j}$ obviously holds locally in (t, x), we have only to use the fact that on the support of $1-\chi(x, t|\xi|)$, $|t|\sim|t|+1/|\xi|$. This concludes the proof of point 4. Point 5. is a consequence of the preceding remarks. Point 6. can be proved using the same arguments as in Proposition 1.11 (i), (ii) of [5].

DEFINITION 2.2. *Let $a\in S^{m,k}$. Define:*

$$a(t, x, D_x)f(t, x)=\int e^{ix\cdot\xi}a(t, x, \xi)\hat{f}(t, \xi)\,đ\xi, \tag{2.6}$$

for $f\in C^\infty(\boldsymbol{R}_t; C_0^\infty(\boldsymbol{R}^n))$, where $\hat{f}(t, \xi)=\int e^{-i\xi\cdot y}f(t, y)\,dy$ and $đ\xi=(2\pi)^{-n}d\xi$.

By $OPS^{m,k}$ we denote the space of all operators A which can be written in the form $A=a(t, x, D_x)+R$, for some $a\in S^{m,k}$; here R is a partially regularizing operator, i.e. there exists a smooth kernel $r(t, x, y)\in C^\infty(\boldsymbol{R}_t\times\boldsymbol{R}^n_x\times\boldsymbol{R}^n_y)$ such that:

$$Rf(t, x)=\int r(t, x, y)f(t, y)\,dy,\quad f\in C^\infty(\boldsymbol{R}_t; C_0^\infty(\boldsymbol{R}^n)). \tag{2.7}$$

The function a will be called a symbol for A.

We shall say that an operator $A\in OPS^{m,k}$ is proper iff for every $\Omega\subset\subset\boldsymbol{R}^n$ there exists $\Omega'\subset\subset\boldsymbol{R}^n$ such that:

$$\begin{cases} f\in C^\infty(\boldsymbol{R}; C_0^\infty(\Omega))\Rightarrow Af\in C^\infty(\boldsymbol{R}_t; C_0^\infty(\Omega')) \\ f\in C^\infty(\boldsymbol{R}_t; C_0^\infty(\boldsymbol{R}^n)),\ f|_{\boldsymbol{R}_t\times\Omega'}=0\Rightarrow Af|_{\boldsymbol{R}_t\times\Omega}=0. \end{cases} \tag{2.8}$$

We state as a Lemma some properties of the classes $OPS^{m,k}$.

LEMMA 2.2. *We have:*

1. *Every operator $A\in OPS^{m,k}$ can be uniquely extended as a continuous map:*

$$A: C^\infty(\boldsymbol{R}_t; \mathscr{E}'(\boldsymbol{R}^n))\to C^\infty(\boldsymbol{R}_t; \mathscr{D}'(\boldsymbol{R}^n))$$

2. *For every $A\in OPS^{m,k}$ there exists a proper operator $A'\in OPS^{m,k}$ such that $A-A'$ is partially regularizing.*

3. *For a proper operator $A\in OPS^{m,k}$, the function*

(2.9) $\sigma(A)(t, x, \xi)=e^{-i\langle x\cdot\xi\rangle}A(e^{i\langle\cdot,\xi\rangle})\in S^{m,k}$

and $A=\sigma(A)(t, x, D_x)$; $\sigma(A)(t, x, \xi)$ *will be called the symbol of* A.

4. *Let* $A\in OPS^{m,k}$, $B\in OPS^{m',k'}$, *one of them being a proper operator. Then* $BA\in OPS^{m+m',k+k'}$ *and for any symbol* $c(t, x, \xi)$ *of* BA *the usual asymptotic expansion holds*:

(2.10) $$c(t, x, \xi)\sim\sum_{\alpha}\frac{1}{\alpha!}\partial_{\xi}^{\alpha} b(t, x, \xi)\ D_x^{\alpha}a(t, x, \xi),$$

where a *(resp.* b*) is a symbol for* A *(resp.* B*)* ; (2.10) *should be understood in the following sense*:

(2.10)′ $$\forall M\geq 1,\ c-\sum_{|\alpha|<M}\frac{1}{\alpha!}\partial_{\xi}^{\alpha} b\ D_x^{\alpha}\ a\in S^{m+m'-M,k+k'}.$$

5. *Let* $A\in OPS^{m,k}$ *and denote by* ${}^tA$ *the formal transpose of* A *defined by*

$$\iint Af(t, x)\ g(t, x)\ dt\ dx=\iint f(t, x)\ {}^tAg(t, x)\ dt\ dx,$$
$$f,\ g\in C_0^{\infty}(\boldsymbol{R}_t\times\boldsymbol{R}_x^n).$$

Then ${}^tA\in OPS^{m,k}$ *and for any symbol* $b(t, x, \xi)$ *for* ${}^tA$ *the usual asymptotic expansion holds*:

(2.11) $$b(t, x, \xi)\sim\sum_{\alpha}\frac{1}{\alpha!}\ \partial_{\xi}^{\alpha} D_x^{\alpha}\ a(t, x, -\xi),$$

where a *is a symbol of* A.

6. *Let* $A\in OPS^{m,k}$ *be a proper operator. Suppose that for every* $\Omega\subset\subset \boldsymbol{R}_t\times\boldsymbol{R}_x^n$ *there exist two constants* C, $\delta>0$ *such that*

(2.12) $$|a(t, x, \xi)|\geq C|\xi|^m(|t|+1/|\xi|)^k,$$

for $(t, x)\in\Omega$, $|\xi|\geq\delta$, *where* a *is the symbol of* A.

Then there exists a proper operator $B\in OPS^{-m,-k}$ *for which* $AB-id$ *and* $BA-id$ *are partially regularizing.*

PROOF.

1. We recall (see, e.g. Treves [23]) that a distribution $u\in\mathscr{D}'(\boldsymbol{R}_t\times\boldsymbol{R}_x^n)$ belongs to $C^{\infty}(\boldsymbol{R}_t;\mathscr{D}'(\boldsymbol{R}_x^n))$ iff for every $\psi(t, x)\in C_0^{\infty}(\boldsymbol{R}_t\times\boldsymbol{R}_x^n)$, the following relation holds:

(2.13) $\forall k\in \boldsymbol{Z}_+$, $\exists\sigma_k\in\boldsymbol{R}$, such that $\psi u\in H^k(\boldsymbol{R}_t; H^{\sigma_k}(\boldsymbol{R}_x^n))$.

Write $A=a(t, x, D_x)+R$, $a\in S^{m,k}$, R partially regularizing. It is straightforward to prove that R maps $C^{\infty}(\boldsymbol{R}_t; \mathscr{E}'(\boldsymbol{R}^n))$ into $C^{\infty}(\boldsymbol{R}_t\times\boldsymbol{R}_x^n)$; therefore we can suppose $A=a(t, x, D_x)$. Given $f\in C^{\infty}(\boldsymbol{R}_t; C_0^{\infty}(\boldsymbol{R}^n))$ and putting $g(t, x)=v(x)a(t, x, D_x)\ f(t, x)$, with $v\in C_0^{\infty}(\boldsymbol{R}^n)$, we can prove by a standard argument that the following inequalities hold:

$$\begin{aligned}(2.14)\qquad &\sum_{j=0}^{r}\|\partial_t^j g(t,x);L^2([-T,T];H^{s-(m+k_-+j)}(\boldsymbol{R}^n))\|\\ &\le C\sum_{j=0}^{r}\|\partial_t^j f(t,x);L^2([-T,T];H^s(\boldsymbol{R}^n))\|,\end{aligned}$$

for every $s\in\boldsymbol{R}$, $T>0$, $r\in\boldsymbol{Z}_+$, with a constant C independent of f. The proof is finished using (2.14) and the characterization (2.13). Point 2. is proved as in the classical case.

To prove 3., arguing as in the classical case, we need only to show that for every $a\in S^{m,k}$ and for every $\varphi(x)\in C_0^\infty(\boldsymbol{R}^n)$, the function $a_\varphi(t,x,\xi)=e^{-i\langle x,\xi\rangle}a(t,x,D_x)\ [e^{-i\langle\cdot,\xi\rangle}\varphi(\cdot)]\in S^{m,k}$ with asymptotic expansion $a_\varphi\sim\sum_\alpha\frac{1}{\alpha!}\partial_\xi^\alpha a\, D_x^\alpha\varphi$.

We prove that a_φ satisfies the estimate (2.1) when $j=0$, $\alpha=\beta=0$ (the estimates for the derivatives can be obtained analogously).

For $|\xi|\ge1$, write

$$\begin{aligned}&a_\varphi(t,x,\xi)\\ &=\Big(\int_{|\xi+\zeta|\le1}+\int_{|\xi+\zeta|\ge1}\Big)(e^{ix\cdot\zeta}a(t,x,\xi+\zeta)\hat{\varphi}(\zeta))\,đ\zeta=I_1+I_2.\end{aligned}$$

For $|\xi+\zeta|\le1$ we have, locally, $|a(t,x,\xi+\zeta)|\le$const.; therefore

$$\begin{aligned}|I_1|&\le const.\int_{|\xi+\zeta|\le1}(1+|\xi+\zeta|)^{m-|k|}|\hat{\varphi}(\zeta)|\,d\zeta\\ &\le const.\ (1+|\xi|)^{m-|k|}\le const.\ |\xi|^m(|t|+1/|\xi|)^k.\end{aligned}$$

Write I_2 in the form

$$\begin{aligned}I_2&=\Big(\int_{\substack{|\xi+\zeta|\ge1\\|\zeta|\le|\xi|/2}}+\int_{\substack{|\xi+\zeta|\ge1\\|\zeta|\ge|\xi|/2}}\Big)(e^{ix\cdot\zeta}a(t,x,\xi+\zeta)\hat{\varphi}(\zeta))\,đ\zeta\\ &=J_1+J_2.\end{aligned}$$

Since locally $|a(t,x,\zeta+\xi)|\le$const. $(1+|\zeta+\xi|)^m(|t|+1/|\zeta+\xi|)^k$, and $|\xi+\zeta|\sim|\xi|$ if $|\xi+\zeta|\ge1$, $|\zeta|\le|\xi|/2$, we easily obtain

$$|J_1|\le\text{const. }(1+|\xi|)^m(|t|+1/|\xi|)^k.$$

To estimate J_2 we remark that for $|\xi+\zeta|\ge1$, $|\zeta|\ge|\xi|/2$, one has

$$(|t|+1/|\xi+\zeta|)^k(|t|+1/|\xi|)^{-k}\le\begin{cases}\text{const. }|\xi|^k\le\text{const. }|\zeta|^k, & \text{if } k\ge0\\ \text{const. }|\xi+\zeta|^{-k}\le\text{const. }|\zeta|^{-k}, & \text{if } k<0.\end{cases}$$

As a consequence

$$|J_2|\le\text{const. }(1+|\xi|)^m(|t|+1/|\xi|)^k.$$

The proof that

$$a_\varphi(t,x,\xi)-\sum_{|\alpha|<M}\frac{1}{\alpha!}\partial_\xi^\alpha a\,D_x^\alpha\varphi\in S^{m-M,k},\ \forall M\ge1,$$

can be obtained arguing as above.

Points 4. and 5. follow by combining standard arguments and integral estimates of the above type.

To prove 6. we observe that (2.12) implies the existence of a symbol $b \in S^{-m,-k}$ such that $ba - 1 \in S^{-\infty,0}$. Take any proper operator $B \in OPS^{-m,-k}$ with symbol b, then $BA - id = R \in OPS^{-1,0}$. By taking $S \in OPS^{0,0}$, $S \sim \sum_{j\geq 0}(-1)^j R^j$, we obtain $SBA - id \in OPS^{-\infty,0}$. The conclusion follows.

By Lemma 2.2, 1., every operator $A \in OPS^{m,k}$ maps continuously $C^\infty(\boldsymbol{R}_t; \mathscr{E}'(\boldsymbol{R}^n))$ into $C^\infty(\boldsymbol{R}_t; \mathscr{D}'(\boldsymbol{R}^n))$. The problem we want to analyze here is the relation between $WF(Af)$ and $WF(f)$ for $f \in C^\infty(\boldsymbol{R}_t; \mathscr{E}'(\boldsymbol{R}^n))$.

We recall (see [13]) that if $f \in C^\infty(\boldsymbol{R}_t; \mathscr{E}'(\boldsymbol{R}^n))$ and

$$WF(f) \cap \{(t, x; \tau, 0) \in T^*\boldsymbol{R}^{1+n} \mid \tau \neq 0\} = \phi, \tag{2.15}$$

then for every symbol $a(t, x, \xi) \in S^m_{\rho,\delta}((\boldsymbol{R}_t \times \boldsymbol{R}^n_x) \times \boldsymbol{R}^n_\xi)$, $0 \leq \delta < \rho \leq 1$, we have:

$$WF(a(t, x, D_x)\, f) \subset WF(f). \tag{2.16}$$

In particular:

$$WF(a(t, x, D_x)\, f) \cap \{(0, x; \tau, 0) \mid \tau \in \boldsymbol{R}\backslash 0)\} = \phi. \tag{2.17}$$

Unfortunately, due to Lemma 2.1, 3. our classes $OPS^{m,k}$ are only imbedded into the classes $OPS^{m+k}_{1,1}$ which are not microlocal!

As a consequence, we expect that even if we apply our operators to distributions verifying (2.15), relation (2.17) may be violated, i.e. our operators allow singularities to arise on the conormal bundle to $t=0$. Actually this happens as the following simple example shows.

Consider the operator $\varphi(t|D_x|) \in OPS^{0,0}$, where $\varphi \in C^\infty_0(\boldsymbol{R})$, $\varphi \equiv 1$ near the origin. Define $v(t, x) = \varphi(t|D_x|)(1_t \otimes \delta_x)$, i.e.

$$v(t, x) = \int e^{ix\cdot\xi} \varphi(t|\xi|)\, d\!\!{}^-\xi. \tag{2.18}$$

It can be easily recognized that v is a distribution homogeneous of degree $-n$, which is C^∞ for $t \neq 0$. From Theorem 8.1.8 of Hörmander [14], we know that the points $(t=0, x=0; \tau=\pm 1, \xi=0) \notin WF(v)$ iff $(\pm 1, 0) \notin$ support $(\hat{v}(\tau, \xi))$. Since $\hat{v}(\tau, \xi)$ is given by

$$\hat{v}(\tau, \xi) = \int e^{-it\tau} \varphi(t|\xi|)\, dt = |\xi|^{-1} \hat{\varphi}(\tau/|\xi|), \ \xi \neq 0, \tag{2.19}$$

the points $(\pm 1, 0) \notin$ support $(\hat{v})$ only if $\hat{\varphi}$ has compact support, which is false.

In order to get some control of the action of our operators $OPS^{m,k}$ over $t=0$, we will use the notion of ≪boundary wave front set≫ introduced by Chazarain [8] and Melrose-Sjöstrand [16]. The analysis of the singularities will be carried over in a sub-class of $C^\infty(\boldsymbol{R}_t; \mathscr{D}'(\boldsymbol{R}^n))$, called here the ≪regular distributions≫.

DEFINITION 2.3. *Let M be an n-dimensional C^∞-manifold without boundary. Let $f \in C^\infty(]-T, T[; \mathscr{D}'(M))$, $0 < T \leq +\infty$.*

We say that a point $(x_0, \xi_0) \in T^*M \setminus 0$ does not belong to the boundary wave front set of f, $(x_0, \xi_0) \notin \partial WF(f)$, iff there exists a proper pdo $B(x, D_x) \in OPS^0(M)$, elliptic near (x_0, ξ_0), such that $Bf \in C^\infty(]-\varepsilon, \varepsilon[\times M)$ for some $\varepsilon \in]0, T[$.

By $\mathscr{D}'_r(]-T, T[\times M)$ we denote the space of all distributions $f \in C^\infty(]-T, T[; \mathscr{D}'(M))$ for which:

$$WF(f) \cap \{(t, x, \tau, \xi) \in T^*(]-T, T[\times M) \setminus 0 \mid \xi = 0, \ t \neq 0\} = \phi. \tag{2.20}$$

Such an f will be called a regular distribution on $]-T, T[\times M$.

For $f \in \mathscr{D}'_r(]-T, T[\times M)$ we define:

$$\widetilde{WF}(f) = \partial WF(f) \cup WF(f|_{\{t\neq 0\} \times M}). \tag{2.21}$$

We put $\mathscr{E}'_r(]-T, T[\times M) = \mathscr{D}'_r(]-T, T[\times M) \cap C^\infty(]-T, T[; \mathscr{E}'(M))$.

We recall that when M is an open subset of $\boldsymbol{R}^n$, the condition $(x_0, \xi_0) \notin \partial WF(f)$ is equivalent to the following property:

There exists a conical neighborhood $U \times \Gamma \subset M \times \boldsymbol{R}^n \setminus 0$ of (x_0, ξ_0) such that for some $\varepsilon \in]0, T[$ and for every $\varphi \in C_0^\infty(U)$, we have:

$$\sup_{\substack{|t| \leq \varepsilon \\ \xi \in \Gamma}} |\langle \partial_t^j f(t, x), e^{-ix\cdot\xi} \varphi(x) \rangle| (1 + |\xi|)^N < +\infty, \tag{2.22}$$

for every $j, N \in \boldsymbol{Z}_+$.

The following result holds.

THEOREM 2.1. *Let $A \in OPS^{m,k}$; then:*

i) $A: \mathscr{E}'_r(\boldsymbol{R}_t \times \boldsymbol{R}^n) \to \mathscr{D}'_r(\boldsymbol{R}_t \times \boldsymbol{R}^n)$

ii) $\widetilde{WF}(Af) \subset \widetilde{WF}(f)$, $\forall f \in \mathscr{E}'_r(\boldsymbol{R}_t \times \boldsymbol{R}^n)$.

PROOF. We can suppose that $A = a(t, x, D_x)$ for some $a \in S^{m,k}$ (because partially regularizing operators map $C^\infty(\boldsymbol{R}_t; \mathscr{E}'(\boldsymbol{R}^n))$ into $C^\infty(\boldsymbol{R}_t \times \boldsymbol{R}^n)$). We note that $Af|_{\{t\neq 0\} \times \boldsymbol{R}^n} = a(t, x, D_x)(f|_{\{t\neq 0\} \times \boldsymbol{R}^n})$ and that for $t \neq 0$ we have $a(t, x, \xi) \in S^m_{1,0}(((\boldsymbol{R}_t \setminus 0)\boldsymbol{R}^n_x) \times \boldsymbol{R}^n_\xi)$. Taking into account (2.20), from classical results we obtain:

$$WF(Af|_{\{t\neq 0\} \times \boldsymbol{R}^n}) \subset WF(f|_{\{t\neq 0\} \times \boldsymbol{R}^n}). \tag{2.23}$$

Relation (2.23) and Lemma 2.2, 1., imply assertion i). To prove ii), in view of (2.23) we have only to show that $(x_0, \eta_0)\notin\partial WF(Af)$ when $(x_0, \eta_0)\notin\partial WF(f)$. Let $B(x, D_x)\in OPS^0(\boldsymbol{R}^n)$ be a proper operator whose symbol $b(x, \eta)$ has the properties:

1) cone supp (b) is contained in a conical neighborhood $U\times\Gamma\subset T^*\boldsymbol{R}^n\backslash 0$ of (x_0, η_0); 2) $b(x, \eta)=1$ in a conical neighborhood $U'\times\Gamma'\subset U\times\Gamma$ of (x_0, η_0); 3) $B(x, D_x)f\in C^\infty(]-\varepsilon, \varepsilon[\times\boldsymbol{R}^n)$, for some $\varepsilon>0$.

Write $BA=AB+[B, A]$; from Lemma 2.2, 4., it follows that $[B, A]\in OPS^{m-1,k}$ with a symbol $c(t, x, \xi)$ having the following asymptotic expansion

$$c(t, x, \xi)\sim\sum_{|\alpha|\geq 1}\frac{1}{\alpha!}(\partial_\xi^\alpha b(x, \xi)D_x^\alpha a(t, x, \xi) -\partial_\xi^\alpha a(t, x, \xi)D_x^\alpha b(x, \xi)).$$

Take $B'(x, D_x)\in OPS^0(\boldsymbol{R}^n)$ (proper), elliptic near (x_0, η_0) and with symbol supported in a conical neighborhood $U''\times\Gamma''\subset\subset U'\times\Gamma'$ of (x_0, η_0). Then $B'BAf=B'ABf+B'[B, A]f$. Now, $B'ABf\in C^\infty(]-\varepsilon, \varepsilon[\times\boldsymbol{R}^n)$ by hypothesis, and $B'[B, A]\in OPS^{-\infty,k}$ because all the terms in the asymptotic expansion of its symbol vanish. Therefore $B'[B, A]f\in C^\infty(\boldsymbol{R}_t\times\boldsymbol{R}^n)$. This ends the proof.

We need to exploit some more structure for our symbols. This will be done by selecting suitable sub-classes of $S^{m,k}$.

DEFINITION 2.4. *By $\boldsymbol{S}^k$, $k\in\boldsymbol{R}$, we denote the space of all functions $\varphi(x, \xi', z)\in C^\infty(\boldsymbol{R}_x^n\times S_{\xi'}^{n-1}\times\boldsymbol{R}_z)$ for which there is a sequence $(\varphi_{-j})_{j\geq 0}$, $\varphi_{-j}(x, \xi')\in C^\infty(\boldsymbol{R}_x^n\times S_{\xi'}^{n-1})$, such that*

$$\varphi(x, \xi', z)\sim\sum_{j\geq 0}\varphi_{-j}(x, \xi')z^{k-j},\quad z\to\infty. \tag{2.24}$$

The above formula has the following meaning:

For every $\Omega\subset\subset\boldsymbol{R}^n$, M, $p\in\boldsymbol{Z}_+$, $\alpha\in\boldsymbol{Z}_+^n$, and for every family $\theta_1,\dots,\theta_q$ of smooth vector fields on S^{n-1}, there is a positive constant C such that:

$$\left|\theta_1\dots\theta_q\partial_z^p\partial_x^\alpha\left[\varphi-\sum_{j<M}\varphi_{-j}z^{k-j}\right]\right|\leq C(1+|z|)^{k-M-p}, \tag{2.25}$$

for $x\in\Omega$, $\xi'\in S^{n-1}$, $z\in\boldsymbol{R}$.

By $\Sigma^{m,k}$, m, $k\in\boldsymbol{R}$, we denote the space of all functions $a(t, x, \xi)\in C^\infty(\boldsymbol{R}_t\times\boldsymbol{R}_x^n\times\boldsymbol{R}_\xi^n)$ such that there exists $\hat{a}(x, \xi', z)\in\boldsymbol{S}^k$ for which:

$$a(t, x, \xi)=|\xi|^{m-k}\,\hat{a}(x, \xi/|\xi|, t|\xi|), \tag{2.26}$$

for all t, x and $|\xi|\geq\delta>0$ (δ depending on a).

By $\hat{\Sigma}^{m,k}$, m, $k\in\boldsymbol{R}$, we denote the space of all functions $a(t, x, \xi)\in\boldsymbol{S}^{m,k}$ for which there exists a sequence $(a_j)_{j\geq 0}$, $a_j\in\Sigma^{m,k+j}$, such that for every $M\geq$

1 *we have*:

(2,27) $$\left(a-\sum_{j<M} a_j \right) \in S^{m,k+M}.$$

We shall write $a \sim \sum_{j\geq 0} a_j$.

As usual we list in a lemma some properties of the above defined classes.

LEMMA 2.3. *We have*:

1. $\Sigma^{m,k} \subset S^{m,k}$ *and* $\Sigma^{m,k} \subset \Sigma^{m',k'}$, *if* $k' \geq k$, $m-k=m'-k'$.

2. *If* $a \in \Sigma^{m,k}$, *then* $\partial_t a \in ^{m,k-1}$, $ta \in \Sigma^{m,k+1}$, $\partial_{x_j} a \in \Sigma^{m,k}$, $\partial_{\xi_j} a \in \Sigma^{m-1,k}$, $j=1,\dots,n$.

3. *Let* $b(t, x, \xi) \in S^m_{cl}((\boldsymbol{R}\times\boldsymbol{R}^n)\times\boldsymbol{R}^n)$ *with an symptotic expansion* $b(t, x, \xi) \sim \sum_{j\geq 0} b_{m-j}(t, x, \xi)$ *where the* b_{m-j} *are positively homogenous of degree* $m-j$ *in* ξ.

Then $b \in \hat{\Sigma}^{m,0}$ *with asymptotic expansion*:

$$(2.28)\quad \begin{cases} b \sim \sum_{j\geq 0} b'_j \\ b'_j(t, x, \xi) \\ = \omega(|\xi|)|\xi|^{m-j} \sum_{r=0}^{j} \frac{1}{(j-r)!}(\partial_t^{j-r} b_{m-1})\left(0, x, \frac{\xi}{|\xi|}\right)(t|\xi|)^{j-r},\ j\geq 0, \end{cases}$$

where $\omega \in C^\infty(\boldsymbol{R}_+)$, $\omega(z)=0$ *for* $0 \leq z \leq 1/2$, $\omega(z)=1$, *for* $z \geq 1$.

PROOF. Points 1. and 2. are left as an exercise. To prove 3. we note that $b \in S^{m,0}$ as a consequence of Lemma 2.1, 4. For every $M \geq 1$ write

$$b-\sum_{j<M} b'_j = \left(b-\sum_{j<M} \omega(|\xi|) b_{m-j} \right) + \sum_{j<M} \omega(|\xi|) \left(b_{m-j}(t, x, \xi) - \sum_{h=0}^{m-j-1} \frac{t^h}{h!} \partial_t^h b_{m-j}(0, x, \xi) \right) = I_1 + I_2 .$$

Now $I_1 \in S^{m-M}_{1,0}((\boldsymbol{R}\times\boldsymbol{R}^n)\times\boldsymbol{R}^n) \subset S^{m-M,0} \subset S^{m,M}$.

Furthermore, we can write:

$$\begin{cases} I_2 = \sum_{j<M} t^{M-j} c_j(t, x, \xi), \\ c_j(t, x, \xi) = \frac{\omega(|\xi|)}{(M-j-1)!} \int_0^1 (1-\sigma)^{M-j-1} (\partial_t^{M-j} b_{m-j})\ (\sigma, x, \xi) d\sigma. \end{cases}$$

Now $c_j \in S^{m-j}_{1,0}((\boldsymbol{R}\times\boldsymbol{R}^n)\times\boldsymbol{R}^n) \subset S^{m-j,0}$, so that $t^{M-j} c_j \in S^{m-j,M-j} \subset S^{m,M}$ and the assertion follows.

To the classes $\Sigma^{m,k}$ and $\hat{\Sigma}^{m,k}$ we associate the related classes of operators according to the following definition.

DEFINITION 2.5. *By* $OP\Sigma^{m,k}$ *we denote the class of all operators of the form*:

(2.29) $$A = a(t, x, D_x) + R$$

for some $a \in \Sigma^{m,k}$ *(called a symbol for A) and with R partially regularizing.*

By $OP\hat{\Sigma}^{m,k}$ *we denote the class of all operators of the form*:

$$A=a(t,\ x,\ D_x)+R+R' \tag{2.30}$$

for some $a\in\hat{\Sigma}^{m,k}$ *(called a symbol for A) and with* $R'\in OPS^{m,\infty}$, *R partially regularizing.*

Operators in the classes $OP\hat{\Sigma}^{m,k}$ can be composed as the following lemma proves.

LEMMA 2.4. *Let* $A\in OP\hat{\Sigma}^{m,k}$, $B\in OP\hat{\Sigma}^{m',k'}$, *one of them being a proper operator. Then* $BA\in OP\hat{\Sigma}^{m+m',k+k'}$. *Furthermore, if* $a\sim\sum_{j\geq 0} a_j$, $a_j\in\Sigma^{m,k+j}$, *is a symbol for a and* $b\sim\sum_{j\geq 0} b_j$, $b_j\in\Sigma^{m',k'+j}$, *is a symbol for B, then BA has a symbol* $c\sim\sum_{j\geq 0} c_j$ *where*

$$c_j=\sum_{|\alpha|+j'+j''=j}\frac{1}{\alpha\,!}\partial_\xi^\alpha\, b_{j'}\, D_x^\alpha\, a_{j''}\in\Sigma^{m+m',k+k'+j}. \tag{2.31}$$

PROOF. By Lemma 2.3, 1., 2. we know that $D_x^\alpha a_{j''}\in\Sigma^{m,k+j''}$ and $\partial_\xi^\alpha b_{j'}\in \Sigma^{m'-|\alpha|,k'+j'}\subset\Sigma^{m',k'+j'+|\alpha|}$ so the (2.31) is correct. By Lemma 2.2, 4. we know that $c\in S^{m+m',k+k'}$ with asymptotic expansion given by (2.10). For every $M\geq 1$, write

$$c-\sum_{j<M} c_j=\left(c-\sum_{|\alpha|<M}\frac{1}{\alpha\,!}\partial_\xi^\alpha\, b\, D_x^\alpha a\right)$$
$$+\left(\sum_{|\alpha|<M}\frac{1}{\alpha!}\partial_\xi^\alpha b D_x^\alpha a-\sum_{j<M} c_j\right)=I+J.$$

Now $I\in S^{m+m'-M,k+k'}\subset S^{m+m',k+k'+M}$; moreover:

$$J=\sum_{|\alpha|<M}\frac{1}{\alpha!}\partial_\xi^\alpha b D_x^\alpha a-\sum_{j<M} c_j$$
$$=\sum_{|\alpha|<M}\frac{1}{\alpha\,!}\partial_\xi^\alpha\left(b-\sum_{l<M} b_l\right)D_x^\alpha\left(a-\sum_{s<M} a_s\right)$$
$$+\sum_{|\alpha|<M}\frac{1}{\alpha!}\partial_\xi^\alpha\left(b-\sum_{l<M} b_l\right)\sum_{s<M} D_x^\alpha a_s+\sum_{|\alpha|<M}\ \sum_{l<M}\frac{1}{\alpha\,!}\partial_\xi^\alpha b_l D_x^\alpha\left(a-\sum_{s<M} a_s\right)$$
$$+\sum_{\substack{|\alpha|<M,l<M,s<M\\ |\alpha|+l+s\geq M}}\frac{1}{\alpha!}\partial_\xi^\alpha b_l D_x^\alpha a_s=J_1+J_2+J_3+J_4.$$

It is easily verified that al the J's belong to $S^{m+m',k+k'+M}$.

The Lemma is proved.

The classes $OP\hat{\Sigma}^{m,k}$ will be used in chapter 3 in order to 《decouple》 a Fuchsian hyperbolic system. However, to construct parametrices for such systems some extra classes of symbols and operators are needed. The heavy definitions which follow are strongly motivated by the construction performed by Hanges in [12].

DEFINITION 2. 6. *By* $HS^{m,k}$, m, $k\in\boldsymbol{R}$, *we denote the space of all functions* $a(\rho, t, x, \xi)\in C^\infty(]0, 1]\times\boldsymbol{R}_t\times\boldsymbol{R}^n_x\times\boldsymbol{R}^n_\xi)$ *such that for every* $\Omega\subset\subset\boldsymbol{R}_t\times\boldsymbol{R}^n_x$, j, $p\in\boldsymbol{Z}_+$, α, $\beta\in\boldsymbol{Z}^n_+$, ε, $\delta>0$, *there exists a constant* $C>0$ *for which*:

(2.32) $$|\rho^\varepsilon(\rho\partial_\rho)^p\partial_t^j\partial_x^\alpha\partial_\xi^\beta a(\rho, t, x, \xi)|\leq C|\xi|^{m-|\beta|}(|t|+1/|\xi|)^{k-j},$$
for $(t, x)\in\Omega$, $\rho\in]0, 1]$, $|\xi|\geq\delta$.

We put

(2.33) $$HS^{-\infty,k}=\bigcap_m HS^{m,k},\quad HS^{m,\infty}=\bigcap_k HS^{m,k}.$$

By HS^k, $k\in\boldsymbol{R}$, *we denote the space of all functions* $\varphi(\rho, x, \xi', z)\in C^\infty(]0, 1]\times\boldsymbol{R}^n_x\times S^{n-1}_{\xi'}\times\boldsymbol{R}_z)$ *such that for every* $\Omega\subset\subset\boldsymbol{R}^n_x$, p, $l\in\boldsymbol{Z}_+$, $\alpha\in\boldsymbol{Z}^n_+$, $\varepsilon>0$ *and for every system* $\theta_1, \dots, \theta_q$ *of smooth vector fields on* S^{n-1}, *there exists a constant* $C>0$ *for which*:

(2.33) $$|\theta_1\dots\theta_q\rho^\varepsilon(\rho\partial_\rho)^p\partial_z^l\partial_x^\alpha\varphi(\rho, x, \xi', z)|\leq C(1+|z|)^{k-l},$$
for $x\in\Omega$, $\rho\in]0, 1]$, $\xi'\in S^{n-1}$, $z\in\boldsymbol{R}$.

By $H\Sigma^{m,k}$ *we denote the space of all functions* $a\in HS^{m,k}$ *for which there exists* $\hat{a}\in HS^k$ *such that*:

(2.34) $$a(\rho, t, x, \xi)=|\xi|^{m-k}\hat{a}(\rho, x, \xi/|\xi|, t|\xi|),$$

for all ρ, t, x, *and for* $|\xi|\geq\delta>0$ (δ *depending on* a).

Finally, by $H\hat{\Sigma}^{m,k}$ *we denote the space of all functions* $a\in HS^{m,k}$ *for which there is a sequence* $(a_j)_{j\geq 0}$, $a_j\in H\Sigma^{m,k+j}$, *such that for every* $M>1$, *we have*:

(2.35) $$\Big(a-\sum_{j<M}a_j\Big)\in HS^{m,k+M}.$$

we shall write $a\sim\sum_{j\geq 0}a_j$.

To the above defined classes of symbols we relate the corresponding operators.

DEFINITION 2. 7. *An operator* $R: C^\infty(\boldsymbol{R}_t; C^\infty_0(\boldsymbol{R}^n))\to C^\infty(\boldsymbol{R}_t\times\boldsymbol{R}^n)$ *will be called partially regularizing of Hardy type (H. p. r. in the sequel) iff*

(2.36) $$Rf(t, x)=\int_0^1\int r(\rho, t, x, y)\, f(\rho t, y)\, d\rho\, dy,\ f\in C^\infty(\boldsymbol{R}_t; C^\infty_0(\boldsymbol{R}^n)),$$

for some kernel $r\in C^\infty(]0, 1]\times\boldsymbol{R}_t\times\boldsymbol{R}^{2n}_{(x,y)})$ *such that for every* $\Omega\subset\subset\boldsymbol{R}^n$, p, $j\in\boldsymbol{Z}_+$, α, $\beta\in\boldsymbol{Z}^n_+$, ε, $\delta>0$ *there exists a constant* $C>0$ *for which*:

(2.37) $$\sup_{\substack{(x,y)\in\Omega\times\Omega \\ |t|\le\delta,\rho\in]0,1]}} |\rho^{\varepsilon}(\rho\partial_\rho)^j\partial_t^p\ \partial_x^\alpha\ \partial_y^\beta\ r(\rho,\ t,\ x,\ y)|\le C.$$

By $OPHS^{m,k}$ *(resp.* $OPH\Sigma^{m,k}$*) we denote the class of all operators A of the form:*

(2.38) $$A=a(\rho,\ t,\ x,\ D_x)+R,$$

where R *is H. p. r.,* $a\in HS^{m,k}$ *(resp.* $a\in H\Sigma^{m,k}$*) and* $a(\rho,\ t,\ x,\ D_x)$ *is defined as follows:*

(2.39) $$a(\rho,\ t,\ x,\ D_x)\ f(t,\ x)$$
$$=\int_0^1\int e^{ix\cdot\xi}a(\rho,\ t,\ x,\ \xi)\ \hat{f}(\rho t,\ \xi)d\rho\, d\!\!{}^-\xi,\ \ f\in C^\infty(\boldsymbol{R}_t\,;\ C_0^\infty(\boldsymbol{R}^n)).$$

By $OPH\hat{\Sigma}^{m,k}$ *we denote the class of all operators A of the form:*

(2.40) $$A=a(\rho,\ t,\ x,\ D_x)+r'(p,\ t,\ x,\ D_x)+R,$$

for some $a\in H\hat{\Sigma}^{m,k}$ *(called a symbol for A),* $r'\in HS^{m,\infty}$ *and R is H. p. r.*

Some relevant properties of the above defined operators are listed in the following lemma, whose proof is left to the reader.

LEMMA 2.5. *We have:*

1. *Every operator* $A\in OPHS^{m,k}$ *can be continuously extended as an operator from* $C^\infty(\boldsymbol{R}_t\,;\ \mathscr{E}'(\boldsymbol{R}^n))$ *into* $C^\infty(\boldsymbol{R}_t\,;\ \mathscr{D}'(\boldsymbol{R}^n))$.

2. *Let* $A\in OPS^{m,k}$, $B\in OPHS^{m',k'}$, *one of them being a proper operator; then both AB and BA belong to* $OPHS^{m+m',k+k'}$. *Furthermore, if a(resp. b) is a symbol for A(resp. B) and c(resp. c') is a symbol for BA(resp. AB), then:*

(2.41) $$\begin{cases} c\sim\sum_\alpha\frac{1}{\alpha!}\partial_\xi^\alpha bD_x^\alpha a \\ c'\sim\sum_\alpha\frac{1}{\alpha!}\partial_\xi^\alpha aD_x^\alpha b. \end{cases}$$

Moreover, if $A\in OP\hat{\Sigma}^{m,k}$, *with* $a\sim\sum_{j\ge0} a_j$, $a_j\in\Sigma^{m,k+j}$, *and* $B\in OPH\hat{\Sigma}^{m',k'}$, *with* $b\sim\sum_{j\ge0} b_j$, $b_j\in H\Sigma^{m',k'+j}$, *then both AB and BA belong to* $OPH\hat{\Sigma}^{m+m',k+k'}$ *and* (2.41) *becomes:*

(2.41)′ $$\begin{cases} c\sim\sum_{j\ge0}\left(\sum_{|\alpha|+j'+j''=j}\frac{1}{\alpha!}\partial_\xi^\alpha\ b_{j'}\ D_x^\alpha\ a_{j''}\right) \\ c'\sim\sum_{j\ge0}\left(\sum_{|\alpha|+j'+j''=j}\frac{1}{\alpha!}\partial_\xi^\alpha\ a_{j'}\ D_x^\alpha\ b_{j''}\right) \end{cases}$$

Concerning point 1. we should add that inequalities (2.14) hold. We point out that the analysis of $\widetilde{WF}(Af)$ in terms of $\widetilde{WF}(f)$ for operators $A \in OPHS^{m,k}$ is postponed to Chapter 5 where we shall have to deal with a more general situation.

GENERAL REMARK. In order to symplify notation we have defined our symbol classes $S^{m,k}, \Sigma^{m,k}, \hat{\Sigma}^{m,k}$, etc. (and the related operators) on the whole $\boldsymbol{R}_t \times \boldsymbol{R}^n$. However, with trivial modifications we can (and in fact, we shall) consider the same classes as defined in a slab $]-T, T[\times \boldsymbol{R}^n$, $T>0$. In this case we shall add a subscript T to the name of the classes (e. g. $S_T^{m,k}$, $\Sigma_T^{m,k}$, etc.). We emphasize that *all* the results stated in this Chapter carry over (with obvious modifications) to this more general situation.

3. Decoupling of a Hyperbolic Fuchsian System.

In Chapter 1 we have shown how an equation $Pu=f$, with $P \in F_0^m(\boldsymbol{R}\times \boldsymbol{R}^n)$ can be reduced to an equivalent (mod. C^∞) singular system of the form:

$$\mathscr{P} v = I_N t\partial_t v - tA(t, x, D_x)v - B(t, x, D_x)v = g, \tag{3.1}$$

where $A \in OPS^1_{cl}(\boldsymbol{R}^n; N\times N)$, $B \in OPS^0_{cl}(\boldsymbol{R}^n; N\times N)$ are suitable matrices of classical pdo's, depending smoothly on $t \in \boldsymbol{R}$.

The main property of the matrix A consists in the fact that its principal symbol has purely imaginary eigenvalues and can be smoothly diagonalized.

In the classical hyperbolic case, i. e. if we had $I_N\partial_t - A$ in place of $I_N t\partial_t - tA$, a general procedure to study such a system would be to decouple the system, that is to put B (via some intertwining elliptic operator) in a block diagonal form and then study the so obtained decoupled equations (see e. g. Taylor [22], Chap. 9, § 1).

In our situation the standard decoupling procedure cannot be applied due to the presence of the t-degeneracy.

The main result of this Chapter is that the system (3. 1) can be 《decoupled》 at least for large values of $t|\xi|$. The precise meaning of this assertion is clarified in the statement of Theorem 3.1.

Let us fix our hypotheses.

h_1) A is in a block diagonal form:

$$A = \begin{pmatrix} A_1 & & \square \\ & A_2 & \\ & & \ddots \\ \square & & & A_\nu \end{pmatrix}, \quad A_j \in OPS^1_{cl}(\boldsymbol{R}^n; N_j\times N_j), \ j=1, \dots, \nu \tag{3.2}$$

$N_1 + \dots + N_\nu = N$, with:

$$A_j(t, x, D_x) = \sqrt{-1}\,\lambda_j(t, x, D_x) I_{N_j}, \tag{3.3}$$

the λ_j being real-valued smooth functions defined on $\boldsymbol{R}_t \times \boldsymbol{R}^n_x \times \boldsymbol{R}^n_\xi \backslash 0$, positively homogeneous of degree 1 in ξ and such that:

(3.4) $$i \neq j \Rightarrow \lambda_i(t, x, \xi) \neq \lambda_j(t, x, \xi), \text{ for every } t, x, \xi \neq 0.$$

h_2) $B \in OPS^0_{cl}(\boldsymbol{R}^n; N \times N)$, with symbol $b(t, x, \xi)$ having an asymptotic expansion $b(t, x, \xi) \sim \sum_{r \geq 0} b_{-r}(t, x, \xi)$.

It is convenient to write B in a block form $B = (B_{ij})_{i,j=1,\dots,\nu}$, $B_{ij} \in OPS^0_{cl}(\boldsymbol{R}^n; N_i \times N_j)$ and denote by $b_{ij}(t, x, \xi)$ the symbol of B_{ij}, whose expansion will be written as $b_{ij} \sim \sum_{r \geq 0} b_{ij,-r}$.

According to Lemma 2.3, 3. and formula (2.28), $B \in OP\hat{\Sigma}^{0,0}(N \times N)$ with asymptotic expansion (in $\hat{\Sigma}^{0,0}$) of the symbol b given by:

(3.5) $$\begin{cases} b(t, x, \xi) \sim \sum_{r \geq 0} b'_r(t, x, \xi) \\ b'_r(t, x, \xi) = |\xi|^{-r} \hat{b}_r(x, \xi/|\xi|, t|\xi|), \ r \geq 0. \\ \hat{b}_r(x, \xi', z) = \sum_{j=0}^{r} \frac{1}{(r-j)!} \partial_t^{r-j} b_{-j}(0, x, \xi') z^{r-j} \end{cases}$$

In the block decomposition of B we write $b'_{ij,r}$ and $\hat{b}_{ij,r}$, $i, j = 1, \dots, \nu$.

We are ready to state the main theorem.

THEOREM 3.1. *Le $\mathscr{P}$ be the system (3.1) satisfying hypotheses h_1) and h_2). For every $\omega \subset\subset \boldsymbol{R}^n$ there exist $\delta, \delta' > 0$, $\delta < \delta'$, depending on ω and there exist:*

1. *$Q \in OP\hat{\Sigma}^{0,0}(N \times N)$, proper, with symbol:*

$$q \sim \sum_{j \geq 0} q_j, \ q_j \in \Sigma^{0,j}(N \times N), \ q_j(t, x, \xi) = |\xi|^{-j} \hat{q}_j(x, \xi/|\xi|, t|\xi|),$$
$$\hat{q}_j \in \boldsymbol{S}^j(N \times N), \ j \geq 0.$$

Denoting by $\hat{q}_j \sim \sum_{k \geq 0} \hat{q}_{j,-k}(x, \xi') z^{j-k}$ the asymptotic expansion of $\hat{q}_j$ in $\boldsymbol{S}^j$, we have:

i) *$\hat{q}_0(x, \xi', z)$ is an invertible matrix for every $(x, \xi', z) \in \boldsymbol{R}^n \times S^{n-1} \times \boldsymbol{R}$. Furthermore, $q_0(x, \xi, z) = I_N$ for $(x, \xi') \in \omega \times S^{n-1}$ and $|z| \leq \delta$.*

ii) *For $j \geq 1$, $\hat{q}_j(x, \xi', z) = \square$ for $(x, \xi') \in \omega \times S^{n-1}$ and $|z| \leq \delta$.*

iii) *For every $j \geq 0$, $k \geq 1$, $\hat{q}_{j,-k}$ is an extra-diagonal matrix, i.e. all the diagonal blocks $N_l \times N_l$, $l = 1, \dots, \nu$, vanish.*

Moreover, $\hat{q}_{j,0} = \square$ for every $j \geq 1$.

2. *$\tilde{B} \in OP\hat{\Sigma}^{0,0}(N \times N)$, proper, with symbol:*

$$\tilde{b} \sim \sum_{j \geq 0} \tilde{b}_j, \ \tilde{b}_j \in \Sigma^{0,j}(N \times N), \ \tilde{b}_j(t, x, \xi) = |\xi|^{-j} \hat{\tilde{b}}_j(x, \xi/|\xi|, t|\xi|),$$
$$\hat{\tilde{b}}_j \in \boldsymbol{S}^j(N \times N), \ j \geq 0.$$

Denoting by $\tilde{\tilde{b}}_j \sim \sum_{k\geq 0} \tilde{\tilde{b}}_{j,-k}(x, \xi') z^{j-k}$ *the asymptotic expansion of* $\tilde{\tilde{b}}_j$ *in* $\boldsymbol{S}^j$, *we have*:

i) $\tilde{\tilde{b}}_j(x, \xi', z) = \hat{b}_j(x, \xi', z)$ (*see* (3.5)), *for every* $(x, \xi') \in \omega \times S^{n-1}$, $|z| \leq \delta$. *and for every* $j \geq 0$.

ii) $\tilde{b}(t, x, \xi)$ *and* $\tilde{b}_j(t, x, \xi)$, *for all* $j \geq 0$, *are in block diagonal form for* $(x, \xi/|\xi|) \in \omega \times S^{n-1}$ *and* $|t||\xi| \geq \delta'$.

iii) *For every* $(x, \xi') \in \boldsymbol{R}^n \times S^{n-1}$, $\tilde{\tilde{b}}_{0,0}(x, \xi')$ *is in diagonal block form and precisely*:

$$\tilde{b}^{(j,j)}_{0,0}(x, \xi') = \begin{cases} \square, & i \neq j \\ \hat{b}_{jj,0}(x, \xi'), & i = j \end{cases} \qquad ,i,\ j = 1, \dots, \nu.$$

(*see* (3.5)).

Such that, putting:

$$(3.6) \qquad \tilde{\mathscr{P}} = I_N\, t\, \partial_t - tA(t, x, D_x) - \tilde{B}(t, x, D_x),$$

we have:

$$(3.7) \qquad \mathscr{P} Q - Q \tilde{\mathscr{P}} \text{ is a partially regularizing operator.}$$

The proof of the Theorem is based on some preliminary lemmas.

LEMMA 3.1. *Suppose we are given two* $N \times N$ *smooth matrices* $a(x, \xi')$, $b(x, \xi')$ *defined on* $\boldsymbol{R}^n_x \times S^{n-1}_{\xi'}$, *with* a *having the following structure*:

$$(3.8) \qquad a = \begin{pmatrix} a_1 & & \square \\ & \ddots & \\ \square & & a_\nu \end{pmatrix}, \quad a_j(x, \xi') = \sqrt{-1} \begin{bmatrix} \lambda_j(x, \xi') & & \square \\ & \ddots & \\ \square & & \lambda_j(x, \xi') \end{bmatrix}$$

is a $N_j \times N_j$ *diagonal matrix,* $N_1 + \dots + N_\nu = N$.

We suppose that the λ_j *are real-valued and satisfy* $\lambda_i(x, \xi') \neq \lambda_j(x, \xi')$ *for every* x, ξ', *provided* $i \neq j$.

Then for every $\omega \subset\subset \boldsymbol{R}^n$ *there exist* δ, $\delta' > 0$, $\delta < \delta'$, *depending on* ω, *and there exist*:

1. $q(x, \xi', z) \in \boldsymbol{S}^0(N \times N)$ *with* $q \sim \sum_{j\geq 0} q_{-j}(x, \xi') z^{-j}$, $z \to \infty$, *such that*:

i) $q(x, \xi', z)$ *is an invertible matrix for every* x, ξ', z.

ii) $q(x, \xi', z) = I_N$, *for* $(x, \xi') \in \omega \times S^{n-1}$ *and* $|z| \leq \delta$.

iii) $q_0(x, \xi') \equiv I_N$ *and, for* $j \geq 1$, *all the blocks on the diagonal of* q_{-j} *vanish.*

2. $\tilde{b}(x, \xi', z) \in \boldsymbol{S}^0(N \times N)$ *with* $\tilde{b} \sim \sum_{j\geq 0} \tilde{b}_{-j}(x, \xi') z^{-j}$, $z \to \infty$, *such that*:

i) $\tilde{b}(x, \xi', z)$ *is a block diagonal matrix for every* $(x, \xi') \in \omega \times S^{n-1}$

and $|z|\geq\delta'$.

ii) $\tilde{b}(x,\ \xi',\ z)=b(x,\ \xi')$ *for every* $(x,\ \xi')\in\omega\times S^{n-1}$ *and* $|z|\leq\delta$.

iii) *If* $(b_{hk})_{h,k=1,\cdots,\nu}$ (*resp.* $(\tilde{b}_0^{(hk)})_{h,k=1,\cdots,\nu}$) *is the block decomposition of* b (*resp.* $\tilde{b}_0$), *then*:

$$\tilde{b}_0^{(hk)}\equiv\begin{cases}\square & ,\ h\neq k\\ b_{hk} & ,\ h=k.\end{cases}$$

such that the following equation is satisfied:

$$z\partial_z q(x,\ \xi',\ z)-z[a(x,\ \xi'),\ q(x,\ \xi',\ z)] \\ -b(x,\ \xi')q(x,\ \xi',\ z)+q(x,\ \xi',\ z)\tilde{b}(x,\ \xi',\ z)=\square, \tag{3.9}$$

for all $x,\ \xi',\ z$.

PROOF. Suppose we have already constructed two $N\times N$ matrices $\hat{q}(x,\xi',z)\in C^{\infty}(\boldsymbol{R}^n\times S^{n-1}\times\boldsymbol{R}\backslash 0)$, $\hat{b}\in\boldsymbol{S}^0(N\times N)$ such that:

a) $\hat{b}$ satisfies condition 2. iii) and is block diagonal for all $x,\ \xi',\ z$.

b) $\hat{q}(x,\ \xi',\ z)$ has an asymptotic expansion $\sum_{j\geq 0}q_{-j}(x,\ \xi')z^{-j}$, $z\rightarrow\infty$, and satisfies condition 1. iii).

Furthermore, suppose that the following equation is satisfied:

$$z\partial_z\hat{q}-z[a,\ \hat{q}]-b\hat{q}+\hat{q}\hat{b}=\square, \tag{3.10}$$

for all $(x,\ \xi')\in\boldsymbol{R}^n\times S^{n-1}$ and for all $z\neq 0$.

Let us show how this implies the Lemma.

Let $\chi(x,\ z)$ be a cut-off function (see (2.3)) and define:

$$\begin{cases}q(x,\ \xi',\ z)=\chi(x,\ z)I_N+(1-\chi(x,\ z))\hat{q}(x,\ \xi',\ z)\\ \hat{b}_1(x,\ \xi',\ z)=\chi(x,\ z)b(x,\ \xi')+(1-\chi(x,\ z))\hat{b}(x,\ \xi',\ z)\end{cases}$$

Putting $f=z\partial_z q-z[a,\ q]-bq+q\hat{b}_1$, it is easily seen that $f\in\boldsymbol{S}^0(N\times N)$ and for every $\omega\subset\subset\boldsymbol{R}^n$, $f\in C_0^{\infty}(\boldsymbol{R}_z\backslash 0\,;\,C^{\infty}(\omega\times S^{n-1}))$. On the other hand, from b) it follows that we can choose the cut off χ in such a way that for every $\omega\subset\subset\boldsymbol{R}^n$:

$$\sup_{\substack{x\in\omega,\xi'\in S^{n-1}\\ z\in\boldsymbol{R}}}|q(x,\ \xi',\ z)-I_N|<1/2. \tag{3.12}$$

With this choice of χ the matrix $q\in\boldsymbol{S}^0(N\times N)$ satisfies conditions 1. i)-iii) (for suitable δ, δ' depending on ω) and has the same asymptotic expansion of q for $z\rightarrow\infty$. Since $q^{-1}\in\boldsymbol{S}^0(N\times N)$, by defining:

$$\tilde{b}=\hat{b}_1-q^{-1}f \tag{3.13}$$

one can check that $\tilde{b}$ verifies conditions 2. i)-iii) and that eq. (3.9) is satisfied for all $x,\ \xi',\ z$.

To construct $\hat{q}$, $\hat{b}$ satisfying conditions a) and b) we consider two formal power series $\sum_{j\geq 0} q_{-j}(x, \xi')z^{-j}$, $\sum_{j\geq 0} \hat{b}_{-j}(x, \xi')z^{-j}$, where the $\hat{q}_{-j}$, $\hat{b}_{-j}$ are smooth $N\times N$ matrices defined on $\boldsymbol{R}^n\times S^{n-1}$, and try to solve eq. (3.10) at a formal level.

We obtain:

$$(3.14)\qquad \sum_{j\geq 1}(-j)\hat{q}_{-j}z^{-j}-\sum_{j\geq 0}(a\hat{q}_{-j}-\hat{q}_{-j}a)z^{-j+1} -\sum_{j\geq 0} b\hat{q}_{-j}z^{-j}+\sum_{j,k\geq 0}\hat{q}_{-j}\hat{b}_{-k}z^{-(j+k)}=\square .$$

Imposing that the coefficient of z in (3.14) vanishes we have the equation $\hat{q}_0 a-a\hat{q}_0=\square$, *which is solved by taking* $\hat{q}_0\equiv I_N$. Imposing that the coefficient of z^0 in (3.14) vanishes we have the equation:

$$(3.15)\qquad [\hat{q}_{-1}\ ,\ a]=b-\hat{b}_0\ .$$

Write $b=(b_{hk})_{h,k=1,\dots,\nu}$ in block form and define $\hat{b}_0$ as the block diagonal part of b. It follows that $b-\hat{b}_0$ is a block extra-diagonal matrix. Write $\hat{q}_{-1}=(\hat{q}_{-1}^{(hk)})_{h,k=1,\dots,\nu}$ in block form and define (taking into account (3.8)):

$$(3.16)\qquad \hat{q}_{-1}^{(hk)}=\begin{cases}\square & ,\ \text{if } h=k\\ -\sqrt{-1}(\lambda_k-\lambda_h)^{-1}\ b_{hk}\ , & \text{if } h\neq k.\end{cases}$$

With this choice eq. (3.15) is satisfied.

By an induction procedure, suppose we have already constructed $\hat{b}_0,\dots,\hat{b}_{-(l-1)}$ (*in diagonal block form*) *and* $\hat{q}_0$, $\hat{q}_{-1},\dots,\hat{q}_{-l}$ (with $\hat{q}_{-j}$, $j\geq 1$, in extra-diagonal block form). Imposing that the coefficient of z^{-l} in (3.14) vanishes we have the equation:

$$(3.17)\qquad [\hat{q}_{-(l+1)}\ ,\ a]-(l\ I_N+b)\hat{q}_{-l}+\sum_{\substack{j+r=l\\ r<l}}\hat{q}_{-j}\ \hat{b}_{-r}+\hat{b}_{-l}=\square .$$

Putting $\varphi=(lI_N+b)\hat{q}_{-l}-\sum_{\substack{j+r=l\\ r<l}}\hat{q}_{-j}\ \hat{b}_{-r}$, *we define* $\hat{b}_{-l}$ *as the block diagonal part of* φ. *Writing* $\hat{q}_{-(l+1)}=(\hat{q}_{-(l+1)}^{(hk)})_{h,k=1,\dots,\nu}$ in block form, we define:

$$(3.18)\qquad \hat{q}_{-(l+1)}^{(hk)}=\begin{cases}\square & ,\ \text{if } h=k\\ -\sqrt{-1}\ (\lambda_k-\lambda_h)^{-1}\varphi^{(hk)}, & \text{if } h\neq k.\end{cases}$$

Having satisfied eq. (3.10) at formal level, we construct $\hat{b}\in\boldsymbol{S}^0$ $(N\times N)$, $\hat{b}$ block diagonal, with $\hat{b}\sim\sum_{j\geq 0}\hat{b}_{-j}\ z^{-j}$, and $q^*\in\boldsymbol{S}^0$ $(N\times N)$ with $q^*\sim\sum_{j\geq 0}\hat{q}_{-j}\ z^{-j}$. Conditions a) and b) are satisfied by q^*, $\hat{b}$.

Putting $-g=z\partial_z\ q^*-z[a,\ q^*]-bq^*+q^*\hat{b}$, it can be easily seen that $g(x,\ \xi',\ z)\in S^{-\infty}_{1,0}((\boldsymbol{R}^n\times S^{n-1})\times\boldsymbol{R}_z;N\times N)$.

Now we look for a $N\times N$ matrix $\varphi(x,\ \xi',\ z)\in C^\infty(\boldsymbol{R}^n\times S^{n-1}\times\boldsymbol{R}_z\backslash 0)$, rapidly decreasing for $z\to\infty$, such that:

(3.19) $$z\partial_z\varphi-z[a,\ \varphi]-b\varphi+\varphi\hat{b}=g,$$

for all $(x,\ \xi')\in\boldsymbol{R}^n\times S^{n-1}$ and all $z\neq 0$.

Once such a φ is obtained it is enough to put $\hat{q}=q^*+\varphi$. To construct φ we consider for every $M>0$ the space $\mathscr{F}_M$ of all $N\times N$ matrices $v(x,\ \xi',\ z)\in C^\infty(\boldsymbol{R}^n\times S^{n-1}\times\{z\|z|\geq M\};N\times N)$ having the following property: for every $\omega\subset\subset\boldsymbol{R}^n$, $p,\ q,\in\boldsymbol{Z}_+$, $\alpha\in\boldsymbol{Z}^n_+$, and for every family $\theta_1,\dots,\theta_l$ of smooth vector fields on S^{n-1}, we have:

(3.20) $$\sup_{\substack{x\in\omega,\ \xi'\in S^{n-1}\\ |z|\geq M}}|z^p\partial^q_z\ \partial^\alpha_x\theta_1\dots\theta_l\ \mathrm{v}(x,\ \xi',\ \mathrm{z})|<+\infty.$$

Equipped with the seminorms defined by (3.20) $\mathscr{F}_M$ becomes a Fréchet space. Consider the following continuous operators acting on $\mathscr{F}_{\mathrm{M}}$:

(3.21) $$\begin{cases} Lv=bv-v\hat{b}, & v\in\mathscr{F}_M\\ Rv=e^{za}\ ve^{-za}, & v\in\mathscr{F}_M.\end{cases}$$

The continuity of L is trivial while the continuity of R is a consequence of (3.8). Putting $H=R^{-1}\ LR$ and defining $g'=R^{-1}\ g$, eq. (3.19) becomes:

(3.22) $$z\partial_z\psi-H\psi=g',\quad \psi=R\varphi.$$

We now argue for $z\geq M$ (the case $z\leq -M$ can be handled analogously). Changing $z=e^s$, $s\geq\ln\ M$ and calling $\tilde{H}(s)$ the operator H in the new variable s, consider the evolution operator $U(s,\ s';\ x,\ \xi')$ defined by:

(3.23) $$\begin{cases}\dfrac{d}{ds}U(s,\ s')-\tilde{H}(s)\ \ U(s,\ s')=\square\\ U(s,\ s')|_{s-s'}=I_N\end{cases},\quad s'\geq s\geq\ln\ M.$$

One can easily recognize that the matrix function

(3.24) $$\psi(x,\ \xi',\ z)=-\int_z^{+\infty}U(\ln\ z,\ \ln\ z';\ x,\ \xi')g'(x,\ \xi',\ z')\frac{dz'}{z'},$$

solves eq. (3.22) for $z\geq M$.

To prove that ψ is actually in $\mathscr{F}_M$ (for $z\geq M$) we use energy estimates. Precisely, for every $\omega\subset\subset\boldsymbol{R}^n$, we obtain from (3.23)

(3.25) $$\sup_{\substack{x\in\omega\\ \xi'\in S^{n-1}}} \|U(s,\ s';\ x,\ \xi')\|^2\leq e^{2C(s-s')}\quad s'\geq s\geq \ln M,$$

for some constant $C>0$ depending on ω. Thus, for every $p\in\boldsymbol{Z}_+$, $p\geq 1$, we get from (3.24):

(3.26) $$z^p\|\psi(x,\ \xi',\ z)\|\leq \int_z^{+\infty}\left(\frac{z}{z'}\right)^{C+p} z'^p\|g'(x,\ \xi',\ z')\|\frac{dz'}{z'}$$

$$\leq \sup_{\substack{|z|\geq M\\ (x,\ \xi')\in\omega\times S^{n-1}}} \|z^p g'(x,\xi',z)\|\frac{1}{C+p},\quad x\in\omega, |\xi'|=1,\ z\geq M.$$

Taking derivatives of eq. (3.22) and using estimates of the above type, we conclude that $\psi\in\mathscr{F}_M$ and therefore $\varphi=R^{-1}\ \psi\in\mathscr{F}_M$.

As $M>0$ can be arbitrarily chosen we obtain a smooth solution of (3.19) defined for $z\neq 0$ and rapidly decreasing at infinity.

LEMMA 3.2. *Let $a(x,\ \xi')$ and $b(x,\ \xi')$ be as in Lemma 3.1. Suppose we are given $c\in\boldsymbol{S}^m(N\times N)$, $m\in\boldsymbol{Z}_+$, $m\geq 1$, with expansion*

(3.27) $$c\sim\sum_{j=0}^{m} c_{m-j}\ z^{m-j}+\sum_{k>0} c_{-k}\ z^{-k},\quad z\rightarrow\infty.$$

There exist:

1. *$Q\in\boldsymbol{S}^m(N\times N)$, with expansion*

$$Q\sim\sum_{j=0}^{m} Q_{m-j}\ z^{m-j}+\sum_{k>0} Q_{-k}\ z^{-k},\quad z\rightarrow\infty,$$

such that:

i) *$Q(x,\ \xi',\ z)=\square$, for $(x,\ \xi')\in\omega\times S^{n-1}$ and $|z|\leq\delta$, with the same δ (and δ') as in Lemma 3.1.*

ii) *$Q_m\equiv\square$ and all the coefficients in the expansion of Q are extra-diagonal block matrices.*

2. *$\tilde{B}\in\boldsymbol{S}^m(N\times N)$, with expansion*

$$\tilde{B}\sim\sum_{j=0}^{m}\tilde{B}_{m-j}\ z^{m-j}+\sum_{k>0}\tilde{B}_{-k}\ z^{-k},\quad z\rightarrow\infty,$$

such that:

i) *$\tilde{B}(x,\ \xi',\ z)=c(x,\ \xi',\ z)$, for $(x,\ \xi')\in\omega\times S^{n-1}$ and $|z|\leq\delta$.*

ii) *$\tilde{B}(x,\ \xi',\ z)$ is a block diagonal matrix, for $(x,\ \xi')\in\omega\times S^{n-1}$ and $|z|\geq\delta'$.*

Such that the following equation is satisfied for every $x,\ \xi',\ z$:

(3.28) $$z\partial_z Q-z[a,\ Q]-bQ+Q\tilde{b}+q\tilde{B}=c,$$

where q and $\tilde{b}$ are the matrices constructed in Lemma 3.1.

PROOF. Suppose we have already constructed two $N\times N$ matrices

$\hat{Q}(x, \xi', z) \in C^{\infty}(\boldsymbol{R}^n \times S^{n-1} \times \boldsymbol{R}_z \backslash 0)$, $\hat{B} \in \boldsymbol{S}^m(N \times N)$, such that:

a) $\hat{B}$ is a block diagonal matrix for every x, ξ', z.

b) $\hat{Q} \sim \sum_{j=0}^{m} \hat{Q}_{m-j} z^{m-j} + \sum_{k>0} \hat{Q}_{-k} z^{-k}$, $z \to \infty$, and satisfies 1. ii).

Furthermore, suppose the following equation is satisfied:

$$z\partial_z \hat{Q} - z[a, \hat{Q}] - b\hat{Q} + \hat{Q}\tilde{b} + q\hat{B} = c, \tag{3.29}$$

for all, x, ξ' and for $z \neq 0$.

We now show that this implies the Lemma. With the same choice of the cut-off function $\chi(x, z)$ made in Lemma 3.1, define:

$$\begin{cases} Q(x, \xi', z) = (1-\chi(x, z))\hat{Q}(x, \xi', z) \\ B^*(x, \xi', z) = \chi(x, z)\ c(x, \xi', z) + (1-\chi(x, z))\hat{B}(x, \xi', z). \end{cases} \tag{3.30}$$

Q satisfies conditions 1. i), ii). Moreover, putting $f = z\partial_z Q - z[a, Q] - bQ + Q\tilde{b} + qB^* - c$, we can see that for very $(x, \xi') \in \omega \times S^{n-1}$, $f(x, \xi', z) = \square$ for $|z| \leq \delta$ and for $|z| \geq \delta'$. Defining:

$$\tilde{B} = B^* - q^{-1} f, \tag{3.31}$$

one can see that $\tilde{B}$ verifies 2. i), ii) and that eq. (3.28) is satisfied.

To construct $\hat{Q}$ and $\hat{B}$ we proceed as in the proof of Lemma 3.1 trying to solve eq. (3.28) at a formal power series level. We obtain:

$$\sum_{j\geq 0} (m-j)\hat{Q}_{m-j}\, z^{m-j} - \sum_{j\geq 0} [a, \hat{Q}_{m-j}]\, z^{m-j+1} - \sum_{j\geq 0} b\hat{Q}_{m-j}\, z^{m-j} + \sum_{j,\, r\geq 0} \hat{Q}_{m-j}\, \tilde{b}_{-r}\, z^{m-(j+r)} + \sum_{j,\, r\geq 0} q_{-j}\, \hat{B}_{m-r}\, z^{m-(j+r)} - \sum_{j\geq 0} c_{m-j}\, z^{m-j} = \square. \tag{3.32}$$

Imposing that the coefficient of z^{m+1} vanishes we have the equation $[a, \hat{Q}_m] = \square$, which is solved by $\hat{Q}_m \equiv \square$. From now on the proof proceeds exactly as in Lemma 3.1. We leave the details to the reader.

REMARK 3.1. It will be crucial to keep in mind that, due to property 1. ii), the matrix Q constructed in the above Lemma actually belongs to $\boldsymbol{S}^{m-1}(N \times N)$.

PROOF of THEOREM 3.1.

By Lemma 2.3, the operator $tA(t, x, D_x)$ belongs to $OP\hat{\Sigma}^{1,1}(N \times N)$ and its symbol $tA(t, x, \xi)$ has the asymptotic expansion:

$$\begin{cases} tA(t, x, \xi) \sim \sum_{j\geq 0} a_{1+j}(t, x, \xi) \\ a_{1+j}(t, x, \xi) = |\xi|^{-j}\, \hat{a}_{1+j}(x, \xi/|\xi|,\ t|\delta|) \qquad ,j \geq 0. \\ \hat{a}_{1+j}(x, \xi', z) = \dfrac{1}{j!}\, (\partial_t^j A)(0, x, \xi') z^{1+j} \end{cases} \tag{3.33}$$

For sake of convenience we will write:

$$a(x,\xi')=\hat{a}_1(x,\xi')=A(0,x,\xi'). \tag{3.34}$$

Let $q_0(t,x,D_x)\in OP\Sigma^{0,0}(N\times N)$ be a proper operator such that in the asymptotic expansion $\sum_{k\geq 0} q_{0,-k}(x,\xi')z^{-k}$ of $\hat{q}_0$ we have $q_{0,0}\equiv I_N$. With this choice, it is easy to verify that the commutator $[tA,q_0]\in OP\hat{\Sigma}^{0,0}(N\times N)$.

If $\tilde{b}_0(t,x,D_x)\in OP\Sigma^{0,0}(N\times N)$ is a proper operator, define:

$$\tilde{\mathscr{P}}_0=I_N\, t\partial_t-tA(t,x,D_x)-\tilde{b}_0(t,x,D_x). \tag{3.35}$$

Then the term in $\Sigma^{0,0}(N\times N)$ of the symbol of $\mathscr{P}q_0-q_0\mathscr{P}_0$ is given by:

$$(t\partial_t q_0)(t,x,\xi)-t[a(x,\xi),\ q_0(t,x,\xi)]-b_0(t,x,\xi)q_0(t,x,\xi)+q_0(t,x,\xi)\tilde{b}_0(t,x,\xi). \tag{3.36}$$

We apply Lemma 3.1 (with a defined by (3.34) and $b=\hat{b}_0(x,\xi')$ as given by (3.5)) and obtain two symbols q_0, $\tilde{b}_0\in\Sigma^{0,0}(N\times N)$ such that

$$\mathscr{P}q_0-q_0\tilde{\mathscr{P}}_0\in OP\hat{\Sigma}^{0,1}(N\times N). \tag{3.37}$$

We now proceed by induction. Suppose we have constructed $\hat{q}_j$, $\hat{\tilde{b}}_j\in S^j(N\times N)$, $j=0,\dots,M$ with the following properties:

For every $\omega\subset\subset\boldsymbol{R}^n$ there exist δ, $\delta'>0$, $\delta<\delta'$, such that:

α) i) $\hat{q}_0(x,\xi',z)$ is an invertible matrix for every x, ξ', z.

ii) $\hat{q}_0(x,\xi',z)=I_N$ and, for $j=1,\dots,M$, $\hat{q}_j(x,\xi',z)=\square$, for every $(x,\xi')\in\omega\times S^{n-1}$ and $|z|\leq\delta$.

iii) If $\sum_{k\geq 0} q_{j,-k}(x,\xi')z^{j-k}$ is the asymptotic expansion of $\hat{q}_j$, then:

$$\begin{cases} q_{0,0}(x,\xi')\equiv I_N, \\ q_{j,0}(x,\xi')\equiv\square,\ j\geq 1, \\ q_{j,-k}(x,\xi')\ \text{is}\equiv\text{a block extra-diagonal matrix},\ j\geq 0,\ k\geq 1. \end{cases}$$

β) i)

$$\hat{\tilde{b}}_j(x,\xi',z)=\begin{cases}\hat{b}_j(x,\xi',z), & \text{for } |z|\leq\delta \\ \text{a block diagonal matrix}, & \text{for } |z|\geq\delta\end{cases},\quad j=0,\dots,M,$$

for every $(x,\xi')\in\omega\times S^{n-1}$, with $\hat{b}_j$ given by (3.5).

ii) If $\sum_{k\geq 0}\tilde{b}_{j,-k}(x,\xi')z^{j-k}$ is the asympotic expansion of $\hat{\tilde{b}}_j$, then

$$\tilde{b}_{0,0}^{(hk)}(x,\xi')\equiv\begin{cases}\hat{b}_{hh,0}(x,\xi'), & \text{for } h=k \\ \square, & \text{for } h\neq k\end{cases},\quad h,\ k=1,\dots,v.$$

Futhermore, if $q_j(t,x,D_x)\in OP\Sigma^{0,j}(N\times N)$ (resp. $\tilde{b}_j(t,x,D_x)\in$

$OP\Sigma^{0,j}(N\times N))$ denotes a proper operator with symbol $q_j\ (t, x, \xi)=|\xi|^{-j}\ \hat{q}_j\ (x, \xi/|\xi|,\ t|\xi|)$(resp. $\tilde{b}_j(t, x, \xi)=|\xi|^{-j}\ \hat{\tilde{b}}_j\ (x, \xi/|\xi|,\ t|\xi|))$ putting:

$$(3.38)\qquad \tilde{\mathscr{P}}_j=I_N t\partial_t-tA(t, x, D_x)-\sum_{k=0}^{j}\tilde{b}_k(t, x, D_x),\ j=0, 1, \ldots,$$

we have:

γ) i) $\mathscr{P}\left(\sum_0^M {}_k q_k(t, x, D_x)\right)-\left(\sum_0^M {}_k q_k(t, x, D_x)\right)\tilde{\mathscr{P}}_M\in OP\hat{\Sigma}^{0,M+1}(N\times N)$.

ii) Denoting by $\psi_{M+1}(t,\ x,\ \xi)$ the term bolonging to $\Sigma^{0,M+1}(N\times N)$ in the symbol of the above operator, we have:

$$(3.39)\qquad \hat{\psi}_{M+1}(x, \xi', z)=-\hat{b}_{M+1}(x, \xi', z),\ \text{for every}\ (x, \xi')\in\omega\times S^{n-1},\ |z|\leq\delta.$$

We look for two proper operators $q_{M+1}(t, x, D_x)$, $\tilde{b}_{M+1}(t, x, D_x)\in OP\Sigma^{0,M+1}(N\times N)$, such that

$$\mathscr{P}\left(\sum_0^{M+1}{}_k q_k\right)-\left(\sum_0^{M+1}{}_k q_k\right)\tilde{\mathscr{P}}_{M+1}\in \mathrm{OP}\hat{\Sigma}^{0,M+2}(N\times N).$$

We use the notation:

$$(3.40)\qquad J_l=\mathscr{P}\left(\sum_0^{l}{}_k q_k\right)-\left(\sum_0^{l}{}_k q_k\right)\tilde{\mathscr{P}}_l,\ l=0, 1, \ldots.$$

Choosing the principal term in the asymptotic expansion of $\hat{q}_{M+1}(x, \xi', z)$ identically equal to the zero matrix, we obtain that $J_{M+1}\in OP\hat{\Sigma}^{0,M+1}(N\times N)$ and the term in $\Sigma^{0,M+1}$ of its symbol is given by:

$$(3.41)\qquad t\partial_t q_{M+1}-t[a,\ q_{M+1}]-b_0 q_{M+1}+q_{M+1}\tilde{b}_0+q_0\tilde{b}_{M+1}+\psi_{M+1},$$

where ψ_{M+1} is the term in $\Sigma^{0,M+1}$ of the symbol of J_M.

We apply Lemma 3.2 (with $m=M+1$, a given by (3.34), $b=\hat{b}_0$ defined by (3.5), $\tilde{b}=\hat{\tilde{b}}_0$, $q=\hat{q}_0$, and $c=-\hat{\psi}_{M+1}(x, \xi', z)$).

From the Lemma we obtain two symbols $\hat{q}_{M+1}$, $\hat{\tilde{b}}_{M+1}\in S^{M+1}(N\times N)$ such that conditions α) and β) are satisfied by $\hat{q}_0, \ldots, \hat{q}_{M+1}$ and $\hat{\tilde{b}}_0, \ldots, \hat{\tilde{b}}_{M+1}$. By the same Lemma, condition γ) i) is satisfied with M replaced by $M+1$. We have to verify that condition γ) ii) is satisfied, when $M+1$ is replaced by $M+2$.

A simple calculation yields:

$$(3.42)\qquad \psi_{M+2}=g+h+u+v,$$

where h (resp. u [resp. v]) is the term in $\Sigma^{0,M+2}$ in the symbol of J_M (resp. $q_1 \tilde{b}_{M+1}$[resp. $q_{M+1} \tilde{b}$]), and g is the term in $\Sigma^{0,M+2}$ in the symbol of $\mathscr{P} q_{M+1} - q_{M+1} \tilde{\mathscr{P}}_0$.

The structure of the $\hat{q}$'s implies immediately that $\hat{u}(x, \xi', z) = \hat{v}(x, \xi', z) = \hat{g}(x, \xi', z) = \square$ for every $(x, \xi') \in \omega \times S^{n-1}$ and $|z| \leq \delta$.
We can write

$$h = \sum_0^M {}_l \left(h'_l + \sum_1^M {}_j \; h''_{l,j} \right),$$

where h'_l (resp. $h''_{l,j}$) is the term in $\Sigma^{0,M+2}$ in the symbol of $\mathscr{P} q_l - q_l \tilde{\mathscr{P}}_0$ (resp. $q_l \tilde{b}_j$).

As a consequence of Lemma 3.2 and by the inductive hypotheses, we have $\hat{h}''_{l,j} (x, \xi', z) = \square$ for every $(x, \xi') \in \times S^{n-1}$ and $|z| \leq \delta$.

On the other hand, we have

$$h'_l = -\sum_{|\alpha|+j=M+2-l} \frac{1}{\alpha!} (\partial_\xi^\alpha a_{1+j} D_x^\alpha q_l - \partial_\xi^\alpha q_l D_x^\alpha a_{1+j} + \partial_\xi^\alpha b_j D_x^\alpha q_l - \partial_\xi^\alpha q_l D_x^\alpha \tilde{b}_0).$$

The structure of the $\hat{q}$'s implies that:

$$\hat{h}'_l (x, \xi', z) = \begin{cases} \square, & \text{if } l > 0 \\ -\hat{b}_{M+2} (x, \xi', z), & \text{if } l = 0, \end{cases}$$

for every $(x, \xi') \in \omega \times S^{n-1}$ and $|z| \leq \delta$.

This proves our claim.

Having constructed the two sequences $(q_j)_{j \geq 0}$, $(\tilde{b}_j)_{j \geq 0}$, let $q(t, x, D_x)$ and $\tilde{b}(t, x, D_x)$ be two proper operators in $OP\hat{\Sigma}^{0,0}$ $(N \times N)$ with symbols $q \sim \sum_{j \geq 0} q_j$, $\tilde{b} \sim \sum_{j \geq 0} \tilde{b}_j$.

Defining

$$\mathscr{P}^{\#} = I_N \, t\partial_t - tA(t, x, D_x) - \tilde{b}(t, x, D_x), \tag{3.43}$$

it is a simple matter to recognize that:

$$\mathscr{P} q(t, x, D_x) - q(t, x, D_x) \mathscr{P}^{\#} = r(t, x, D_x) \in OPS^{0,\infty}(N \times N). \tag{3.44}$$

We have now to exorcise the term $r(t, x, D_x)$ (which is not a partially regularizing operator!).

To start with, let us point out that the above construction implies that:

$$q(t, x, \xi) = q_0(t, x, \xi) + q'(t, x, \xi), \tag{3.45}$$

where $q' \in \hat{\Sigma}^{-1,0}(N \times N)$ and $q_0 \in \Sigma^{0,0}(N \times N)$ is a nonsingular matrix.

It will be convenient to consider two more classes of symbols which are either flat or have polar singularities at $t = 0$.

(3.46) $S_f^m=\{\varphi(t,\ x,\ \xi)\in S_{1,0}^m((\boldsymbol{R}\times\boldsymbol{R}^n)\times\boldsymbol{R}^n)\mid\ \forall\Omega\subset\subset\boldsymbol{R}_t\times\boldsymbol{R}_x^n,$
$M,\ j\in\boldsymbol{Z}_+,\ \alpha,\ \beta\in\boldsymbol{Z}_+^n,\ \exists c>0:$
$|\partial_t^j\partial_x^\alpha\partial_\xi^\beta\varphi(t,\ x,\ \xi)|\le c|t|^M(1+|\xi|)^{m-|\beta|},$
$(t,\ x)\in\Omega,\ \xi\in\boldsymbol{R}^n\}.$

(3.47) $S_P^m=\{\varphi(t,\ x,\ \xi)|\forall k\in\boldsymbol{Z}_+,\ \alpha,\ \beta\in\boldsymbol{Z}_+^n,\ (t\partial_t)^k\partial_x^\alpha\partial_\xi^\beta q\in$
$C^{(0)}(\boldsymbol{R}_t\times\boldsymbol{R}_x^n\times\boldsymbol{R}_\xi^n)$, and $\forall\Omega\subset\subset\boldsymbol{R}_t\times\boldsymbol{R}_x^n,\ \exists c>0:$
$|(t\partial_t)^k\partial_x^\alpha\partial_\xi^\beta\varphi(t,x,\xi)|\le c(1+|\xi|)^{m-|\beta|},\ (t,x)\in\Omega,\ \xi\in\boldsymbol{R}^n\}.$

Since $r(t,\ x,\ \xi)\in S^{0,\infty}(N\times N)$, property 5. in Lemma 2.1 can be restated as:

(3.48) $\begin{cases}\chi(x,\ t|\xi|)r(t,\ x,\ \xi)\in S_{1,0}^{-\infty}((\boldsymbol{R}\times\boldsymbol{R}^n)\times\boldsymbol{R}^n)\\ (1-\chi(x,\ t|\xi|))r(t,\ x,\ \xi)\in S_f^0,\end{cases}$

for every cut-off χ. Furthermore, it is easy to verify that:

(3.49) $a(t,\ x,\ \xi)\in S^{m,0}\Rightarrow(1-\chi(x,\ t|\xi|))a(t,\ x,\ \xi)\in S_P^m,$

for every cut-of χ.

We leave to the reader to verify that the following composition properties hold:

(3.50) $OPS_{1,0}^m\circ OPS_f^{m'}\subset OPS_f^{m+m'},\ OPS_{1,0}^m\circ OPS_P^{m'}\subset OPS_P^{m+m'},$
$OPS_P^m\circ OPS_f^{m'}\subset OPS_f^{m+m'},$

with the usual asymptotic expansions for product symbol.

We shall now prove the existence of two matrices $s(t,x,\xi)\in S_f^{-1}(N\times N)$ and $v(t,\ x,\ \xi)\in S_f^0(N\times N)$ (in a block diagonal form) such that, for some cut-off χ, we have:

(3.51) $\mathscr{P}s(t,\ x,\ D_x)-s(t,\ x,\ D_x)\ (\mathscr{P}^\#-v(t,\ x,\ D_x))$
$+q(t,\ x,\ D_x)v(t,\ x,\ D_x)+(1-\chi(x,\ t|D_x|))r(t,\ x,\ D_x)$
$\in OPS_f^{-\infty}(N\times N).$

Then, defining:

(3.52) $\begin{cases}Q(t,\ x,\ D_x)=q(t,\ x,\ D_x)+(1-\chi(x,\ t|D_x|))s(t,\ x,\ D_x)\\ \tilde{B}(t,\ x,\ D_x)=\tilde{b}(t,\ x,\ D_x)+(1-\chi(x,\ t|D_x|))v(t,\ x,\ D_x)\\ \tilde{\mathscr{P}}=I_Nt\partial_t-tA(t,\ x,\ D_x)-\tilde{B}(t,\ x,\ D_x),\end{cases}$

it can be verified that both Q and $\tilde{B}$ have all properties listed in Theorem 3.1 and that the conclusion (3.7) holds.

To construct s and v we observe that as a consequence of the structure of q_0, there exists a cut-off function $\chi(x,z)$ such that $(1-\chi(x,t|\xi|))\ q_0^{(hh)}$

(t, x, ξ) is a non singular matrix on the support of $1-\chi$, for every $h=1, \ldots, \nu$.

Let us define:

$$(3.53)\quad v_0(t, x, \xi)=(\chi(x, t|\xi|)-1)\begin{pmatrix} q_0^{(11)}(t, x, \xi)^{-1}r^{(11)}(t, x, \xi) & & \square \\ & q_0^{(22)}(t, x, \xi)^{-1}r^{(22)}(t, x, \xi) & \\ & \ddots & \\ \square & & q_0^{(\nu\nu)}(t, x, \xi)^{-1}r^{(\nu\nu)}(t, x, \xi) \end{pmatrix}$$
$$\stackrel{\text{def.}}{=}(\chi(x, t|\xi|)-1)\ \Delta_0(t, x, \xi).$$

We obviously have $v_0 \in S_f^0(N\times N)$, in a block diagonal form.

Moreover, $qv_0+(1-\chi(x, t|\xi|))r=\theta_0+q'v_0$, where q' is given by (3.45), so that $q'v_0 \in S_f^{-1}(N\times N)$, and $\theta_0 \in S_f^0(N\times N)$ is a block extra-diagonal matrix. It is easy to see that there exists a unique matrix $s_{-1} \in S_f^{-1}(N\times N)$ such that $t[A(t, x, \xi), s_{-1}(t, x, \xi)]=\theta_0(t, x, \xi)$. With the above definition of v_0 and s_{-1} we have:

$$(3.54)\quad \begin{aligned} &\mathscr{P}s_{-1}(t, x, D_x)-s_{-1}(t, x, D_x)[\mathscr{P}^{\#}-v_0(t, x, D_x)] \\ &+q(t, x, D_x)v_0(t, x, D_x)+(1-\chi(x, t|D_x|))r(t, x, D_x) \\ &\in OPS_f^{-1}(N\times N). \end{aligned}$$

Going on in this way we can construct two sequences of matrices $s_{-1-j} \in S_f^{-1-j}(N\times N)$ and $v_{-j} \in S_f^{-j}(N\times N)$ (in a block diagonal form), $j\geq 0$, such that for every $M\geq 1$ we have:

$$(3.55)\quad \begin{aligned} &\mathscr{P}\left(\sum_0^M{}_j s_{-1-j}(t, x, D_x)\right)-\left(\sum_0^M{}_j s_{-1-j}(t, x, D_x)\right)\left[\mathscr{P}^{\#}-\sum_0^M{}_j v_{-j}(t, x, D_x)\right] \\ &+q(t, x, D_x)\left(\sum_0^M{}_j v_{-j}(t, x, D_x\right) \\ &+(1-\chi(x, t|D_x|))r(t, x, D_x)\in OPS_f^{-M-1}(N\times N). \end{aligned}$$

Choosing $s \in S_f^{-1}(N\times N)$, $s\sim\sum_{j\geq 0} s_{-1-j}$, and $v \in S_f^0(N\times N)$ (in a block diagonal form), $v\sim\sum_{j\geq 0} v_{-j}$, we obtain (3.51).

This ends the proof of the Theorem.

REMARK 3.2. By (3.45), (3.52), using Lemma 2.2, 6. it follows that the operator $Q(t, x, D_x)$ has a two sided parametrix in $OP\hat{\Sigma}^{0,0}(N\times N)$, i.e. there exists $Q^{-1}(t, x, D_x) \in OP\hat{\Sigma}^{0,0}(N\times N)$ such that both $Q^{-1}Q-\mathrm{id}$ and $QQ^{-1}-\mathrm{id}$ are partially regularizing operators.

4. Parametrices.

In this Chapter we consider a singular hyperbolic system of the type $\tilde{\mathscr{P}}$ constructed in Theorem 3.1 and we will construct, under suitable conditions, a right and a left parametrix for $\tilde{\mathscr{P}}$.

To simplify the notation we drop the $\sim$. Let us fix precisely the hypotheses we shall assume in this Chapter. We consider a system:

$$\mathscr{P}=I_N t\partial_t - tA(t, x, D_x) - B(t, x, D_x), \tag{4.1}$$

where the matrix A is given by (3.2) and satisfies hypothesis h_1) of Chapter 3.

Moreover, we assume that the functions λ_j are independent of (t, x) for x outside of a compact subset of $\boldsymbol{R}^n$.

We suppose that the matrix $B\in OP\hat{\Sigma}^{0,0}(N\times N)$ and that for every $\omega\subset\subset \boldsymbol{R}^n_x$ there exists $\delta'>0$ (δ' depending on ω) such that if $b(t, x, \xi)\sim \sum_{j\geq 0} b_j(t, x, \xi)$, $b_j\in\Sigma^{0,j}(N\times N)$, is a symbol for B, we have:

(4.2) $\begin{cases} b(t, x, \xi) \text{ and } b_j(t, x, \xi), \text{ for all } j\geq 0, \text{ are in block diagonal form} \\ \text{for every } (x, \xi/|\xi|)\in\omega\times S^{n-1} \text{ and for } |t||\xi|\geq\delta'. \\ \operatorname{Re} b_0(t, x, \xi)\leq -I_N, \ \forall t, x, \xi\neq 0. \end{cases}$

We now define the ≪phases≫ involved in the parametrices we shall construct.

Due to the hypotheses on the λ_j's, we know (cfr. e.g. [10]) that there exists a $T>0$ such that for every $j=1,\dots,\nu$ there is a unique real function $\varphi_j(t, s, x, \xi)\in C^\infty([-T, T]\times[-T, T]\times\boldsymbol{R}^n\times\boldsymbol{R}^n\setminus 0)$, positively homogeneous of degree one in ξ, which solves the Cauchy problem:

$$\begin{cases} \partial_t\varphi_j(t, s, x, \xi)=\lambda_j(t, x, d_x\varphi_j(t, s, x, \xi)) \\ \varphi_j|_{t=s}=x\cdot\xi \end{cases}, \quad j=1,\dots,\nu. \tag{4.3}$$

For any $\rho\in[0, 1]$ we put:

$$\psi_j(\rho, t, x, \xi)=\varphi_j(t, \rho t, x, \xi), \qquad j=1,\dots,\nu. \tag{4.3$'$}$$

The following Lemmas will be of a crucial importance in the sequel.

LEMMA 4.1. *Let $\psi(\rho, t, x, \xi)$ denote any of the ψ_j's defined in* (4.3)$'$. *Then:*

1. $\psi\in H\hat{\Sigma}^{1,0}_T$, *with asymptotic expansion* $\sum_{k\geq 0}\psi^{(k)}(\rho, t, x, \xi)$, *where:*

$$\psi^{(0)}=x\cdot\xi,\ \psi^{(1)}=(1-\rho)\lambda(0, x, \xi/|\xi|)(t|\xi|),$$

$$\psi^{(k)}=\frac{1}{k!}(\partial_t^k\psi)(\rho, 0, x, \xi/|\xi|)|\xi|^{1-k}(t|\xi|)^k,\ k\geq 1.$$

2. *For any cut-off function* χ.

$$e^{-ix\cdot\xi}\chi(x,\ t|\xi|)e^{i\psi(\rho,t,x,\xi)}\in H\hat{\Sigma}_T^{0,0}.$$

PROOF. Point 1. is straightforward. To prove 2., let $\chi'(x, z)$ be a cut-off function with $\chi'\chi=\chi$ and write

$$e^{-ix\xi}\chi(x,\ t|\xi|)e^{i\psi(\rho,t,x,\xi)}=\chi(x,\ t|\xi|)e^{i\psi^{(1)}(\rho,t,x,\xi)}\cdot$$
$$\chi'(x,\ t|\xi|)e^{i[\psi(\rho,t,x,\xi)-x\cdot\xi-\psi^{(1)}(\rho,t,x,\xi)]}.$$

A simple computation shows that $\chi e^{i\psi^{(1)}}\in H\Sigma^{0,0}$. On the other hand, putting $\theta=\psi-x\cdot\xi-\psi^{(1)}$, we have $\frac{(i\theta)^k}{k!}\in H\hat{\Sigma}_T^{k,2k}$, $k\geq 0$, with an expansion $\frac{(i\theta)^k}{k!}\sim\sum_{j\geq 0}\theta_j^{(k)}$, $\theta_j^{(k)}\in H\Sigma_T^{k,2k+j}$. Since actually $\chi'\theta_j^{(k)}\in H\Sigma_T^{-k-j,0}$, we can construct an $f(\rho,\ t,\ x,\ \xi)\in H\hat{\Sigma}_T^{0,0}$ such that

$$f\sim\sum_{r\geq 0}\Big(\sum_{k+j=r}\chi'(x,\ t|\xi|)\theta_j^{(k)}(\rho,\ t,\ x,\ \xi)\Big). \tag{4.4}$$

Since $\chi'e^{i\theta}-f\in HS_T^{-\infty,0}$, the conclusion follows and we have the asymptotic expansion:

$$\begin{aligned}&e^{-ix\cdot\xi}\chi(x,\ t|\xi|)e^{i\psi(\rho,t,x,\xi)}\\&\quad\sim\sum_{r\geq 0}\Big(\sum_{k+j=r}\chi(x,\ t|\xi|)e^{i\psi^{(1)}(\rho,t,x,\xi)}\ \theta_j^{(k)}(\rho,\ t,\ x,\ \xi)\Big).\end{aligned}\tag{4.5}$$

LEMMA 4.2. *Let $\psi(\rho, t, x, \xi)$ denote any of the ψ_j's defined in (4.3)'. Let $h(\rho,\ t,\ x,\ \xi)\in HS_T^{m,k}$ and let $q(t,\ x,\ D_x)\in OPS_T^{m',k'}$ be a proper operator with symbol $q(t,\ x,\ \xi)$. Put*:

$$c(\rho,\ t,\ x,\ \xi)=e^{-i\psi(\rho,t,x,\xi)}\ q(t,\ x,\ D_x)[e^{i\psi(\rho,\ t,\cdot,\ \xi)}h(\rho,\ t,\ \cdot,\ \xi)]. \tag{4.6}$$

Then we have:

1. $c\in HS_T^{m+m',k+k'}$ *with the following asymptotic expansion*:

$$c\sim\sum_\alpha\frac{1}{\alpha!}(\partial_\xi^\alpha q)(t,x,d_x\psi(\rho,t,x,\xi))D_y^\alpha[h(\rho,t,y,\xi)e^{i\Psi(\rho,t,x,y,\xi)}]_{x=y}, \tag{4.7}$$

where:

$$\begin{aligned}\Psi(\rho,\ t,\ x,\ y,\ \xi)&=\psi(\rho,\ t,\ y,\ \xi)-\psi(\rho,\ t,\ x,\ \xi)\\&\quad-\langle y-x,\ d_x\psi(\rho,\ t,\ x,\ \xi)\rangle.\end{aligned}$$

The expansion (4.7) *means that there is a sequence $\sigma_M\uparrow+\infty$ as $M\to+\infty$ such that $c-\sum_{|\alpha|<M}\frac{1}{\alpha!}(\partial_\xi^\alpha q)(t,\ x,\ d_x\psi)D_y^\alpha[h\ e^{i\Psi}]_{y=x}\in HS_T^{m+m'-\sigma_M,k+k'}$, for every $M\geq 1$.*

2. If $h\in H\hat{\Sigma}_T^{m,k}$, *$h\sim\sum_{j\geq 0}h_j$, and $q\in\hat{\Sigma}_T^{m',k'}$, $q\sim\sum_{j\geq 0}q_j$, then the symbol* c

defined in (4.6) *belongs to* $H\hat{\Sigma}_T^{m+m', k+k'}$, *with an asymptotic expansion* $c \sim \sum_{\mu\geq 0} c_\mu$, $c_\mu \in H\Sigma_T^{m+m', k+k'+\mu}$, *where*

$$(4.8) \qquad c_\mu = \sum_{\substack{|\alpha|+v=\mu \\ v\geq 0}} \sum_{\substack{j+j'+|\beta|+l+s=v \\ l,s\geq 0}} (\partial_\xi^{\alpha+\beta} q_j)(t, x, \xi) \cdot \mathcal{M}_{\beta,l,s}^{(\psi),\alpha}(\rho, t, x, \xi; D_x) h_{j'}(\rho, t, x, \xi),$$

for some differential operators $\mathcal{M}_{\beta,l,s}^{(\psi),\alpha}$ *depending only on* ψ *which have the form :*

$$(4.9) \begin{cases} \mathcal{M}_{\beta,l,s}^{(\psi),\alpha}(\rho, t, x, \xi; D_x) = \sum_{\substack{\beta'\leq\alpha \\ |\beta'|\geq 2}} d_{\beta,l,s,\beta'}^{(\psi),\alpha}(\rho, t, x, \xi)\, D_x^{\alpha-\beta'}, \\ \text{with} \\ d_{\beta,l,s,\beta'}^{(\psi),\alpha} \in H\Sigma_T^{[|\alpha|/2]+|\beta|,\ [|\alpha|/2]+|\beta|+l+s}, \ ([|\alpha|/2] = \text{integral} \\ \text{part of } |\alpha|/2). \end{cases}$$

3. *Under the same conditions of* 2., *the symbol* c *can be rewritten as :*

$$(4.10) \qquad c(\rho, t, x, \xi) = q(t, x, d_x\psi(\rho, t, x, \xi))\, h(\rho, t, x, \xi) + c'(\rho, t, x, \xi),$$

with

i) $q(t, x, d_x\psi) h(\rho, t, x, \xi) \in H\hat{\Sigma}_T^{m+m',k+k'}$,

ii) $c' \in H\hat{\Sigma}_T^{m+m'-1,k+k'}$ *with an expansion* $\sum_{\mu\geq 1} c'_\mu$ *where the* c'_μ *are given by* (4.8) ($|\alpha|\geq 1$).

The proof of Lemma 4.2 is quite technical and will be omitted here. To give some comments concerning the proof we note that the proof of point 1. can be performed as in the classical case with only minor modifications. As far as points 2. and 3. are concerned one has to insert in each term $(\partial_\xi^\alpha q)(t, x, d_x\psi) D_y^\alpha [h(\rho, t, y, \xi) e^{i\psi(\rho,t,x,y,\xi)}]_{y=x}$ the asymptotic expansion of $d_x\psi$ as given by Lemma 4.1.

Remark 4.1. With the notation of Lemma 4.2, 2. suppose that for every $\omega \subset\subset \boldsymbol{R}^n$ there exists a $\delta' > 0$ such that $q(t, x, \xi) = q_j(t, x, \xi) = 0$, for every $x \in \omega$ and $|t||\xi| \geq \delta'$ (for every $j \geq 0$).

As a consequence, the symbols c_μ given by (4.8) have the same support property so that they actually belong to $H\Sigma_T^{m+m'-\mu,k+k'}$. We can thus construct a symbol $\tilde{c} \in H\hat{\Sigma}_T^{m+m',k+k'}$ with the same support property of the c_μ's such that $c - \tilde{c} \in HS_T^{-\infty,k+k'}$.

We are now in a position to state the first result concerning the existence of a right parametrix for the system $\mathscr{P}$.

THEOREM 4.1. *Let $\mathscr{P}$ be the system* (4.1) (*satisfying condition* (4.2)). *Put* :

$$
(4.11)\qquad e^{i\psi(\rho,t,x,\xi)} = \begin{pmatrix} e^{i\varphi_1} I_{N_1} & & & \square \\ & e^{i\varphi_2} I_{N_2} & & \\ & & \ddots & \\ \square & & & e^{i\varphi_\nu} I_{N_\nu} \end{pmatrix}
$$

where the ψ_j's are given by (4.3)$'$.

Then there exists a matrix $h(\rho,\ t,\ x,\ \xi)\in H\hat{\Sigma}_T^{0,0}(N\times N)$ such that if we define the operator :

$$
(4.12)\qquad E(h\,;f)=\int_0^1\!\!\int e^{i\psi(\rho,t,x,\xi)}h(\rho,\ t,\ x,\ \xi)\hat{f}(\rho t,\ \xi)\ d\rho \bar{d}\xi,
$$
$$
f\in C^\infty(]-T,\ T[\,;C_0^\infty(\boldsymbol{R}^n))^N,
$$

then :

$$
(4.13)\qquad \mathscr{P}E(h,\cdot)-id : C^\infty(]-T,\ T[\,;\ \mathscr{E}'(\boldsymbol{R}^n))^N \to C^\infty(]-T,\ T[\times \boldsymbol{R}^n)^N.
$$

PROOF. Let h be any matrix in $H\hat{\Sigma}_T^{0,0}(N\times N)$ which satisfies the initial condition

$$
(4.14)\qquad h(1,\ t,\ x,\ \xi)=I_N\,.
$$

Letting $t\partial_t$ act on $E(h,\cdot)$ we obtain :

$$
(4.15)\qquad t\partial_t E(h\,;f)=f+E(\sqrt{-1}((t\partial_t-\rho\partial_\rho)\psi)h\,;f)
$$
$$
+E((t\partial_t-\rho\partial_\rho-1)h\,;f),
$$

where we used the notation :

$$
(4.16)\qquad (t\partial_t-\rho\partial_\rho)\psi=\begin{pmatrix} (t\partial_t-\rho\partial_\rho)\psi_1 I_{N_1} & & \square \\ & (t\partial_t-\rho\partial_\rho)\psi_2 I_{N_2} & \\ & & \ddots \\ \square & & (t\partial_t-\rho\partial_\rho)\psi_\nu I_{N_\nu} \end{pmatrix}
$$

Lemma 4.2 yields $tA(t,\ x,\ D_x)E(h\,;f)=E(q\,;f)$ for a matrix $q\in H\hat{\Sigma}_T^{1,1}(N\times N)$. Writing h and q in a block form, we have :

$$
(4.17)\qquad q^{(\sigma,\sigma')}(\rho,\ t,\ x,\ \xi)=\sqrt{-1}\ t\lambda_\sigma(t,\ x,\ d_x\psi_\sigma(\rho,\ t,\ x,\ \xi))h^{(\sigma,\sigma')}(\rho,\ t,\ x,\ \xi)
$$
$$
+q'^{(\sigma,\sigma')}(\rho,\ t,\ x,\ \xi),\ 1\leq\sigma,\ \sigma'\leq\nu,
$$

where $q'^{(\sigma,\sigma')}\in H\hat{\Sigma}_T^{0,1}(N_\sigma\times N_{\sigma'})$.

Form (4.3), (4.3)′ we get

$$t\partial_t\psi_\sigma-\rho\partial_\rho\psi_\sigma=t\lambda_\sigma(t,\ x,\ d_x\psi_\sigma),\ \sigma=1,\dots,\nu. \tag{4.18}$$

As a partial conclusion, from (4.15) and (4.17), taking into account (4.18), we obtain:

$$(t\partial_t-tA(t,\ x,\ D_x))E(h;f)=f+E((t\partial_t-\rho\partial_\rho-1)h;f)-E(q';f). \tag{4.19}$$

Using Lemma 4.2 once more, we have $B(t,\ x,\ D_X)E(h;f)=E(p;f)$ for a matrix $p\in H\hat{\Sigma}_T^{0,0}(N\times N)$ whose blocks are given by:

$$p^{(\sigma,\sigma')}=e^{-i\varphi_\sigma}B^{(\sigma,\sigma)}(t,x,D_x)[e^{i\varphi_\sigma}h^{(\sigma,\sigma')}] + \sum_{\substack{\sigma''=1\\ \sigma''\neq\sigma}}^{\nu} e^{i(\varphi_{\sigma''}-\varphi_\sigma)}\{e^{-i\varphi_{\sigma''}}B^{(\sigma,\sigma'')}(t,x,D_x)[e^{i\varphi_{\sigma''}}h^{(\sigma'',\sigma')}]\}, \quad 1\leqq\sigma,\ \sigma'\leqq\nu \tag{4.20}$$

We now use an argument which exploits in an essential way the first hypothesis in (4.2). Let $\chi(x,\ z)$ be a cut-off function such that the matrices $(1-\chi(x,\ t|\xi|))b(t,\ x,\ \xi)$ and $(1-\chi(x,\ t|\xi|))b_j(t,\ x,\ \xi)$, $j\geq 0$, are in block diagonal form for every $t,\ x,\ \xi$. Writing B_χ(resp. $B_{1-\chi}$) as the operator with symbol χb (resp. $(1-\chi)b$), we obtain that $B_{1-\chi}\in OP\hat{\Sigma}_T^{0,0}(N\times N)$ is block diagonal. From (4.20) it follows that

$$p^{(\sigma,\sigma')}=e^{-i\varphi_\sigma}B^{(\sigma,\sigma)}(t,x,D_x)[e^{i\varphi_\sigma}h^{(\sigma,\sigma')}] + \sum_{\substack{\sigma''=1\\ \sigma''\neq\sigma}}^{\nu\ \nu} e^{i(\varphi_{\sigma''}-\varphi_\sigma)}\{e^{-i\varphi_{\sigma''}}B_\chi^{(\sigma,\sigma'')}(t,x,D_x)[e^{i\varphi_{\sigma''}}h^{(\sigma'',\sigma')}]\}. \tag{4.21}$$

From the Remark 4.1. it follows that, putting:

$$g_{\sigma,\sigma'',\sigma'}=e^{-i\varphi_{\sigma''}}B_\chi^{(\sigma,\sigma'')}(t,x,D_x)(e^{i\varphi_{\sigma''}}h^{(\sigma'',\sigma')}), \tag{4.22}$$

we have $g_{\sigma,\sigma'',\sigma'}\in H\hat{\Sigma}_T^{0,0}(N_\sigma\times N_{\sigma'})$ (modulo a symbol in $HS_T^{-\infty,0}(N_\sigma\times N_{\sigma'})$) and moreover,

for every $\omega\subset\subset\boldsymbol{R}^n$ we have $g_{\sigma,\sigma'',\sigma'}(\rho,\ t,\ x,\ \xi)=\square$ for $x\in\omega$, $|t||\xi|\geq\delta'$, $\rho\in]0,\ 1]$.

Using Lemma 4.1 we can conclude that $e^{i(\varphi_{\sigma''}-\varphi_{\sigma'})}g_{\sigma,\sigma'',\sigma'}\in H\hat{\Sigma}_T^{0,0}(N_\sigma\times N_{\sigma'})$, with asymptotic expansion:

$$\sum_{\mu\geq 0}\left[\sum_{\mu'+\mu''=\mu}\left(\sum_{k+j=\mu'}\chi'(x,\ t|\xi|)\,e^{it|\xi|(1-\rho)(\lambda_{\sigma''}(0,x,\xi/|\xi|)-\lambda_\sigma(0,x,\xi/|\xi|))}\cdot\theta_{j,\sigma'',\sigma}^{(k)}(\rho,\ t,\ x,\ \xi)g_{\sigma,\sigma'',\sigma',\mu''}(\rho,\ t,\ x,\ \xi)\right)\right], \tag{4.23}$$

where: $g_{\sigma,\sigma'',\sigma'}\sim\sum_{\mu''\geq 0}g_{\sigma,\sigma'',\sigma',\mu''}$, $\chi'\chi=\chi$ and $\sum_{j\geq 0}\theta_{j,\sigma'',\sigma}^{(k)}\sim\frac{1}{k!}(i(\psi_{\sigma''}-\psi_\sigma-\psi_{\sigma''}^{(1)}+$

$\psi_\sigma^{(1)}))^k$.

To work with more compact notation, we put:

$$(4.24)\quad \begin{cases} k^{(\sigma,\sigma')}=e^{-i\psi_\sigma}B^{(\sigma,\sigma)}[e^{i\psi_\sigma}h^{(\sigma,\sigma')}],\quad k=(k^{(\sigma,\sigma')})_{1\leq\sigma,\sigma'\leq\nu} \\ g_{\sigma''}^{(\sigma,\sigma')}=e^{i(\psi_{\sigma''}-\psi_\sigma)}g_{\sigma,\sigma'',\sigma'},\quad 1\leqq\sigma,\sigma'',\sigma'\leqq\nu \end{cases}$$

Therefore, (4.21) can be rewritten as:

$$(4.24)\qquad p^{(\sigma,\sigma')}=k^{(\sigma,\sigma')}+\sum_{\substack{\sigma''=1\\ \sigma''\neq\sigma}}^{\nu}g_{\sigma''}^{(\sigma,\sigma')},\quad p=(p^{(\sigma,\sigma')})_{1\leq\sigma,\sigma'\leq\nu}.$$

In conclusion, for any $h\in H\hat{\Sigma}_T^{0,0}(N\times N)$ which satisfies (4.14), we obtain:

$$(4.25)\qquad \mathscr{P}E(h;f)-f=E((t\partial_t-\rho\partial_\rho-1)h-q'-p;f)+Rf,$$

where $q'=(q'^{(\sigma,\sigma')})_{1\leq\sigma,\sigma'\leq\nu}$ is defined in (4.17), p is given by (4.24) and R is a partially regularizing operator of Hardy type.

From the preceding remarks and from Lemma 4.2 we have:

$$(4.26)\qquad v(\rho,t,x,\xi)\overset{\text{def}}{=}(t\partial_t-\rho\partial_\rho-1)h-q'-p\in H\hat{\Sigma}_T^{0,0}(N\times N),$$

with asymptotic expansion $\sum_{j\geq 0}v_j(\rho,t,x,\xi)$, where:

$$(4.27)\quad \begin{cases} v_0(\rho,t,x,\xi)=\hat{v}_0\left(\rho,x,\frac{\xi}{|\xi|},t|\xi|\right),\quad \hat{v}_0\in HS^0(N\times N), \\ \hat{v}_0(\rho,x,\xi',z)=(z\partial_z-\rho\partial_\rho-1)\hat{h}_0(\rho,x,\xi',z) \\ \qquad -\hat{b}_0'(x,\xi',z)\hat{h}_0(\rho,x,\xi',z) \\ \qquad -\Lambda^-(\rho,x,\xi',z)\chi(x,z)\hat{b}_0''(x,\xi',z)\Lambda^+(\rho,x,\xi',z)\hat{h}_0(\rho,x,\xi',z), \end{cases}$$

where we used the notation:

$$(4.28)\quad \begin{cases} \hat{b}_0=\hat{b}_0'+\hat{b}_0'' \\ \hat{b}_0'^{(hk)}=\begin{cases}\hat{b}_0^{(hh)}, & \text{if } h=k,\\ \square, & \text{if } h\neq k,\end{cases}\quad \hat{b}_0''^{(hk)}=\begin{cases}\square & \text{if } h=k\\ \hat{b}_0^{(hk)}, & \text{if } h\neq k\end{cases},\quad h,\ k=1,\cdots,\nu \\ \Lambda^{\pm}(\rho,x,\xi',z)=\begin{pmatrix} e^{\pm iz(1-\rho)\lambda_1(0,x,\xi')}I_{N_1} & & \square \\ & e^{\pm iz(1-\rho)\lambda_2(0,x,\xi')}I_{N_2} & \\ \square & & \ddots\ e^{\pm iz(1-\rho)\lambda_\nu(0,x,\xi)}I_{N_\nu} \end{pmatrix} \end{cases}$$

A natural way to obtain (4.13) is to impose the conditions:

$$v_0=\square,\ \hat{h}_0|_{\rho=1}=I_N \text{ and, for } j\geq 1,\ v_j=\square,\ \hat{h}_j|_{\rho=1}=\square.$$

These conditions lead to the following transport equations:

$$(4.29)_j \begin{cases} (z\partial_z-\rho\partial_\rho-1)\hat{h}_j(\rho, x, \xi', z)-\hat{b}'_0(x, \xi', z)\hat{h}_j(\rho, x, \xi', z) \\ -\Lambda^-(\rho, x, \xi', z)\chi(x, z)\hat{b}''_0(x, \xi', z)\Lambda^+(\rho, x, \xi', z)\hat{h}_j(\rho, x, \xi', z) \\ =f_j(\rho, x, \xi', z),\ \hat{h}_j(\rho, x, \xi', z)|_{\rho=1}=\begin{cases} I_N, & \text{if } j=0 \\ \square, & \text{if } j\geq 1, \end{cases} \end{cases}$$

where f_j is a suitable symbol in $HS^j(N\times N)$ depending on $\hat{h}_0, \dots, \hat{h}_{j-1}$, $\hat{b}_0, \dots, \hat{b}_{j-1}$ and $f_0\equiv\square$.

To solve transport equations $(4.29)_j$ we use the following Lemma.

LEMMA 4.3. *Suppose we are given two smooth $N\times N$ matrices (written in block form) $C(x, \xi', z)=(C^{(hk)}(x, \xi', z))_{h,k=1,\dots,\nu}$, $C'(\rho, x, \xi', z)=(C'^{(hk)}(\rho, x, \xi', z))_{h,k=1,\dots,\nu}$, $x\in\boldsymbol{R}^n$, $\xi'\in S^{n-1}$, $z\in\boldsymbol{R}$, $\rho\in[0, 1]$, such that*: 1. *$C^{(hk)}=\square$ for $h\neq k$, and $C^{(hh)}(x, \xi', z)\in\boldsymbol{S}^0(N_h\times N_h)$, $h=1,\dots,\nu$.*
2. *For every $\omega\subset\subset\boldsymbol{R}^n$ there exists a $\delta'>0$ such that $C'(\rho, x, \xi', z)=\square$ for every $(\rho, x, \xi')\in[0, 1]\times\omega\times S^{n-1}$* and $|z|\geq\delta'$.
3. *There exists $\gamma\geq 0$ such that*

$$(4.30) \qquad \mathrm{Re}\,[C(x, \xi', z)+C'(\rho, x, \xi', z)]\leq-\gamma I_N,$$

for every ρ, x, ξ', z.

Furthermore, let $\psi(x, \xi', z)\in\boldsymbol{S}^k(N\times N)$ and $g(\rho, x, \xi', z)\in HS^k(N\times N)$, $k\in\boldsymbol{R}$. Then there exists a unique matrix $f(\rho, x, \xi', z)\in HS^k(N\times N)$ such that

$$(4.31) \qquad \begin{cases} (z\partial_z-\rho\partial_\rho)f-[C+C']f=g \\ f|_{\rho=1}=\psi. \end{cases}$$

PROOF. Put $\rho=e^{-s}$, $z=z_0e^s$, $s\geq 0$, $z_0\in\boldsymbol{R}$, and let $F(s, z_0; x, \xi')$ be the unique smooth matrix solution of the Cauchy problem:

$$\begin{cases} \partial_sF-[C(x, \xi', z_0e^s)+C'(e^{-s}, x, \xi', z_0e^s)]F=g(e^{-s}, x, \xi', z_0e^s) \\ F|_{s=0}=\psi(x, \xi', z_0) \end{cases}$$

Defining $f(\rho, x, \xi', z)=F(-\ln\rho, z\rho; x, \xi')$ and using standard energy estimates we obtain that the following inequality holds for every $\varepsilon>0$:

$$(4.32) \qquad \|f(\rho, x, \xi', z\|^2\leq\|\psi(x, \xi', \rho z)\|^2 + \frac{1}{\varepsilon}\int_\rho^1\sigma^{2\gamma-\varepsilon}\left\|g\left(\frac{\rho}{\sigma}, x, \xi', \sigma z\right)\right\|^2\frac{d\sigma}{\sigma}.$$

As a consequence, for every $\omega\subset\subset\boldsymbol{R}^n$, M, $\varepsilon'>0$, we have

(4.33)
$$\sup_{\substack{(x,\xi')\in\omega\times S^{n-1}\\ \rho\in]0,1],|z|\le M}} \rho^{\varepsilon'}\|f(\rho, x, \xi', z)\|<+\infty.$$

By taking derivatives of eq. (4. 31) one can verify in the same way that on every compact interval $|z|\le M$ we have the estimates:

(4.34)
$$\sup_{\substack{(x,\xi')\in\omega\times S^{n-1}\\ \rho\in]0,1],|z|\le M}} \rho^{\varepsilon'}\|(\rho\partial_\rho)^j\partial_z^k\partial_x^\alpha\theta_1\cdots\theta_q f\|<+\infty,$$

for every $j,\ k\in\boldsymbol{Z}_+,\ \alpha\in\boldsymbol{Z}_+^n,\ \varepsilon'>0,\ \omega\subset\subset\boldsymbol{R}^n$ and for every family $\theta_1,\dots,\theta_q$ of smooth vector fields on S^{n-1}.

To estimate f for $z\to\infty$ we fix $\omega\subset\subset\boldsymbol{R}^n$ and take $|z|\ge\delta'$ so that $C'(\rho, x, \xi', z)\equiv\square$ for $x\in\omega$, $|\xi'|=1$, $|z|\ge\delta'$.

Let us consider $z\ge\delta'$ (the case $z\leqq-\delta'$ can be handled analogously).

Put $J_\pm=\{(\rho, x, \xi', z)|\ z\ge\delta',\ \pm(\rho z-\delta')\ge 0\}$, $J=J_+\cup J_-$. On J, f satisfies the boundary problem:

(4.35)
$$\begin{cases}(z\partial_z-\rho\partial_\rho)f-Cf=g, \text{ in } J\\ f|_{\rho=1}=\psi,\\ f|_{z=\delta'}=\varphi\end{cases}$$

for some smooth matrix $\varphi(\rho, x, \xi')$ satisfying estimates of the form (4. 34).

In the region J_+ we put $\rho=e^{-s}$, $z=z_0e^s$, $s\ge0$, $z_0\ge\delta'$, and arguing as above we obtain the estimates:

(4.36)
$$\sup_{J_+}(1+z)^{-k+l}\rho^{\varepsilon}\|(\rho\partial_\rho)^j\partial_z^l\partial_x^\alpha\theta_1\dots\theta_q f\|<+\infty.$$

In the region J_- we put $\rho=\rho_0e^{-s}$, $z=\delta'e^s$, $s\ge0$, $\rho_0\in]0, 1]$, and arguing as above we obtain, for every $\varepsilon>0$, the estimate:

(4.37)
$$\|f(\rho, x, \xi', z)\|^2\le\|\varphi(\rho z/\delta', x, \xi')\|^2 + \frac{1}{\varepsilon}\int_{\delta'}^{z}(z/z')^{-2\gamma+\varepsilon}\|g(\rho z/z', x, \xi', z')\|^2dz'/z'.$$

As a consequence, since $\rho z\le\delta'\le z'$, for every $\varepsilon'>0$ we obtain:

(4.38)
$$\sup_{J_-}(1+z)^{-k}\rho^{\varepsilon'}\|f(\rho, x, \xi', z)\|<+\infty.$$

Higher order estimates on J_-, analogous to (4. 36), follow by taking derivatives of (4. 35).

Lemma 4.3 allows to solve the transport equations $(4.29)_j$, $j\ge0$, by identifying $C(x, \xi', z)=I_N+\hat b_0'(x, \xi', z)$ and $C'(\rho, x, \xi', z)=\Lambda^-(\rho, x, \xi', z)\chi(x, z)\hat b_0''(x, \xi', z)\Lambda^+(\rho, x, \xi', z)$ and noting that hypothesis 3. in the Lemma is a consequence of the second basic assumption in (4. 2).

Once the symbols $\hat h_j$, $j\ge0$, are constructed we take a symbol $h\in$

$H\hat{\Sigma}_T^{0,0}(N\times N)$ such that:

$$(4.39)\qquad \begin{cases} h(\rho,\ t,\ x,\ \xi)\sim\sum_{j\geq 0}|\xi|^{-j}\hat{h}_j(\rho,\ x,\ \xi/|\xi|,\ t|\xi|) \\ h|_{\rho=1}\equiv I_N\ . \end{cases}$$

With the above definition of h we have, for every $f\in C^\infty(]-T,\ T[\,;\ C_0^\infty(\boldsymbol{R}^n))^N$:

$$(4.40)\qquad \mathscr{P}E(h\,;f)=f+Rf+E(q\,;f),$$

where R is H. p. r. (Cfr. Def. 2.7) and q is a suitable matrix belonging to $HS_T^{0,\infty}(N\times N)$.

To get rid of the term $E(q\,;.)$ we observe that for every cut-off function χ the operator $f\to E(\chi(x,\ t|\xi|)q\,;f)$ is H. p. r. and therefore we are left with the term $E(p\,;\bullet)$ with $p(\rho,\ t,\ x,\ \xi)=(1-\chi(x,\ t|\xi|))q(\rho,\ t,\ x,\ \xi)$. It is obvious that $p\in HS_{T,f}^0(N\times N)$ (the definition of HS_f^m and HS_p^m is analogous to the definitions (3.46) and (3.47) of S_f^m and S_p^m respectively, the only modification being the usual ρ-behaviour of the symbols).

We shall try to construct a matrix $r\in HS_{T,f}^0(N\times N)$ for which

$$(4.41)\qquad \mathscr{P}E(r\,;\bullet)=-E(p\,;\bullet),$$

modulo some H. p. r. operator. Since $HS_{T,f}^0\subset HS_T^{0,\infty}$, adding r to the matrix h we will obtain that $\mathscr{P}E(h+r\,;\bullet)-$ identity is a H. p. r. operator and the conclusion of the theorem follows.

To start with, we look for a matrix $r_0\in HS_{T,f}^0(N\times N)$ such that

$$(4.42)\qquad \mathscr{P}E(r_0\,;\bullet)=-E(p\,;\bullet)+E(p_{-1}\,;\bullet),$$

for some symbol $p_{-1}\in HS_{T,f}^{-1}(N\times N)$.

Supposing that $r_0|_{\rho=1}\equiv\square$, and arguing as in the first part of the proof one can show that:

$$(4.43)\qquad \mathscr{P}E(r_0\,;\bullet)=E(c_0\,;\bullet)+E(p_{-1}\,;\bullet).$$

for some symbols $p_{-1}\in HS_{T,f}^{-1}(N\times N)$, $c_0\in HS_{T,f}^0(N\times N)$ with

$$(4.44)\qquad \begin{cases} c_0=(c_0^{(\sigma,\sigma')})_{1\leq\sigma,\sigma'\leq\nu} \\ c_0^{(\sigma,\sigma')}(\rho,\ t,\ x,\ \xi)=(t\partial_t-\rho\partial_\rho)r_0^{(\sigma,\sigma')}(\rho,\ t,\ x,\ \xi) \\ -tL_\sigma(\rho,\ t,\ x,\ \xi\,;\partial_x)r_0^{(\sigma,\sigma')}(\rho,\ t,\ x,\ \xi) \\ \qquad -(I_{N_\sigma}+b^{(\sigma,\sigma)}(t,\ x,\ \xi))r_0^{(\sigma,\sigma')}(\rho,\ t,\ x,\ \xi), \end{cases}$$

where:

$$(4.45)\qquad L_\sigma(\rho,\ t,\ x,\ \xi\,;\partial_x)=\langle(d_\xi\lambda_\sigma)(t,\ x,\ d_x\psi_\sigma(\rho,\ t,\ x,\ \xi)),\ \partial_x\rangle$$

$$+\sum_{|\alpha|=2}\frac{1}{\alpha!}(\partial_\xi^\alpha\lambda_\sigma)(t,\ x,\ d_x\psi_\sigma(\rho,\ t,\ x,\ \xi))\partial_x^\alpha\psi_\sigma(\rho,\ t,\ x,\ \xi).$$

We now recall that if $(x^{(\sigma)}(t\,;s,\ y,\ \eta),\ \xi^{(\sigma)}(t\,;s,\ y,\ \eta))$ is the integral curve of the Hamiiltonian vector field $H_{-\lambda_\sigma}$ satisfying $x^{(\sigma)}|_{t=s}=y,\ \xi^{(\sigma)}|_{t=s}=\eta$, then for all $\sigma=1,\dots,\nu$ the map $\boldsymbol{R}^n\ni y\to x^{(\sigma)}(t\,;s,\ y,\ \eta)$ is smoothly invertible for every $(t,\ s)\in[-T,\ T]\times[-T,\ T]$ and for every $\eta\neq 0$ (the existence of a $T>0$ for which the above property holds is guaranteed by the hypothesis that the $\lambda_\sigma(t,\ x,\ \xi)$ are independent of t and x for large x).

Putting

$$\tilde{r}_0^{(\sigma,\sigma')}(\rho,\ t,\ y,\ \eta)=r_0^{(\sigma,\sigma')}(\rho,\ t,\ x^{(\sigma)}(t\,;\rho t,\ y,\ \eta),\ \eta),\quad 1\leq\sigma,\sigma'\leq\nu, \tag{4.46}$$

and imposing that c_0 in (4.44) is equal to $-p$, yields the equations:

$$\begin{cases}(t\partial_t-\rho\partial_\rho)\tilde{r}_0^{(\sigma,\sigma')}(\rho,t,\ y,\ \eta)-[(d_\sigma(\rho,\ t,\ y,\ \eta)+1)I\\ +b^{(\sigma,\sigma)}(t,\ x^{(\sigma)}(t\,;\rho t,\ y,\ \eta),\ \eta)]\tilde{r}_0^{(\sigma,\sigma')}(\rho,\ t,\ y,\ \eta)\\ =-p^{(\sigma,\sigma')}(\rho,\ t,\ x^{(\sigma)}(t\,;\rho t,\ y,\ \eta),\ \eta)\\ \tilde{r}_0^{(\sigma,\sigma')}|_{\rho=1}=\square\qquad\qquad ,\ 1\leq\sigma,\ \sigma'\leq\nu,\end{cases} \tag{4.47}$$

where:

$$\begin{aligned}&d_\sigma(\rho,\ t,\ y,\ \eta)\\ &=t\sum_{|\alpha|=2}\frac{1}{\alpha\,!}(\partial_\xi^\alpha\lambda_\sigma)(t,\ x^{(\sigma)}(t\,;\rho t,\ y,\ \eta),\ \xi^{(\sigma)}(t\,;\rho t,\ y,\ \eta))\\ &\cdot(\partial_x^\alpha\psi_\sigma)(\rho,\ t,\ x^{(\sigma)}(t\,;\rho t,\ y,\ \eta),\ \eta).\end{aligned} \tag{4.48}$$

To solve (4.47) we apply the following lemma.

LEMMA 4.4. *Let* $\Phi(\rho,\ t,\ y,\ \eta)\in HS_T^{0,0}(N\times N)$. *For every* $g\in HS_{T,f}^{-k}(N\times N)$, $k\geq 0$, *there exists a unique* $\varphi\in HS_{T,f}^{-k}(N\times N)$ *such that*:

$$\begin{cases}(t\partial_t-\rho\partial_\rho)\varphi-\Phi\varphi=g\\ \varphi|_{\rho=1}=\square.\end{cases} \tag{4.49}$$

PROOF. Putting $\rho=e^{-s},\ t=t_0e^s,\ s\geq 0,\ |t_0|<T$, and using standard energy estimates we obtain that the following inequality holds for every $\Omega\subset\subset]-T,\ T[\times\ \boldsymbol{R}^n$ and $\varepsilon>0$:

$$\|\varphi(\rho,\ t,\ y,\ \eta)\|^2\leq\frac{1}{\varepsilon}\int_\rho^1(\rho/\mu)^{-C-\varepsilon}\|g(\mu,\ \rho t/\mu,\ y,\ \eta)\|^2d\mu/\mu, \tag{4.50}$$

for all $(\rho,\ t,\ y,\ \eta)\in]0,1]\times\Omega\times\boldsymbol{R}^n$, with a suitable positive constant C. As a consequence of (4.50), if $M>0$ is large enough we get:

$$\sup_{\substack{\rho\in]0,1],(t,x)\in\Omega\\ \eta\in\boldsymbol{R}^n}}|t|^{-M}(1+|\eta|)^k\rho^{\varepsilon'}\|\varphi(\rho,\ t,\ y,\ \eta)\|<+\infty, \tag{4.51}$$

for every $\varepsilon'>0$. Estimates of the above form for higher derivative of φ can be obtained by taking derivatives of eq. (4.49). This proves the Lemma.

Iterating the procedure, we construct a matrix $r_{-1}\in HS_{T,f}^{-1}(N\times N)$, with $r_{-1}|_{\rho=1}=\square$, such that $\mathscr{P}E(r_{-1};\cdot)=-E(p_{-1};\cdot)+E(p_{-2};\cdot)$ for a suitable symbol $p_{-2}\in HS_{T,f}^{-2}(N\times N)$. Going on in this way, claim (4.41) is proved. This completes the proof of the Theorem.

In Theorem 4.1 we have constructed a right parametrix for the system $\mathscr{P}$. The existence of a left parametrix for $\mathscr{P}$ will be proved following more or less the same procedure as in Theorem 4.1. There are, however, some important differences which we think it is convenient to put in evidence.

First of all we define our phases. In the rest of this Chapter the function $-\varphi_j(t, 0, y, \eta)$, where φ_j is defined in (4.3), will be denoted by $\varphi_j(t, y, \eta)$, $j=1, \cdots \nu$, and we will put:

$$(4.52)\qquad \psi_j(\rho, t, y, \eta)=\varphi_j(\rho t, y, \eta),\ j=1, \dots, \nu.$$

As in Lemma 4.1. one can prove that $\psi_j\in H\hat{\Sigma}_T^{1,0}$ and that for every cut-off function χ, $e^{iy\cdot\eta}\chi(y, \rho t|\eta|)e^{i\psi_j}\in H\hat{\Sigma}_T^{0,0}$, $j=1, \dots, \nu$.

We have a modified version of Lemma 4.2 if in (4.6) we consider $q(\rho t, x, D_x)$ in place of $q(t, x, D_x)$ and ψ is now one of the phases defined in (4.52). As a consequence, in formulas (4.8) the coefficients $(\partial_\xi^{\alpha+\beta}q_j)(t, x, \xi)$ must be replaced by $(\partial_\xi^{\alpha+\beta}q_j)(\rho t, x, \xi)$; moreover, the operators $\mathscr{M}_{\beta,l,s}^{(\psi),\alpha}$ defined in (4.9) depend only on the argument ρt.

We now define the operators involved in the construction of the left parametrinx. Put

$$(4.53)\qquad e^{i[\psi(\rho,t,y,\eta)+x\cdot\eta]}=\begin{pmatrix} e^{i(\psi_1(\rho,t,y,\eta)+x\cdot\eta)}I_{N_1} & & \square \\ & \ddots & \\ \square & & e^{i(\psi_\nu(\rho,t,y,\eta)+x\cdot\eta)}I_{N_\nu}\end{pmatrix}$$

For a given matrix $h(\rho, t, y, \eta)\in H\hat{\Sigma}_T^{0,0}(N\times N)$, we define the operator:

$$(4.54)\qquad F(h;f)=\int_0^1\iint h(\rho, t, y, \eta)e^{i[\psi(\rho,t,y,\eta)+x\cdot\eta]}f(\rho t, y)d\rho\ dy\ đ\eta,$$
$$f\in C^\infty(]-T, T[\,; C_0^\infty(\boldsymbol{R}^n))^N.$$

The integral in (4.54) should be interpreted as an oscillatory integral which makes sense integrating by parts with respet to y since for some positive constant C we have:

$$(4.55)\qquad |d_y\psi_j(\rho, t, y, \eta)|\geq C|\eta|,$$

for all $(\rho, t, y, \eta)\in[0, 1]\times[-T, T]\times\boldsymbol{R}^n\times(\boldsymbol{R}^n\backslash 0)$, $j=1, \dots, \nu$.

We can now state the second main result of this Chapter.

THEOREM 4.2. *Let $\mathscr{P}$ be the system* (4.1) (*satisfying condition* (4.2)). *Then there exists a matrix* $h(\rho, t, y, \eta)\in H\hat{\Sigma}_T^{0,0}(N\times N)$ *such that for every* $f\in C^\infty(]-T, T[\,; C_0^\infty(\boldsymbol{R}^n))^N$ *we have*:

$$F(h\,;\mathscr{P}f)=I(f)+Rf, \tag{4.56}$$

where R is a H. p. r. operator (*cfr. Def.* 2.7) *and* $I(\cdot)$ *is the Fourier integral operator*:

$$I(f)(t, x)=\iint e^{i[\psi(1,t,y,\eta)+x\cdot\eta]}f(t, y)\ dy\ \bar{d}\eta. \tag{4.57}$$

PROOF. Let $h\in H\hat{\Sigma}_T^{0,0}(N\times N)$ be a matrix satisfying the initial condition (4.14). We have:

$$\begin{aligned}F(h\,; t\partial_t f)=I(f)+F(-\sqrt{-1}\ h(\rho\partial_\rho\psi)\,;f)+\\ F((-\rho\partial_\rho-1)h\,;f),\end{aligned} \tag{4.58}$$

where we used the notation:

$$\rho\partial_\rho\psi=\begin{pmatrix}\rho\partial_\rho\psi_1 I_{N_1} & & \square\\ & \ddots & \\ \square & & \rho\partial_\rho\psi_\nu I_{N_\nu}\end{pmatrix}. \tag{4.59}$$

Using Lemma 4.2 (or rather its modified version) we obtain:

$$F(h\,; tA(t, y, D_y)f)=F(q\,;f), \tag{4.60}$$

for some matrix $q\in H\hat{\Sigma}_T^{1,1}(N\times N)$, $q=(q^{(\sigma,\sigma')})_{1\le\sigma,\sigma'\le\nu}$, with:

$$\begin{aligned}&q^{(\sigma,\sigma')}(\rho, t, y, \eta)\\ &=\sqrt{-1}\ e^{-i\psi_{\sigma'}(\rho,t,y,\eta)}\,t\rho\,{}^t\lambda_\sigma(t\rho, y, D_y)[e^{i\psi_\sigma(\rho,t,\cdot,\eta)}\,h^{(\sigma,\sigma')}(\rho, t, \cdot\,;\eta)],\end{aligned} \tag{4.61}$$

where ${}^t\lambda_{\sigma'}(t, y, D_y)\in OP\hat{\Sigma}_T^{1,0}$ denotes the transpose of the operator $\lambda_{\sigma'}(t, y, D_y)$ (which we suppose to be proper). We can write:

$$\begin{aligned}q^{(\sigma,\sigma')}(\rho, t, y, \eta)=\sqrt{-1}\ t\rho\lambda_{\sigma'}(\rho t, y, -d_y\psi_{\sigma'}(\rho, t, y, \eta))\\ h^{(\sigma,\sigma')}(\rho, t, y, \eta)+q'^{(\sigma,\sigma')}(\rho, t, y, \eta),\end{aligned} \tag{4.62}$$

for a suitable symbol $q'^{(\sigma,\sigma')}\in H\hat{\Sigma}_T^{0,1}(N_\sigma\times N_{\sigma'})$.

As a consequence of (4.3) we have $(\rho\partial_\rho\psi_\sigma)(\rho t, y, \eta)=t\rho\lambda_\sigma(t\rho, y, -d_y\psi_\sigma)$, $\sigma=1, \cdots, \nu$. As a partial conclusion, from (4.62), (4.58), we get:

$$\begin{aligned}F(h\,;(t\partial_t-tA(t, y, D_y))f)=I(f)+F((-\rho\partial_\rho-1)h\,;f)\\ -F(q'\,;f),\ q'=(q'^{(\sigma,\sigma')})_{1\le\sigma,\sigma'\le\nu}.\end{aligned} \tag{4.63}$$

Using again (the modified version of) Lemma 4.2, we obtain

(4.64) $$F(h;B(t,y,D_y)f)=F(p;f), \text{ with:}$$

(4.65) $$p_{j,k}^{(\sigma,\sigma')}(t,\rho,y,\eta)=e^{-i\phi_{\sigma'}(\rho,t,y,\eta)}\sum_{\sigma''=1}^{\nu}\sum_{l=1}^{N_{\sigma''}}{}^tB_{l,k}^{(\sigma'',\sigma')}(\rho t,y,D_y)$$
$$\cdot[h_{j,l}^{(\sigma,\sigma'')}(\rho,t,\cdot,\eta)e^{i\phi_{\sigma''}(\rho,t,\cdot,\eta)}],\ 1\le\sigma,\ \sigma'\le\nu,$$

where $p_{j,k}^{(\sigma,\sigma')}$ is the (j,k)-entry in the block $p^{(\sigma,\sigma')}$, $j=1,\dots,N_\sigma$, $k=1,\dots,N_{\sigma'}$, and ${}^tB_{l,k}^{(\sigma'',\sigma')}\in OPH\hat{\Sigma}_T^{0,0}$ denotes the transpose of the operator $B_{l,k}^{(\sigma'',\sigma')}$.

Using Remark 4.1 and proceeding as in the proof of Theorem 4.1, we obtain

(4.66) $$F(h;\mathscr{P}f)=I(f)-((\rho\partial_\rho+1)h-q'+p;f),$$

modulo a H.p.r. operator.

Moreover, putting:

(4.67) $$w(\rho,t,y,\eta)=(\rho\partial_\rho+1)h(\rho,t,y,\eta)$$
$$+q'(\rho,t,y,\eta)+p(\rho,t,y,\eta),$$

we have $w\in H\hat{\Sigma}_T^{0,0}(N\times N)$, $w\sim\sum_{j\ge0}w_j$, and

(4.68) $$\begin{cases} w_0(\rho,t,y,\eta)=\hat{w}_0(\rho,y,\eta/|\eta|,t|\eta|),\ \hat{w}_0\in HS^0(N\times N),\\ \hat{w}_0(\rho,y,\eta',z)=(\rho\partial_\rho+1)\hat{h}_0(\rho,y,\eta',z)+\hat{h}_0(\rho,y,\eta',z)\hat{b}_0'(y,\eta',\rho z)\\ +\hat{h}_0(\rho,y,\eta',z)\Gamma^+(y,\eta',\rho z)\chi(y,\rho z)\hat{b}_0''(y,\eta',\rho z)\Gamma^-(y,\eta',\rho z),\end{cases}$$

where $\hat{b}_0'$, $\hat{b}_0''$ have been defined in (4.28), χ is the same cut-off as in (4.27) and $\Gamma^\pm$ (y,η',z) are as follows:

(4.69) $$\Gamma^\pm(y,\eta',z)=\begin{pmatrix} e^{\pm iz\lambda_1(0,y,\eta')}I_{N_1} & & \square\\ & \ddots & \\ \square & & e^{\pm iz\lambda_\nu(0,y,\eta')}I_{N_\nu}\end{pmatrix}$$

A natural way to obtain (4.56) is to impose the conditions: $w_0=\square$, $\hat{h}_0|_{\rho=1}=I_N$ and, for $j\ge1$, $w_j=\square$, $\hat{h}_j|_{\rho=1}=\square$. These conditions lead to the following transport equations:

$(4.70)_j$ $$\begin{cases} (\rho\partial_\rho+1)\hat{h}_j(\rho,y,\eta',z)+\hat{h}_j(\rho,y,\eta',z)\hat{b}_0'(y,\eta',\rho z)\\ +\hat{h}_j(\rho,y,\eta',z)\Gamma^+(y,\eta',\rho z)\chi(y,\rho z)\hat{b}_0''(y,\eta',\rho z)\Gamma^-(y,\eta',\rho z)\\ \quad =f_j(\rho,y,\eta',z),\\ \hat{h}_j(\rho,y,\eta',z)|_{\rho=1}=\begin{cases} I_N, & \text{if } j=0\\ \square, & \text{if } j\ge1\end{cases}\end{cases}$$

where $f_j(\rho,y,\eta',z)$ is a suitable symbol in $HS^j(N\times N)$ depending on $\hat{h}_0,\dots,\hat{h}_{j-1}$, $\hat{b}_0,\dots,\hat{b}_{j-1}$, and $f_0\equiv\square$.

To solve equations $(4.70)_j$ we use the following Lemma.

LEMMA 4.5 *Let $C(y, \eta', z) \in S^0(N \times N)$ be a $N \times N$ matrix with the following properties :*

i) *Re $C(y, \eta', z) \leq \square$, for every y, η', z.*

ii) *Writing $C = (C^{(\sigma,\sigma')})_{1 \leq \sigma, \sigma' \leq \nu}$ in block form, for every $\omega \subset\subset \mathbf{R}^n$ there exists $\delta' > 0$ such that $C^{(\sigma,\sigma')}(y, \eta', z) = \square$ for all $(y, \eta') \in \omega \times S^{n-1}$ and $|z| \geq \delta'$.*

Then for every $f \in HS^k(N \times N)$, $\varphi \in S^k(N \times N)$, $k \in \mathbf{R}$, there exists a unique $h \in HS^k(N \times N)$ satisfying :

$$(4.71) \qquad \begin{cases} \rho\partial_\rho h + hC = f \\ h|_{\rho=1} = \varphi. \end{cases}$$

Moreover, if $f(\rho, y, \eta', z)$ and $\varphi(y, \eta', z)$ are in block diagonal form for every $(y, \eta') \in \omega \times S^{n-1}$, $\rho|z| \geq \delta'$, then the same property holds for h.

PROOF. As far as existence and uniqueness of h (satisfying (4.71)) is concerned the proof proceeds as in Lemma 4.3. By the uniqueness, in any region $y \in \omega$, $|\eta'| = 1$, $\rho|z| \geq \delta'$, the extra diagonal blocks in h solve equations like (4.71) with zero data, hence they vanish.

To apply Lemma 4.5 in solving $(4.70)_j$ we make the identification:

$C(y, \eta', z) = I_N + \hat{b}_0'(y, \eta', z) + \Gamma^+(y, \eta', z)\chi(y, z)\hat{b}_0''(y, \eta', z)$
$\cdot\Gamma^-(y, \eta', z)$

(properties i), ii) are satisfied as a consequence of (4.2)).

Once the symbols $\hat{h}_j$, $j \geq 0$, are constructed we take a symbol $h \in H\hat{\Sigma}_T^{0,0}(N \times N)$ such that:

$$(4.72) \qquad \begin{cases} h(\rho, t, y, \eta) \sim \sum_{j \geq 0} |\eta|^{-j} \hat{h}_j(\rho, y, \eta/|\eta|, t|\eta|) \\ h|_{\rho=1} \equiv I_N \end{cases}$$

REMARK 4.2 By Lemma 4.5, all the symbols h_j, $j \geq 0$, and h can be chosen to be in a block diagonal form for every $y \in \omega$, $\eta \in \mathbf{R}^n$, $\rho|t||\eta| \geq \delta'$.

With the above definition of h we have, for every $f \in C^\infty(]-T, T[; C_0^\infty(\mathbf{R}^n))^N$:

$$(4.73) \qquad F(h; \mathscr{P}f) = I(f) + F(q; f) + Rf,$$

where R is H.p.r. (cfr. Def. 2.7), $I(f)$ is defined as in (4.57), and q is a suitable symbol belonging to $HS_T^{0,\infty}(N \times N)$.

To get rid of the term $F(q; \cdot)$ one proceed as in the proof of Theorem 4.1, determining a matrix $r \in HS_{T,f}^0(N \times N) \subset HS_T^{0,\infty}(N \times N)$, which is block diagonal, satisfying $F(r; \mathscr{P}f) = -F((1-\chi(y, \rho t|\eta|)q; f)$, modulo a H.p.r. operator.

This completes the proof of Theorem 4.2.

We explicitly remark that the F.i.o. defined by (4.57) can be supposed to be invertible, i.e. there exists a continuous operator

$$I^{-1}: C^\infty(]-T,\ T[;\ C_0^\infty(\boldsymbol{R}^n))^N \to C^\infty(]-T,\ T[\times \boldsymbol{R}^n)^N$$

such that $II^{-1}-\mathrm{id}$ and $I^{-1}I-\mathrm{id}$ map $C^\infty(]-T,\ T[;\ \mathscr{E}'(\boldsymbol{R}^n))^N$ into $C^\infty(]-T,\ T[\times \boldsymbol{R}^n)^N$.

As a consequence of (4.56) we have:

$$(4.74)\qquad I^{-1}F(h;\mathscr{P}f)=f+\mathscr{R}f,\ \forall f\in C^\infty(]-T,\ T[;\ C_0^\infty(\boldsymbol{R}^n)^N,$$

where $\mathscr{R}$ maps $C^\infty(]-T,\ T[;\ \mathscr{E}'(\boldsymbol{R}^n))^N \to C^\infty(]-T,\ T[\times \boldsymbol{R}^n)^N$.

5. Existence and Uniqueness. Propagation of Singularities.

In this Chapter we shall prove the existence and uniqueness result stated in the Introduction, as well as some propagation results for the $\widetilde{WF}$ of a (regular distribution) solution of a Fuchsian hyperbolic Cauchy problem.

As a preliminary step we analyze how the parametrices constructed in Chapter 4 propagate the $\widetilde{WF}$.

Let us consider a system $\mathscr{P}$ of the form (4.1) (satisfying hypotheses (4.2)).

By $\mathscr{E}$ we denote either the right parametrix E, constructed in Theorem 4.1, or the operator $I^{-1}\circ F$, constructed in Theorem 4.2 (cfr. (4.74)).

By $\Phi_j^t: T^*\boldsymbol{R}^n\backslash 0 \to T^*\boldsymbol{R}^n\backslash 0$ we denote the flow out of the Hamiltonian vector field $H_{\gamma_j}=\partial_\xi\lambda_j\cdot\partial_x-\partial_x\lambda_j\cdot\partial_\xi$, $j=1,\dots,\nu$. We have the following result.

THEOREM 5.1 *The operator $\mathscr{E}$ has the following properties:*

1. *$\mathscr{E}$ can be extended as a continuous operator from $C^\infty(]-T,\ T[;\ \mathscr{E}'(\boldsymbol{R}^n))^N$ into $C^\infty(]-T,\ T[;\mathscr{D}'(\boldsymbol{R}^n))^N$.*
2. *$\mathscr{E}: \mathscr{E}'_r(]-T,\ T[\times \boldsymbol{R}^n)^N \to \mathscr{D}'_r(]-T,\ T[\times \boldsymbol{R}^n)^N$ (Cfr. Def. 2.3).*
3. *For every $f\in \mathscr{E}'_r(]-T,\ T[\times \boldsymbol{R}^n)^N$ we have:*

a) $\partial WF(\mathscr{E}f)\subset \partial WF(f)$ *(Cfr. Def. 2.3).*

b)
$$\begin{aligned}
&WF(\mathscr{E}f)|_{\{t\neq 0\}}\\
&\subset\{(t,x,\tau,\xi)\mid t\neq 0,\ (t,x,\tau,\xi)\in WF(f)\}\cup\\
&\bigcup_{j=1}^{\nu}\{(t,x,\lambda_j(t,x,\xi),\xi)\mid \exists s,\ \frac{s}{t}\in]0,1[,\ \exists (y,\eta)\\
&\in T^*\boldsymbol{R}^n\backslash 0,\ (s,y,\lambda_j(s,y,\eta),\eta)\in WF(f),\ (x,\xi)\\
&=\Phi_j^{t-s}(y,\eta)\}\cup\bigcup_{j=1}^{\nu}\{(t,x,\lambda_j(t,x,\xi),\xi)\mid t\neq 0,\ \exists (y,\eta)\\
&\in T^*\boldsymbol{R}^n\backslash 0,\ (y,\eta)\in\partial WF(f),\ (x,\xi)=\Phi_j^t(y,\eta)\}.
\end{aligned}$$

For a vector-valued distribution $g=(g_1, \dots, g_N)$ *we have put*

$$\partial WF(g)=\bigcup_{r=1}^{N}\partial WF(g_r),\quad WF(g)=\bigcup_{r=1}^{N}WF(g_r).$$

PROOF. We will prove the Theorem in the case $\mathscr{E}=I^{-1}\circ$F and we leave to the reader to supply the modifications required in the other case.

We begin by considering scalar operators of the form:

$$(5.1)\quad \begin{cases} Kf(t,x)=\int_0^1\iint e^{i[x\cdot\eta-\varphi(\rho t,y,\eta)]}k(\rho,t,y,\eta)f(\rho t,y)\,d\rho\,dy\,\bar{d}\eta, \\ If(t,x)=\iint e^{i[x\cdot\eta-\varphi(t,y,\eta)]}a(t,y,\eta)f(t,y)dy\,\bar{d}\eta, \end{cases}$$

where $\varphi(t,y,\eta)=\varphi(t,0,y,\eta)$ is any of the phases defined in (4.3) and entering in the expression of the operator F (Cfr. 4.54)), $k\in HS_T^{m,1}$ and $a\in S^m_{1,0}((\boldsymbol{R}\times\boldsymbol{R}^n)\times\boldsymbol{R}^n)$.

Property 1. for the operators (5.1) is straightforward.

Denoting by Φ^t the flow out associated with the phase φ we claim that:

i) $K:\ \mathscr{E}'_r(]-T,T[\times\boldsymbol{R}^n)\to\mathscr{D}'_r(]-T,T[\times\boldsymbol{R}^n)$

ii) $\partial WF(Kf)\subset\partial WF(f),\ \forall f\in\mathscr{E}'_r$

iii) $WF(Kf)|_{\{t\neq0\}}$

$\subset\{(t,x,0,\xi)\mid\exists s,\ \frac{s}{t}\in]0,1],\ \exists(y,\eta)\in T^*\boldsymbol{R}^n\backslash0,$

$(s,y,\lambda(s,y,\eta),\eta)\in WF(f),\ (y,\eta)=\Phi^s(x,\xi)\}\cup$

$\{(t,x,\tau,\xi)\mid t\neq0,\ \tau\in\boldsymbol{R},\ \exists(y,\eta)\in T^*\boldsymbol{R}^n\backslash0,$

$(t,y,\tau+\lambda(t,y,\eta),\eta)\in WF(f),\ (y,\eta)=\Phi^t(x,\xi)\}\cup$

$\{(t,y,0,\xi)\mid t\neq0,\ (x,\xi)\in\partial WF(f)\}=A_1\cup A_2\cup A_3,$

for every $f\in\mathscr{E}'_r$.

iv) $I:\ \mathscr{E}'_r(]-T,T[\times\boldsymbol{R}^n)\to\mathscr{D}'_r(]-T,T[\times\boldsymbol{R}^n)$

v) $\partial WF(If)\subset\partial WF(f),\ \forall f\in\mathscr{E}'_r$

vi) $WF(If)|_{\{t\neq0\}}\subset A_2,\ \forall f\in\mathscr{E}'_r$.

Let us show that the claims i)–vi) imply the Theorem (for $\mathscr{E}=I^{-1}\circ F$). Property 2. is a consequence of i) and iv).

Property 3. a) follows from ii) and v). To prove b) we recall that the F.i.o. I and hence I^{-1} have diagonal form. Writing f$\in\mathscr{E}'_r(]-T,T[\times\boldsymbol{R}^n)^N$ in block form $f=(f_\sigma)_{\sigma=1,\dots,\nu}$, $f_\sigma\in\mathscr{E}'_r(]-T,T[\times\boldsymbol{R}^n)^{N_\sigma}$, and taking into account (4.54) we have, for $\sigma=1,\dots,\nu$:

$$(5.2)\quad \begin{aligned}&(I^{-1}F(h;f))_\sigma\\ &=\sum_{\sigma'=1}^{\nu}I_\sigma^{-1}\int_0^1\iint h^{(\sigma,\sigma')}(\rho,t,y,\eta)e^{i[x\cdot\eta-\varphi_{\sigma'}(\rho t,y,\eta)]}f_{\sigma'}(\rho t,y)\,d\rho\,dy\,\bar{d}\eta\\ &=I_\sigma^{-1}\int_0^1\iint e^{i[x\cdot\eta-\varphi_\sigma(\rho t,y,\eta)]}h^{(\sigma,\sigma)}(\rho,t,y,\eta)\,f_\sigma(\rho t,y)\,d\rho\,dy\,\bar{d}\eta\end{aligned}$$

$$+\sum_{\substack{\sigma'=1\\ \sigma'\neq\sigma}}^{\nu} I_\sigma^{-1}\int_0^1\iint e^{i[x\cdot\eta-\varphi_\sigma(\rho t,y,\eta)]}e^{i(\varphi_\sigma(\rho t,y,\eta)-\varphi_{\sigma'}(\rho t,y,\eta))}h^{(\sigma,\sigma')}(\rho,\ t,\ y,\ \eta)$$
$$\cdot f_{\sigma'}(\rho t,\ y)\ d\rho\ dy\ \bar{d}\eta.$$

By Remark 4.2, the matrix h has been choosen to be in a block diagonal form for any $y\in\omega\subset\subset \boldsymbol{R}^n$, $\eta\in\boldsymbol{R}^n$ and $\rho|t||\eta|\geq\delta'$ ($\delta'>0$, depending on ω). Therefore there exists a cut-off function $\chi(y,\ z)$ for which $h^{(\sigma,\sigma')}(\rho,\ t,\ y,\ \eta)=\chi(y,\ \rho t|\eta|)h^{(\sigma,\sigma')}(\rho,\ t,\ y,\ \eta)$ for every ρ, t, y, η and $\sigma\neq\sigma'$.

By Lemma 4.1 we conclude that $e^{i[\varphi_\sigma-\varphi_{\sigma'}]}h^{(\sigma,\sigma')}\in H\hat{\Sigma}_T^{0,0}(N_\sigma\times N_{\sigma'})$, for $\sigma\neq\sigma'$. As a consequence, it will be enough to study the WF for a scalar composition $I^{-1}\circ K$ with the notation (5.1). From vi) we get

$$(5.3)\qquad WF(I^{-1}f)|_{\{t\neq0\}}\subset\{(t,\ y,\ \tau+\lambda(t,\ y,\ \eta),\ \eta)\,|\,t\neq0,\ \tau\in\boldsymbol{R},\ \exists(x,\ \xi)\in T^*\boldsymbol{R}^n\backslash0,\ (t,\ x,\ \tau,\ \xi)\in WF(f),\ (y,\ \eta)=\Phi^t(x,\ \xi)\}.$$

From (5.3) and iii) the conclusion follows.

We now prove claims i) - vi). With no loss of generality we can suppose that the amplitudes k and a vanish for $|\eta|\leq1$.

It is obvious that the distribution kernel of K, still denoted by K, has support in the set $\{(t,\ x),\ (s,\ y)|\frac{s}{t}\in[0,\ 1],\ x,\ y,\in\boldsymbol{R}^n\}$ and we remark that in the interior of this set the function $x\cdot\eta-\varphi(s,\ y,\ \eta)$ is a non-degenerate phase function. Application of Theorem 2.5.14 of Hörmander[11] yields:

$$(5.4)\qquad WF'(K)\cap\{((t,\ x,\ \tau,\ \xi),\ (s,\ y,\ \tau',\ \eta'))\in(T^*\boldsymbol{R}^{1+n}\times T^*\boldsymbol{R}^{1+n})\backslash0\,|\,\tfrac{s}{t}\in]0,\ 1[\}$$
$$\subset\{((t,\ x,\ 0,\ \eta),\ (s,\ y,\ \varphi'_s(s,\ y,\ \eta),\ d_y\varphi(s,\ y,\ \eta)))\,|\,\eta\neq0,\ \tfrac{s}{t}\in]0,\ 1[,\ x=d_\eta\varphi(s,\ y,\ \eta)\}=J.$$

Putting $\Phi^s(y,\ \eta)=(x(s\,;\ y,\ \eta),\ \xi(s\,;\ y,\ \eta))$, we recall that $\varphi(s,\ x(s\,;\ y,\ \eta),\ \eta)=y\cdot\eta$, $(d_y\varphi)\ (s,\ x(s\,;\ y,\ \eta),\ \eta)=\xi(s\,;\ y,\ \eta)$ and $d_\eta\varphi(s,\ x,\ \eta)=y(s\,;\ x,\ \eta)$, where $y(s\,;\ x,\ \eta)$ denotes the inverse of the mapping $y\rightarrow x(s\,;\ y,\ \eta)$ (see, e.g. Chazarain-Piriou [10]). Using these remarks we obtain:

$$(5.5)\qquad J=\{((t,\ x,\ 0,\ \xi),\ (s,\ y,\ \lambda(s,\ y,\ \eta),\ \eta))\,|\,\frac{s}{t}\in]0,\ 1[,\ (y,\ \eta)=\Phi^s(x,\ \xi)\}.$$

Now we claim that:

$$(5.6)\qquad ((t,\ x,\ \tau,\ 0),\ (t,\ y,\ \tau',\ 0))\in WF'(K),\ t\neq0\Rightarrow\tau=\tau'.$$

Consider a point $P_0=((t_0,\ x_0,\ \tau_0,\ 0),\ (t_0,\ y_0,\ \tau'_0,\ 0))$ with $t_0\neq0$ and $\tau_0\neq$

τ_0'; we want to prove that $P_0 \notin WF'(K)$. Choose $f, g \in C_0^\infty(]-T, T[\times \boldsymbol{R}^n)$ with support close to (t_0, x_0) and (t_0, y_0) respectively (and disjoint from $t=0$). Then

$$\begin{aligned}(5.7)\qquad &\langle K(f(s, y)e^{i(s\tau'+y\cdot\xi')}),\ g(t, x)e^{-i(t\tau+x\cdot\xi)}\rangle \\ &= \int_0^1\int e^{i\psi}f(\rho t, y)\ g(t, x)\ k(\rho, t, y, \eta)\ d\rho\ dt\ dx\ dy\ \bar{d}\eta \\ &= H(\tau, \tau', \xi, \xi'),\end{aligned}$$

where

$$\psi = x\cdot(\eta-\xi) - \varphi(\rho t, y, \eta) + y\cdot\xi' + t(\rho\tau'-\tau).$$

Since $|d_y\varphi(s, y, \eta)| \geq c|\eta|$ for all $(s, y, \eta)\in[-T, T]\times\boldsymbol{R}^n\times\boldsymbol{R}^n\setminus 0$, taking into account that $\tau_0\neq\tau_0'$, we can find a conic neighborhood Γ of $((\tau_0, 0), (\tau_0', 0))$ in $\boldsymbol{R}^{2+2n}$ and choose the support of $g(t, x)\ f(s, y)$ so close to (t_0, x_0, t_0, y_0) in such a way that the following relations hold:

$$(5.8)\qquad \begin{cases} \inf\limits_{\substack{|s|\geq T\\ y\in\boldsymbol{R}^n}} (|\eta-\xi| + |\xi'-d_y\varphi(s, y, \eta)|) \geq \text{const.}|\eta| \\ |\frac{s}{t}\tau'-\tau| \geq \text{const. } (|\tau|+|\tau'|+|\xi|+|\xi'|), \end{cases}$$

for $((\tau, \xi), (\tau', \xi'))\in\Gamma$ and $(t, x, s, y)\in\text{supp }(gf)$.

We first integrate by parts in (5.7) using the operator

$$L = \frac{1}{\sqrt{-1}}\sum_1^n{}_j\ [|\xi-\eta|^2 + |\xi'-d_y\varphi(\rho t, y, \eta)|^2]^{-1} \cdot((\eta_j-\xi_j)\partial_{x_j} + (\xi_j'-\partial_{y_j}\varphi)\ \partial_{y_j}),$$

and then integrate by parts with respect to t using the operator $(\sqrt{-1}\ (\rho\tau'-\tau))^{-1}\partial_t$. As a consequence, we obtain that $H(\tau, \tau', \xi, \xi') = 0((|\tau|+|\tau'|)^{-N})$, on Γ, for every $N\geq 0$, and this proves that $P_0\notin WF'(K)$.

The next claim is:

$$\begin{aligned}(5.9)\qquad &((t, x, \tau, \xi), (t, y, \tau', \xi'))\in WF'(K),\ t\neq 0, |\xi|+|\xi'|>0 \Rightarrow \\ &x = d_\eta\varphi(t, y, \eta),\ \xi=\eta,\ \xi' = d_y\varphi(t, y, \eta),\ \tau' = \tau + \varphi_t'(t, y, \eta).\end{aligned}$$

Consider a point $P_0 = ((t_0, x_0, \tau_0, \xi_0), (t_0, y_0, \tau_0', \xi_0'))$ with $t_0\neq 0$ and $|\xi_0| + |\xi_0'|>0$. We can find: an open covering $\Omega_1\cup\Omega_2$ of S_η^{n-1}, a neighborhood $U\times V$ of (x_0, y_0), a positive small ε and a conic neighborhood Γ of $((\tau_0, \xi_0), (\tau_0', \xi_0'))$ in $\boldsymbol{R}^{2+2n}$, such that:

$$(5.10)\qquad \begin{cases} x - d_\eta\varphi(s, y, \eta)\neq 0,\ (x, y)\in U\times V, |s-t_0|<\varepsilon,\ \eta/|\eta|\in\Omega_1 \\ |\xi|+|\xi'|\geq\text{const. }(|\xi|+|\xi'|+|\tau|+|\tau'|),\text{ in }\Gamma \\ \left(\xi, \xi', \tau-\frac{s}{t}\tau'\right) \neq \left(\eta, d_y\varphi(s, y, \eta), \frac{s}{t}\varphi_s'(s, y, \eta)\right), \\ (x, y)\in U\times V, |t-t_0|<\varepsilon, |s-t_0|<\varepsilon, \eta/|\eta|\in\Omega_2, ((\tau, \xi), (\tau', \xi'))\in\Gamma. \end{cases}$$

Consider (5.7) with supp $(f)\subset\{(s, y)\mid |s-t_0|<\varepsilon,\ y\in V\}$, supp $(g)\subset\{(t, x)\mid |t-t_0|<\varepsilon,\ x\in U\}$ and $((\tau, \xi), (\tau', \xi'))\in\Gamma$ and write $H(\tau, \tau', \xi, \xi')=H_1+H_2$, with

(5.11)
$$H_j(\tau, \tau', \xi, \xi') = \int_0^1\int e^{i\psi}\chi_j\left(\frac{\eta}{|\eta|}\right)f(\rho t, y)\ g(t, x)\ k(\rho, t, y, \eta) \cdot d\rho\ do\ dx\ dy\ \bar{d}\eta.$$

where χ_1, χ_2 is a smooth partition of the unity related to the covering $\Omega_1\cup\Omega_2$.

Now, in H_1 we first integrate by parts using the operator

$$\frac{1}{\sqrt{-1}}\sum_1^n{}_j|x-d_\eta\varphi(\rho t, y, \eta)|^{-2}(x_j-\partial_{\eta_j}\varphi(\rho t, y, \eta))\partial_{\eta_j}$$

and then integrate by parts using the operator$-(|\xi^2+|\xi'|^2)^{-1}(\Delta_x+\Delta_y)$. It is easily seen that $H_1(\tau, \tau', \xi, \xi')=0((|\ \xi\ |+|\ \xi'|)^{-N})$, on Γ, for every $N\geq 0$.

In H_2 we integrate by parts using the operator

$$\frac{1}{\sqrt{-1}}\left[\sum_1^n{}_j\ \gamma((\eta_j-\xi_j)\partial_{x_j}+(\xi'_j-\partial_{y_j}\varphi(\rho t, y, \eta))\partial_{y_j}) +\gamma(\rho(\tau'-\varphi'_s(\rho t, y, \eta))-\tau)\partial_t\right],$$

where $\gamma^{-1}=|\eta-\xi|^2+|\xi'-d_y\varphi(\rho t, y, \eta|^2+|\rho(\tau'-\varphi'_s(\rho t, y, \eta))-\tau|^2$.

As a consequence, we get $H_2(\tau, \tau', \xi, \xi')=0((|\xi|+|\xi'|)^{-N})$, on Γ, for every $N\geq 0$ and claim (5.9) follows.

As a partial conclusion, from (5.9), (5.6), (5.5) and (5.4) we get:

(5.11)
$$\begin{aligned}&WF'(K)\cap\{((t, x, \tau, \xi), (s, y, \tau', \xi'))\in(T^*\boldsymbol{R}^{1+n}\times T^*\boldsymbol{R}^{1+n})\setminus 0\mid \frac{s}{t}\\ &\in]0, 1]\}\subset\{((t, x, 0, \xi), (s, x', \lambda(s, x', \xi'))\mid \frac{s}{t}\in]0, 1[, (x', \xi')\\ &=\Phi^s(x, \xi)\}\cup\{((t, x, \tau, \xi), (t, x', \tau+\lambda(t, x', \xi'), \xi'))\mid\ t\neq 0,\\ &\tau\in \boldsymbol{R}, (x', \xi')=\Phi^t(x, \xi)\}\cup\{((t, x, \tau, 0), (t, x', \tau, 0))\mid\ t\neq 0,\\ &\tau\neq 0\}.\end{aligned}$$

We now prove the claim ii). Precisely, let $f\in\mathscr{E}'_r(]-T, T[\times\boldsymbol{R}^n)$ and suppose that $(x_0, \xi_0)\in T^*\boldsymbol{R}^n\backslash\partial WF(f)$; we have to show that $(x_0, \xi_0)\notin\partial\mathrm{WF}(\mathrm{K}f)$. By the hypothesis we can find a proper pdo $B(x, D_x)\in OPS^0(\boldsymbol{R}^n)$ with symbol $b(x, \xi)$ supported in a conic neighborhood $U\times\Gamma$ of (x_0, ξ_0), $b\equiv 1$ on some $U'\times\Gamma'\subset\subset U\times\Gamma$, such that $B(x, D_x)\ f\in C^\infty([-\varepsilon, \varepsilon]; C_0^\infty(\boldsymbol{R}^n))$, for some $\varepsilon\in]0, T[$.

Possibly, by multiplying f by a cut-off function $\chi(t)$ supported in$]-\varepsilon, \varepsilon[$, we can suppose that $B(x, D_x)\chi f\in C^\infty(]-T, T[; C_0^\infty(\boldsymbol{R}^n))$.

We have $BK\chi f=KB\chi f+[B,\ K]\chi f$, with $KB\chi F\in C^{\infty}(]-T,\ T[\times\boldsymbol{R}^{n})$.

By Lemma 4.2 we can write $BK\chi=K'$, with K' defined as K with an amplitude $k'(\rho,\ t,\ x,\ y,\ \eta)=b(x,\ \eta)\ \chi(\rho t)\ k(\rho,\ t,\ y,\ \eta)$, modulo a H. p. r. operator. On the other hand, by the same Lemma, we can write $KB\chi=K''$, with K'' defined as K with an amplitude
$k''(\rho,\ t,\ y,\ \eta)=e^{i\varphi(\rho t,y,\eta)\,t}B(y,\ D_{y})\ [e^{-i\varphi(\rho t,\cdot,\eta)}\chi(\rho t)\ k(\rho,\ t,\cdot,\eta)]\in HS_{T}^{m,r}$, where ${}^{t}B$ is the transpose of the operator B.

We now observe that is possible to find a conic neighborhood $U_{1}\times\Gamma_{1}\subset\subset U'\times\Gamma'$ of $(x_{0},\ \xi_{0})$ such that for some $\delta\in]0,\ \varepsilon[$ we have $(y,\ d_{y}\varphi(s,\ y,\ \eta))\in U'\times\Gamma'$ for all $(y,\ \eta)\in U_{1}\times\Gamma_{1}$ and $|s|\leq\delta$. As a consequence, the amplitude of $[B,\ K]\ \chi$ has zero asymptotic expansion for all $(x,\ y,\ \eta)\in U'\times U_{1}\times\Gamma_{1}$, and for all $\rho,\ t$. Take now $\tilde{B}(x,\ D_{x})\in OPS^{0}(\boldsymbol{R}^{n})$ with symbol supported in $U_{2}\times\Gamma_{2}\subset\subset U_{1}\times\Gamma_{1}$, $\tilde{b}\equiv 1$ in a conical neighborhood of $(x_{0},\ \xi_{0})$ contained in $U_{2}\times\Gamma_{2}$. Using Lemma 4.2 once more, the operator $\tilde{B}\ [B,\ K]\ \chi$ has an amplitude with zero asymptotic expansion for $(x,\ y,\ \eta)\in\boldsymbol{R}^{n}\times U_{1}\times(\boldsymbol{R}^{n}\backslash 0)$, and for all $\rho,\ t$. Choose $\zeta\in C_{0}^{\infty}(U_{1})$, $\zeta\equiv 1$ in U_{2} and write

$$\tilde{B}[B,\ K]\chi f=\tilde{B}[B,\ K]\chi(\zeta f)+\tilde{B}[B,\ K]\chi((1-\zeta)f).$$

Since the amplitude of $\tilde{B}[B,\ K]\chi\zeta$ has zero asymptotic expansion for $(x,\ y,\ \eta)\in\boldsymbol{R}^{n}\times\boldsymbol{R}^{n}\times(\boldsymbol{R}^{n}\backslash 0)$ (and for all $\rho,\ t$), we get $\tilde{B}[B,\ K]\ \chi(\zeta f)\in C^{\infty}(]-T,\ T[\times\boldsymbol{R}^{n})$. On the other hand, if ε is small enough, we can suppose that $|x-d_{\eta}\varphi(\rho t,\ y,\ \eta)|\geq C>0$ on the support of the amplitude of the operator $\tilde{B}[B,\ K]\chi(1-\zeta)$. Integrating by parts using the operator

$$\frac{1}{\sqrt{-1}}\sum_{1}^{n}{}_{j}|x-d_{\eta}\varphi(\rho t,\ y,\ \eta)|^{-2}(x_{j}-\partial_{\eta_{j}}\varphi(\rho t,\ y,\ \eta))\partial_{\eta_{j}},$$

we can easily conclude that $\tilde{B}[B,\ K]\ \chi((1-\zeta)f)\in C^{\infty}(]-T,\ T[\times\boldsymbol{R}^{n})$. This proves claim ii).

Claim v) is proved arguing as above and we omit the details.

Claim iv) and vi) follow from Theorem 2. 5. 14 of Hörmander [13] by remarking that $I\ f|_{\{t\neq 0\}}=I(f|_{\{t\neq 0\}})$.

Therefore we are left with the proof of i) and iii).

Let us prove that for every $f\in\mathscr{E}_{r}'(]-T,\ T[\times\boldsymbol{R}^{n})$ and for every $\chi\in C_{0}^{\infty}(]-T,\ T[)$, $\chi(t)=1$ near $t=0$, we have:

$$(5.12)\qquad WF(K\chi f|_{\{t\neq 0\}})\subset\{(t,\ x,\ \tau,\ \xi)\,|\,\tau=0\}.$$

Fix a point $(t_{0},\ x_{0},\ \tau_{0},\ \xi_{0})$, $t_{0}\neq 0$, $\tau_{0}\neq 0$ and take $\zeta(t),\ \zeta_{1}(t)\in C_{0}^{\infty}(\boldsymbol{R})$, $\zeta_{1}\ \zeta=\zeta$, supported near $t=t_{0}$ with support disjoint from $t=0$ and such that $\zeta_{1}\chi\equiv 0$. Let $\omega(x),\ \omega_{1}(x)\in C_{0}^{\infty}(\boldsymbol{R}^{n})$, $\omega_{1}\ \omega=\omega$, supported near x_{0}, and consider:

$$(5.13)\qquad \langle K\chi f,\ e^{-i(t\tau+x\cdot\xi)}\ \zeta(t)\ \omega(x)\rangle=I(\tau,\xi),$$

with (τ, ξ) in a cone $|\tau| \geq \text{const.} |\xi|$.

We can write:

$$(5.14) \qquad I(\tau, \xi) = \int_{-\infty}^{+\infty} e^{-it\tau} \zeta(t) \int_0^t \langle g(t, s, x), \omega(x) e^{-ix\cdot\xi} \rangle \, ds \, dt,$$

where:

$$(5.15) \qquad g(t, s, x) = \int e^{i[x\cdot\eta - \varphi(s,y,\eta)]} \frac{\zeta_1(t)}{t} \chi(s) \omega_1(x) k\left(\frac{s}{t}, t, y, \eta \right) \cdot f(s, y) \, dy \, \bar{d}\eta.$$

Note that for every $\varepsilon > 0$ the amplitude in (5.15) satisfies uniform estimates of the type

$$(5.16) \qquad \left| s^{\varepsilon} \partial_t^j \partial_x^{\alpha} \partial_y^{\beta} \partial_{\eta}^{\gamma} \left[(\zeta_1(t)/t) \chi(s) \omega_1(x) k\left(\frac{s}{t}, t, y, \eta \right) \right] \right| \leq \text{Const.} \ (1 + |\eta|)^{m - |\gamma|}.$$

Since $f \in C^{\infty}(]-T, T[; \mathscr{E}'(\boldsymbol{R}^n))$, for some $\sigma \in \boldsymbol{R}$ we have $f \in C^{(0)}(]-T, T[; H^{\sigma}(\boldsymbol{R}^n))$. Hence, from the continuity properties of F. i. o's and from (5.16), the following inequalities follow:

$$(5.17) \qquad \sup_{|t| \leq T_1} \int (1 + |\eta|^2)^{\sigma - m} |\partial_t^j \hat{g}(t, s, \eta)|^2 d\eta \leq \text{const.} \ |s|^{-\varepsilon} \sup_{|t| \leq T_1} \int (1 + |\eta|^2)^{\sigma} |\hat{f}(t, \eta)|^2 \, d\eta,$$

for every $s \in]-T, T[$, $0 < T_1 < T$, $\varepsilon > 0$, $j \geq 0$.

As a consequence of the choice of ζ_1 and χ, for every $N \in \boldsymbol{Z}_+$, $(-i\gamma)^N I(\tau, \xi)$ is a linear combination of integrals of the form:

$$(5.18) \qquad I_{k,j}(\tau, \xi) = \int_{-\infty}^{+\infty} e^{-it\tau} \zeta^{(k)}(t) \int_0^t \langle \partial_t^j g(t, s, x), \omega(x) e^{-ix\cdot\xi} \rangle \, ds \, dt, \quad k + j = N.$$

From (5. 17) we obtain, by Parseval identity:

$$|I_{k,j}(\tau, \xi)| \leq \int_{-\infty}^{+\infty} |\zeta^{(k)}(t)| \left| \int_0^t \langle \partial_t^j \hat{g}(t, s, \eta), \hat{\omega}(\xi + \eta) \rangle \, ds \right| dt \leq \text{const.} \ (1 + |\eta|^2)^{|m - \sigma|} \leq \text{const.} \ (1 + \tau^2)^{|m - \sigma|}.$$

As a consequence, for every $N \in \boldsymbol{Z}_+$ we obtain $I(\tau, \xi) = 0 \ (|\tau|^{|m - \sigma| - N})$ on the cones $|\tau| \geq \text{const.} \ |\xi|$. This proves (5.12).

Since the operator $K(1 - \chi)$ maps $\mathscr{D}'_r$ into $\mathscr{D}'_r$ as a consequence of (5. 11) and the above quoted result of Hörmander, claim i) is proved.

To prove claim iii) let $(t, x, 0, \xi)$ be a point $T^*(]-T, T[\times \boldsymbol{R}^n)$ with

$t\neq 0$, $\xi\neq 0$. Suppose that $(x, \xi)\notin\partial WF(f)$ and, for every $s\neq 0$, $\frac{s}{t}\in]0, 1[$, we have $(s, y, \lambda(s, y, \eta), \eta)\notin WF(f)$, $(y, \eta)=\Phi^{s}(x, \xi)$. We can find a cut-off function $\chi(t)\in C_{0}^{\infty}(]-T, T[)$ and a proper operator $B(y, D_{x})\in OPS^{0}(\boldsymbol{R}^{n})$ whose symbol $b(y, \eta)$ is supported in a conic neighborhood $U\times\Gamma$ of (x, ξ) and $b(y, \eta)\equiv 1$ on some $U\times\Gamma\supset\supset U_{1}\times\Gamma_{1}\ni(x, \xi)$, such that $B\chi f\in C^{\infty}(]-T, T[\times\boldsymbol{R}^{n})$.

Therefore $KB\chi f\in C^{\infty}(]-T, T[\times\boldsymbol{R}^{n})$; moreover, $K(I-B)\chi f\in C^{\infty}(]-T, T[\times V)$ for some neighborhood V of x(this is proved as in the last part of the proof of claim ii)). As a consequence, we get that $(t, x, 0, \xi)\notin WF(K\chi f)$.

From the hypotheses on f and from (5.11), application of Hörmander's Theorem yields $(t, x, 0, \xi)\notin WF(K(1-\chi)f)$ and hence $(t, x, 0, \xi)\notin WF(Kf)$. To conclude, let (t, x, τ, ξ) be a point with $t\neq 0$, $\tau\neq 0$, $\xi\neq 0$, and suppose that $(t, y, \tau+\lambda(t, y, \eta), \eta)\notin WF(f)$, $(y, \eta)=\Phi^{t}(x, \xi)$. From (5.12) we get $(t, x, \tau, \xi)\notin WF(K\chi f)$.

A further application of Hörmander's Theorem and (5.11) yields $(t, x, \tau, \xi)\notin WF(K(1-\chi)f)$. This concludes the proof of claim iii) and hence of the Theorem.

We can now prove the following result.

THEOREM 5.2. *Let M be a n-dimensional C^{∞} compact manifold without boundary and let $P\in F_{m-k}^{m}(\boldsymbol{R}\times M)$ be a differential operator satisfying the Fuchs condition* (1.3). *Then for every $f\in\mathscr{D}_{r}'(\boldsymbol{R}\times M)$ and for every $g_{j}\in\mathscr{D}'(M)$, $j=0, 1, \ldots, m-k-1$ there exists a unique distribution $u\in\mathscr{D}_{r}'(\boldsymbol{R}\times M)$ satisfying the Cauchy problem :*

$$(5.19)\qquad \begin{cases} Pu=f, \text{ in } \boldsymbol{R}\times M \\ \partial_{t}^{j}u|_{t=0}=g_{j}, \text{ in } M, \ j=0, 1, \ldots, m-k-1. \end{cases}$$

Moreover, denoting by $\lambda_{j}(t, x, \xi)$, $j=1, \ldots, m$, the hyperbolic roots of P and by Φ_{j}^{t} the Hamiltonian flow in $T^{*}M\backslash 0$ associated to λ_{j}, $j=1, \ldots, m$, the following inclusions hold :

a) $\partial WF(u)\subset\partial WF(f)\cup\bigcup_{j=0}^{m-k-1}WF(g_{j})$.

b) $WF(u)|_{t\neq 0}\subset\{t, x, \tau, \xi)\,|\,t\neq 0, (t, x, \tau, \xi)\in WF(f)\}\cup$
$\bigcup_{j=1}^{m}\{t, x, \lambda_{j}(t, x, \xi), \xi)\,|\,\exists s, \frac{s}{t}\in]0, 1[, \exists(y, \eta)\in T^{*}M\backslash 0,$
$(s, y, \lambda_{j}(s, y, \eta), \eta)\in WF(f), (x, \xi)=\Phi_{j}^{t-s}(y, \eta\}\cup$
$\bigcup_{j=1}^{m}\{(t, x, \lambda_{j}(t, x, \xi), \xi)\,|\,t\neq 0, \exists(y, \eta)\in T^{*}M\backslash 0,$
$(y, \eta)\in\partial WF(f)\cup\bigcup_{l=0}^{m-k-1}WF(g_{l}), (x, \xi)=\Phi_{j}^{t}(y, \eta)\}$.

PROOF. Suppose that the Theorem holds in the case $k=m$ and let us prove it in the case $k<m$. We observe that there exists an operator R: $\mathscr{D}'(M)\rightarrow\mathscr{D}_r'(\boldsymbol{R}\times M)$ such that $Rv|_{t=0}=v$ and $\widetilde{WF}(Rv)=\partial WF(Rv)=WF(v)$, for every $v\in\mathscr{D}'(M)$. Since the property of being a regular distribution is invariant under change of coordinates involving only the ≪spatial variables≫, we can reduce ourselves to the case $M=$ an open subset of $\boldsymbol{R}^n$. Let $\chi\in C_0^\infty(\boldsymbol{R})$, $\chi=1$ near $t=0$, and for every $v\in\mathscr{E}'(\boldsymbol{R}^n)$ put:

$$(5.20)\qquad Rv(t, x)=\int e^{ix\cdot\xi}\chi(t(1+|\xi|^2)^{1/2})\hat{v}(\xi)\,d\xi.$$

Since $\chi(t(1+|\xi|^2)^{1/2})\in S^{0,0}$, application of Theorem 2.1 implies that R maps $\mathscr{E}'(\boldsymbol{R}^n)$ into $\mathscr{D}_r'(\boldsymbol{R}\times\boldsymbol{R}^n)$ and $\widetilde{WF}(Rv)=\partial WF(Rv)\subset WF(v)$.

Furthermore, $\partial_t^j Rv|_{t=0}=\begin{cases} v, & j=0\\ 0, & j>0\end{cases}$, so that $WF(v)\subset\widetilde{WF}(Rv)$.

By glueing together operators of the form (5.20) we can construct the ≪extension≫ operator R with the properties listed above.

Now put $v(t, x)=\sum_{j=0}^{m-k-1}\frac{t^j}{j!}(Rg_j)(t, x)$. Since $\partial_t^j v|_{t=0}=g_j$, we can write $u=v+t^{m-k}$ w for a uniquely defined $w\in\mathscr{D}_r'(\boldsymbol{R}\times M)$.

Using property i) of the class F_{m-k}^m, we get $P(t^{m-k}w)=\tilde{P}w=f-Pv$ for a well defined differential operator $\tilde{P}\in F_0^m(\boldsymbol{R}\times M)$ satisfying F. c. Let $w\in\mathscr{D}_r'(\boldsymbol{R}\times M)$ be the unique solution of the equation $\tilde{P}w=f-Pv$; then $u=v+t^{m-k}w$ is the unique solution of the Cauchy problem (5.19). Moreover, since $\widetilde{WF}(Pv)\subset\partial WF(Pv)\cup\bigcup_{j=0}^{m-k-1}WF(g_j)$, properties a) and b) hold in case $k<m$ provided they are satisfied when $k=m$.

From now on we shall consider the case $k=m$ (no Cauchy data is given at $t=0$!).

Let us prove that for every $x_0\in M$ there exists a distribution $u\in\mathscr{D}_r'(]-\delta, \delta[\times U')$ defined on some open cylinder $]-\delta, \delta[\times U'\ni(0, x_0)$ such that $Pu=f$ in this cylinder. Let (U, φ) be a chart of M, $\varphi: U\rightarrow V=\{y\in\boldsymbol{R}^n||y|<1\}$, $\varphi(x_0)=0$ and denote by $P_V\in F_0^m(\boldsymbol{R}\times V)$ the transformed operator. We observe that $I_{P_v}(\varphi(x);\zeta)=I_P(x;\zeta)$, for every $x\in U$ and $\zeta\in$ C. Let $\theta\in C_0^\infty(\boldsymbol{R})$ with $\theta(s)=1$, $|s|\leq 1/4$, $\theta(s)=0$ for $|s|\geq 1/2$ and with the notation of (1.4), put $P_{V,\theta}(t, y, t\partial_t, D_y)=\sum_{j=0}^{m}\sum_{h=0}^{m-j}t^{m-j-h}A_{m-j-h,j}(\theta(t^2+|y|^2)t, \theta(t^2+|y|^2)y, D_y)(t\partial_t)^h$. Then $P_{V,\theta}\in F_0^m(\boldsymbol{R}\times\boldsymbol{R}^n)$ with $P_{V,\theta}=P_V$ on some open cylinder $]-\delta, \delta[\times V'\ni(0, 0)$. We observe that the coefficients of $P_{V,\theta}$ do not depend on t and y for $|y|\geq 1/2$; moreover, $P_{V,\theta}$ satisfies F. c. and the roots $\zeta_1(y), \ldots, \zeta_m(y)$ of the indicial equation $I_{P_{v,\theta}}(y;\zeta)=0$ are independent of y for $|y|\geq 1/2$.

Given $f\in\mathscr{D}'_r(\boldsymbol{R}\times M)$, denote by $f_V\in\mathscr{D}'_r(\boldsymbol{R}\times V)$ the push-forward of f via φ. Take $\omega\in C_0^\infty(V)$, $\omega\equiv 1$ on V' and put $g_V(t, y)=\omega(y)\ f_V(t, y)$. We try to solve the equation $P_{V,\theta}\ u_V=g_V$.

For sake of simplicity in the following we write $P_{V,\theta}=P$, $g_V=g$, $u_V=u$. Let $p\in\boldsymbol{Z}_+$ be a positive integer to be specified later on. From the property iii) of the classes F_0^m we know that for every $j\geq 0$, $\partial_t^j u|_{t=0}=\sum_{r=0}^{j} L_{j-r,j}^P(y, D_y)\ (\partial_t^r g|_{t=0})$, for some differential operators $L_{j-r,j}^P$ of order $j-r (L_{0,j}^P=1/I_P(y:j))$. Put:

$$v_P(t, y)=\sum_{j=0}^{p-1}\sum_{r=0}^{j}\frac{t^j}{j!}RL_{j-r,j}^P\ (\partial_t^r g|_{t=0}). \tag{5.21}$$

It is easy to check that $u-v_p=t^p w$ and $g-P(v_p)=t^p \mathrm{h}$, for well determined distributions w, $h\in\mathscr{D}'_r(\boldsymbol{R}\times\boldsymbol{R}^n)$. By property ii) of the classes F_0^m we know that $P(t^p w)=t^p P(t, y, t\partial_t+p, D_y)w$, so that we are left with the equation $P(t, y, t\partial_t+p, D_y)w=h\in\mathscr{E}'_r(\boldsymbol{R}\times\boldsymbol{R}^n)$.

We now apply the reduction to a singular system performed in Chapter 1, taking as L in (1.15) the operator $t\partial_t+p-t\ Z(t, y, D_y)-\gamma$, where $\frac{1}{\sqrt{-1}}$ $Z(t, y, \eta)$ *is any of the hyperbolic roots of* P and $\gamma\in\boldsymbol{C}\setminus\boldsymbol{Z}$.

The equation $P(t, y, t\partial_t+p, D_y)\ w=h$ is transformed into a $N\times N$ system $\mathscr{P}w=h$, where $\mathscr{P}$ has the form (1.12), $N=\frac{m(m+1)}{2}$, $h=(\underbrace{0,\dots,0, h,}_{m} 0,\dots,0)$ and the principal symbol of $B(0, y, D_y)$ has the eigenvalues $\zeta_j(y)-p$, $\gamma+(m-j)-p$, $j=1,\dots, m$. We now choose $p\in\boldsymbol{Z}_+$ such that

$$\operatorname{Re}\sigma_0(B)(t, y, \eta)\leq -I_N, \tag{5.22}$$

for all $(t, y, \eta)\in\boldsymbol{R}\times\boldsymbol{R}^n\times(\boldsymbol{R}^n\setminus 0)$ (since the coefficients of P are independent of (t, y) for $|y|\geq 1/2$, the same property is verified by the matrices $A(t, y, D_y)$ and $B(t, y, D_y)$).

Let now $U(t, y, D_y)$, $U^{-1}(t, y, D_y)$ be proper operators of order zero such that $UU^{-1}-\mathrm{id}$, $U^{-1}U-\mathrm{id}$ are partially regularizing and

$$U^{-1}(t, y, D_y)\ A(t, y, D_y)\ U(t, y, D_y)= \tag{5.23}$$

$$=\sqrt{-1}\begin{pmatrix} \lambda_1(t, y, D_y) & & & \square \\ & \ddots & & \\ & & \lambda_m(t, y, D_y) & \\ & \square & & \frac{1}{\sqrt{-1}}Z(t, y, D_y)\ I_{N-m} \end{pmatrix}$$

$$=\mathscr{A}(t, y, D_y).$$

Then $U^{-1}(t, y, D_y)\ w$ satisfies the system:

(5.24) $$(t\partial_t I_N - t\mathscr{A}(t, y, D_y) - \mathscr{B}(t, y, D_y))U^{-1}w = U^{-1}h + h',$$

with $h' \in C^\infty(\boldsymbol{R}_t \times \boldsymbol{R}^n_y)^N$ and $\mathscr{B} = U^{-1}BU$.

We shall continue to denote the system (5.24) by $\mathscr{P}\, U^{-1}\, w$.

Let now $Q(t, y, D_y) \in OP\hat{\Sigma}^{0,0}$ be the decoupling operator constructed in Chapter 3 and denote by $Q^{-1}(t, y, D_y) \in OP\hat{\Sigma}^{0,0}$ a parametrix of Q (Q and Q^{-1} are supposed to be proper operators).

As follows from Theorem 3.1, the vector $Q^{-1}\, U^{-1}\, w$ satisfies the system

(5.25) $$\tilde{\mathscr{P}} Q^{-1}\, U^{-1}\, w = (I_N t\partial_t - t\mathscr{A}(t, y, D_y) - \tilde{\mathscr{B}}(t, y, D_y))\ Q^{-1}\, U^{-1}\, w = Q^{-1}\, U^{-1}\, h + h'',$$

for some $h'' \in C^\infty(\boldsymbol{R}_t \times \boldsymbol{R}^n)^N$.

Since by (5.22) the matrix $\tilde{\mathscr{B}}$ satisfies the hypotheses (4.2), using the right parametrix E constructed in Theorem 4.1 we obtain that the vector $EQ^{-1}\, U^{-1}\, h$ satisfies the system (5.25) modulo C^∞.

Taking the last component of the vector $UQ\ EQ^{-1}\, U^{-1}h$ we obtain a function $u'_v(t, y) \in \mathscr{D}'_r(\boldsymbol{R} \times \boldsymbol{R}^n)$ which satisfies:

(5.26) $$P_{V,\theta}(t, y, t\partial_t, D_y)[v_p + t^p u'_v] = g_v + g'_v,$$

for some $g'_v \in C^\infty(\boldsymbol{R} \times \boldsymbol{R}^n)$. Using the results of Tahara ([20], Theorem 3.1) we finally get a distribution $u_v \in \mathscr{D}'_r(\boldsymbol{R} \times \boldsymbol{R}^n)$ such that $P_{V,\theta}\ u_v = g_v$ in $\boldsymbol{R}_t \times \boldsymbol{R}^n$. Pulling-back u_v to the manifold M we obtain a distribution $u \in \mathscr{D}'_r(]-\delta, \delta[\times U')$ which satisfies the original equation $Pu = f$ on some open cynder $]-\delta, \delta[\times U' \subset \boldsymbol{R} \times M$, containing the point $(0, x_0)$.

The next step consists in showing that if $u \in \mathscr{D}'_r(]-\delta, \delta[\times U')$ and $v \in \mathscr{D}'_r(]-\delta, \delta[\times U'')$ are two local solutions constructed as above, and if $U' \cap U'' \neq \phi$, there is a neighborhood $\mathscr{O}$ of $\{0\} \times (U' \cap U'')$ in $\boldsymbol{R}_t \times (U' \cap U'')$ such that $u|_{\mathscr{O}} = v|_{\mathscr{O}}$.

To prove this claim, let $\bar{x} \in U' \cap U''$ and use a chart (Ω, φ) of M with $\Omega \subset U' \cap U''$, $\varphi: \Omega \rightarrow V = \{y \in \boldsymbol{R}^n \mid |y| < 1\}$, $\varphi(\bar{x}) = 0$.

We push-forward P, u and v onto $\boldsymbol{R} \times V$. We deform P and cut-off the image of $u - v$ as above, obtaining a distribution $w \in \mathscr{E}'_r(\boldsymbol{R} \times \boldsymbol{R}^n)$ satisfying $P_{V,\theta}(t, y, t\partial_t, D_y)\ w = 0$ on some open cylinder $]-T, T[\times V' \subset \boldsymbol{R} \times V$, containing the point $(0, 0)$. Using the same procedure described above we find $w \in \mathscr{D}'_r(\boldsymbol{R} \times \boldsymbol{R}^n)^N$ which has the property that $\tilde{\mathscr{P}} w \in C^\infty(]-T', T'[\times V'')^N$ for some open cylinder $]-T', T'[\times V'' \subset]-T, T[\times V'$. We now use the left parametrix constructed in Theorem 4.2 and the propagation results of Theorem 5.1 to conclude that $w \in C^\infty(]-T'', T''[\times V''')$ for some smaller cylinder.

Hence we can conclude that $w \in C^\infty(]-T'', T''[\times V''')$ and $P_{V,\theta}w=0$ on this cylinder. By the local uniqueness Theorem 1.3. b) we obtain that $u=v$ on some open cylinder $]-\delta(\bar{x}), \delta(\bar{x})[\times\Omega'\subset]-\delta, \delta[\times\Omega$, containing $(0,\bar{x})$.

It follows that we can find a finite open covering $\bigcup_{j=1}^{r} U_j=M$ of M and distribution $u_j\in\mathscr{D}'_r(]-\delta, \delta[\times U_j)$, for some $\delta>0$, such that:

i) $Pu_j=f$, in $]-\delta, \delta[\times U_j$, $j=1, \dots, r$; ii) whenever $U_i\cap U_j\neq\phi$, there is a neighborhood $\mathscr{O}_{ij}$ of $\{0\}\times(U_i\cap U_j)$ in $\boldsymbol{R}\times(U_i\cap U_j)$ for which $u_i|_{\mathscr{O}_{ij}}=u_j|_{\mathscr{O}_{ij}}$.

Let $\{\zeta_j\in C_0^\infty(U_j)\,|\,j=1, \dots, r\}$ be a partition of unity and put u(t, x)$=$ $\sum_1^r{}_j\zeta_j(x)u_j(t, x)$, $(t, x)\in]-\delta, \delta[\times M$. Then $u\in\mathscr{D}'_r(]-\delta, \delta[\times M)$ and $Pu=f$ on some cylinder $]-\delta', \delta'[\times M\,(0<\delta'\leq\delta)$.

To get a solution defined on $\boldsymbol{R}\times M$ it is enough to solve the classdcal Cauchy problem $P\,u_\pm=f$, in $[\delta'/2, +\infty[\times M$ (resp.$]-\infty, -\delta'/2]\times M$) with $\partial_t^j u_\pm|_{t=\pm\delta'/2}=\partial_t^j u|_{t=\pm\delta'/2}$, $j=0, 1, \dots, m-1$.

This proves the existence result.

Now if $u\in\mathscr{D}'_r(\boldsymbol{R}\times M)$ satisfies $Pu=0$ in $\boldsymbol{R}\times M$, using the same arguments as above we conclude that for every $x_0\in M$ there is an open cylinder $]-\delta(x_0), \delta(x_0)[\times U\ni(0, x_0)$ on which $u=0$. As a consequence, $u=0$ on some cylinder $]-\delta, \delta[\times M$. By standard uniqueness results for strictly hyperbolic operators, we conclude that $u=0$ in $\boldsymbol{R}\times M$. The propagation results a) and b) follow from Theorem 5.1 taking into account that for the truncated Taylor expansions v_p of (5.21) we have $\widetilde{WF}(v_p)=\partial WF(v_p)\subset\partial WF(g)$. Details are left to the reader.

Local existence and uniqueness results can be proved using the same tecnique as in the proof of Theorem 5.2.

In the remainder of this Chpater we shall analyze how the propagation relations of Theorem 5.1 can be improved.

We recall that one of the major shortcomings of our calculus in the classes $OPS^{m,k}$ is that operators in these classes do not preserve distributions whose wave front set is disjoint from the conormal bundle of the initial hypersurface $t=0$. This is the main reason which forced us to work in the classes of *regular* distributions $\mathscr{D}'_r(\boldsymbol{R}\times M)$. From our discussion we cannot deduce that if the r. h. s. f in (5.19) has the property $WF(f)\cap N^*M=\phi$ (N^*M being the conormal bundle of $\{0\}xM$) then the same property holds for the solution $u\in\mathscr{D}'_r(\boldsymbol{R}\times M)$. However, this property should be true as suggested by the results of Tahara ([19], Theorem 2.1.3) in the analytic-hyperfunction framework.

Now we shall prove that in fact it holds true even in the C^∞-category as a consequence of the following general result.

THEOREM 5.3. *Let $P=\sum_0^m {}_j t^{k-j}P_{m-j}(t, x, D_t, D_x)$ be a differential operator of order m(the P_{m-j} being of order $m-j$ and $1\leq k\leq m$) defined in $\boldsymbol{R}_t\times\boldsymbol{R}_x^n$ with smooth coefficients.*

Put:

$$\begin{cases} N_\pm^*=\{(t, x, \tau, \xi)\in T^*(\boldsymbol{R}\times\boldsymbol{R}^n)\backslash 0 \mid t=0, \ \xi=0, \ \pm\tau>0\} \\ N^*=N_+^*\cup N_-^*. \end{cases}$$

Suppose that:

$$\sigma_m(P_m)|_{N^*}\neq 0. \tag{5.27}$$

Then, for every $u\in C^\infty(\boldsymbol{R}_t; \mathscr{D}'(\boldsymbol{R}^n))$ we have:

$$WF(u)\cap N^*=WF(Pu)\cap N^*. \tag{5.28}$$

PROOF. We shall prove that if $\rho_0\in N^*\backslash WF(Pu)$ then $\rho_0\notin \mathrm{WF(u)}$.

To be definite, suppose $\rho_0=(0, x_0, 1, 0)\in N_+^*$ (the case $\rho_0\in N_-^*$ can be handled analogously). Let Γ be a closed conic neighborhood of ρ_0 for which $\Gamma\cap N_-^*=\phi$ and $WF(Pu)\cap\Gamma=\phi$. Take $\chi(t, x, D_t, D_x)\in OPS_{1,0}^0$ to be a proper operator whose symbol is supported in Γ and is identically 1 in some open conic neighborhood Γ' of ρ_0, $\Gamma'\subset\subset\Gamma$. By Lemma 5.1 (whose proof is postponed) we have $\chi u\in C^\infty(\boldsymbol{R}_t; \mathscr{D}'(\boldsymbol{R}^n))$. We claim that $WF(P\chi u)\cap N^*=\phi$. Obviously $WF(P\chi u)\cap N_-^*\subset WF(\chi)\cap N_-^*=\phi$; moreover, $P\chi u=\chi Pu+[P, \chi]u$ and $WF(\chi)\subset\Gamma$, while $WF([P, \chi])\subset\Gamma\backslash\Gamma'$. Since $WF(Pu)\cap\Gamma=\phi$ we easily obtain that $WF(\chi Pu)\cap N_+^*=\phi$ and $WF([P, \chi]u)\cap N_+^*=\phi$.

In conclusion, putting $v=\chi u$, we can suppose from the beginning that we have a distribution v with the following properties:

$$v\in C^\infty(\boldsymbol{R}_t; \mathscr{D}'(\boldsymbol{R}^n)), \quad WF(v)\cap N^*\subset N_+^*, \quad WF(Pv)\cap N^*=\phi, \tag{5.29}$$

and we want to show that $WF(v)\cap N_+^*=\phi$.

To prove this claim we shall deeply rely on some constructions performed in B. L. P. [6]. Using Lemma 3.3 [6] we can write $f=Pv=Q_mt^kv+Q_{m-1}t^{k-1}v+\cdots+Q_{m-k}v$ for some differential operators Q_{m-j} of order $m-j$, $j=0, \cdots, k$. Since $\sigma_m(Q_m)=\sigma_m(P_m)$, (5.27) implies the existence of a classical proper pdo Q_m^{-1} such that $Q_m^{-1}Q_m-$ id is regularizing in a conical neighborhood of N^*. As a consequence v satisfies the equation:

$$t^kv+Q_m^{-1}Q_{m-1}t^{k-1}v+\ldots+Q_m^{-1}Q_{m-k}v=g, \tag{5.30}$$

for some g with $WF(g)\cap N^*=\phi$, which we write $g\in\mathscr{D}'_{CN^*}$.

We now reduce eq. (5.30) to a system putting (cfr. formula (3.68) of [6]):

$$(5.31) \qquad v_j=\Lambda^{-(k-j)}t^{\,j-1}v, \ j=1, \cdots, k,$$

where $\Lambda\in OPS^1_{cl}$ is an invertible pdo whose (full) symbol is equal to τ in a conic neighborhood of N^*. Denoting by v the vector $(v_1, \dots, v_k)$ we obtain a system

$$(5.32) \qquad (t\partial_t I_k-A(t, x, D_t, D_x))v=g\in(\mathscr{D}'_{CN^*})^k,$$

for some $A\in OPS^0_{cl}(k\times k)$. By Lemma 5.1 the vector $v\in C^\infty(\boldsymbol{R}_t;\mathscr{D}'(\boldsymbol{R}^n))^k$ and we have $WF(v)\cap N^*\subset N^*_+$.

Using Lemmas 2.6-2.8 and Propositions 2.3, 2.4 of [6] we get that there exist:

i) Two matrices $E, E'\in OPS^0_{1,0}(k\times k)$ which are elliptic on some neighborhood $]-\delta, \delta[\times U\times\Gamma_C(=\{(\tau, \xi)\,|\,\tau>C|\xi|\})$ of $\rho_0=(0, x_0, 1, 0)$.

ii) A $k\times k$ matrix $C(x, D_t, D_x)\in OPS^0_{1,0}(k\times k)$ which is independent of t and in block triangular form $C=(C_{ij})_{i,j=1,\cdots,\nu}$. Moreover:

α) $C_{ij}\equiv\square$ for $i>j$.

β) $C_{ij}(x, D_t, D_x)=C_{jj}(x)$ for a smooth matrix $C_{jj}(x)$ satisfying $\sup_{x\in\boldsymbol{R}^n}\|C_{jj}(x)\|=M<+\infty$,
such that $(t\partial_t I_k-A(t, x, D_t, D_x))E-E'(t\partial_t I_k-C(x, D_t, D_x))$ is a pdo of order $-\infty$ on a smaller neighborhood $]-\delta', \delta'[\times U'\times\Gamma_{C'}$ of ρ_0.

Denote by E^{-1}(resp. E'^{-1}) a parametrix of E(resp. E') such that $EE^{-1}-\mathrm{id}$, $E^{-1}E-\mathrm{id}$ (resp. $E'E'^{-1}-\mathrm{id}$, $E'^{-1}E'-\mathrm{id}$) are of order $-\infty$ on a possibly smaller neighborhood $]-\delta'', \delta''[\times U''\times\Gamma_{C''}$ of ρ_0.

Take a proper pdo $\chi\in OPS^0_{1,0}$, elliptic near ρ_0 and with symbol supported in $]-\delta'', \delta''[\times U''\times\Gamma_{C''}$. Putting $w=E^{-1}\chi v$, we have:

$$(5.33) \qquad (t\partial_t I_k-C(x, D_t, D_x))w=h\in(\mathscr{D}'_{CN^*})^k.$$

By Lemma 5.1 we have $w\in C^\infty(\boldsymbol{R}_t;\mathscr{D}'(\boldsymbol{R}^n))^k$ and $WF(w)\cap N^*\subset N^*_+$. Writing w and h in block form we obtain the equations:

$$(5.34) \qquad \begin{cases} (t\partial_t-C_{\nu\nu}(x))w_\nu=h_\nu \\ (t\partial_t-C_{\nu-1,\nu-1}(x))w_{\nu-1}-C_{\nu-1,\nu}(x, D_t, D_x)w_\nu=h_{\nu-1} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (t\partial_t-C_{11}(x))w_1-\sum_2^\nu{}_j C_{1,j}(x, D_t, D_x)w_j=h_1. \end{cases}$$

Let us consider the first equation in (5.34) and put for simplicity $C_{\nu\nu}(x)=C(x)$. By Proposition 2.5 of [6] there exists two matrices $A_C\in$

$OPS^{M}_{1,\delta}(k\times k)$, $B_C\in OPS^{M+1}_{1,\delta}(k\times k)$, for every $\delta>0$, such that: 1) There exist operators $A_C^{-1}\in OPS^{M}_{1,\delta}$, $B_C^{-1}\in OPS^{M+1}_{1,\delta}$ with $A_CA_C^{-1}-\mathrm{id}$, $B_CB_C^{-1}-\mathrm{id}$ smoothing near N^*_+; furthermore, A_C and B_C are of order $-\infty$ near N^*_-; 2) $A_C(t\partial_t-C(x))B_C-tI$ is of order $-\infty$ near N^*_+.

As a consequence, by putting $\psi_\nu=B_C^{-1}w_\nu$, we obtain the equation

$$t\psi_\nu(t,\ x)=\varphi_\nu(t,\ x)\in(\mathscr{D}'_{CN^*})^{k\nu}. \tag{5.35}$$

Lemma 5.1 can be applied yielding $\psi_\nu\in C^\infty(\boldsymbol{R}_t;\mathscr{D}'(\boldsymbol{R}^n))^{k\nu}$ and $WF(\psi_\nu)\cap N^*\subset N^*_+$.

Since φ_ν and ψ_ν have traces of all order at $t=0$, from (5.35) we obtain $\varphi_\nu(0,\ x)=0$, hence we can write $\varphi_\nu(t,\ x)=t\omega_\nu(t,\ x)$ for a well defined distribution $\omega_\nu\in(\mathscr{D}'_{CN^*})^{k\nu}$ (cfr. Hanges [12]).

It follows that all distribution solutions of eq. (5.35) are of the form $\psi_\nu(t,\ x)=\omega_\nu(t,\ x)+\gamma_\nu(x)\otimes\delta_t$ for some $\gamma_\nu\in\mathscr{D}'(\boldsymbol{R}^n)^{k\nu}$. Since ψ_ν and ω_ν are C^∞ in t, we conclude that $\gamma_\nu=0$.

Thus $\psi_\nu=\omega_\nu\in(\mathscr{D}'_{CN^*})^{k\nu}$ and the same is true for w_ν.

The second eq. is (5.34) is now of the form $(t\partial_t-C_{\nu-1,\nu-1}(x))w_{\nu-1}=(h_{\nu-1}+C_{\nu-1,\nu}(x,\ D_t,\ D_x)w_\nu)\in(\mathscr{D}'_{CN^*})^{k\nu-1}$. We can argue as above and conclude that $WF(w_{\nu-1})\cap N^*=\phi$. Proceeding in this way we finally obtain that $WF(w)\cap N^*=\phi$. Since $w=E^{-1}\chi\ v$ and χ is elliptic near ρ_0, we conclude that $\rho_0\notin WF(v)$ and hence $\rho_0\notin WF(v)$ because $v_1=\Lambda^{-(k-1)}v$ and Λ is elliptic near N^*.

To complete the proof of the above Theorem we need the following lemma.

LEMMA 5.1 *Let $a(t, x, \tau, \xi)$ be a smooth function such that for some $m\in\boldsymbol{R}$ and for every j, $k\in\boldsymbol{Z}_+$, α, $\beta\in\boldsymbol{Z}^n_+$, $K\subset\subset\boldsymbol{R}_t\times\boldsymbol{R}^n_x$, $\epsilon>0$, there exists a constant $C>0$ for which:*

$$|\partial^j_t\partial^k_\tau\partial^\alpha_x\partial^\beta_\xi\ a(t,x,\tau,\xi)|\le C(1+|\tau|+|\xi|)^{m+\epsilon-k-|\beta|}, \tag{5.36}$$

for every $(t, x)\in K$, $(\tau, \xi)\in\boldsymbol{R}\times\boldsymbol{R}^n$.

Denote by $A=a(t, x, D_t, D_x)$ the pdo associated to the symbol a. Then A maps $C^\infty(\boldsymbol{R}_t;\ \mathscr{E}'(\boldsymbol{R}^n))$ into $C^\infty(\boldsymbol{R}_t;\mathscr{D}'(\boldsymbol{R}^n))$.

PROOF. Without loss of generality we can suppose that a satisfies estimates (5.36) uniformly in $(t,\ x)\in\boldsymbol{R}^{1+n}$. Denote by $\mathscr{S}^{m+}$ this class and by OP$\mathscr{S}^{m+}$ the class of the corresponding operators.

We can suppose that $u\in C^\infty(\boldsymbol{R}_t;\ \mathscr{E}'(\boldsymbol{R}^n))\cap\mathscr{E}'(\boldsymbol{R}_t\times\boldsymbol{R}^n_x)$ so that by a well known characterization (see e.g. Treves [23]), we know that for every $k\in\boldsymbol{Z}_+$ there is a $\sigma_k\in\boldsymbol{R}$ such that $u\in H^k(\boldsymbol{R}_t, H^{\sigma_k}(\boldsymbol{R}^n))$.

Let $\mathscr{S}^{p+,q+}$, p, $q\in\boldsymbol{R}$, denote the class of all smooth functions $b(t, \tau, x, \xi)$ which, for every $\epsilon>0$, satisfy estimates of the form:

$$\sup_{(t,x)} |\partial_t^j \partial_\tau^k \partial_x^\alpha \partial_\xi^\beta\, b(t, \tau ; x, \xi)| \le C(1+|\tau|)^{p+\epsilon-k}(1+|\xi|)^{q+\epsilon-|\beta|}.$$

By $OP\mathscr{S}^{p+,q+}$ we denote the class of the corresponding pdo's.

Obviously $(1+|D_t|^2)^{p/2}\in OP\mathscr{S}^{p+,0+}$, $(1+|D_x|^2)^{q/2}\in OP\mathscr{S}^{0+,q+}$; moreover, we have the inclusions $\mathscr{S}^{m+}\subset\mathscr{S}^{m+,m+}$ $(m\ge 0)$, $\mathscr{S}^{m+}\subset\mathscr{S}^{0+,0+}$ $(m<0)$.

Let us suppose $a(t, x, \tau, \xi)\in\mathscr{S}^{m+}$ with $m\ge 0$. We claim that A maps continuously $H^k(\boldsymbol{R}_t ; H^{\sigma_k}(\boldsymbol{R}^n))$ into $H^{k-(m+1)}(\boldsymbol{R}_t ; H^{\sigma_k-(m+1)}(\boldsymbol{R}^n))$, so that the Lemma follows.

To prove this claim it is enough to show that the operator

$$B=(1+|D_t|^2)^{(-(m+1)+k)/2}(1+|D_x|^2)^{(-(m+1)+\sigma_k)/2}$$
$$\cdot A(1+|D_t|^2)^{-k/2}(1+|D_x|^2)^{-\sigma_k/2}$$

maps continuously $L^2(\boldsymbol{R}_t\times\boldsymbol{R}^n)$ into itself.

It can be verified that $B\in OP\mathscr{S}^{-1+,-1+}$. Denoting by $b(t, \tau ; x, \xi)$ a symbol of B, from well known results we have:

$$\|\partial_t^j\partial_\tau^k b(t, \tau, x, D_x)\|_{L^2(\boldsymbol{R}_x^n)\to L^2(\boldsymbol{R}_x^n)} \le \text{const.}\ (1+|\tau|)^{-k}, \tag{5.37}$$

for every j, $k\in\boldsymbol{Z}_+$. From (5. 37) the conclusion follows.

REMARKS 5. 1. 1) The result of the Lemma applies obviously to the vector valued situation.

2) The operators A_C, B_C (as well as their parametrices A_C^{-1}, B_C^{-1}) considered in Theorem 5. 3 actually belong $OP\mathscr{S}^{M+}$and $OP\mathscr{S}^{(M+1)+}$ respectively as a consequence of their symbol structure explained in [6], pag. 90.

COROLLARY. *Under the same hypotheses of Theorem* 5.2, *denoting by* N^*M *the conormal bundle of* $t=0$ *in* $\boldsymbol{R}_t\times M$, *we have* $WF(u)\cap N^*M = WF(f)\cap N^*M$, *where* $u\in\mathscr{D}'_r(\boldsymbol{R}\times M)$ *is the solution of the Cauchy problem* (5.19).

PROOF. It is a trivial consequence of Theorem 5.3 since for $P\in F^m_{m-k}(\boldsymbol{R}\times M)$, having the form (1. 1), we have $\sigma_m(P_m)|_{N^*M}\ne 0$.

In the next result we show that singularities of the initial data g_j in Cauchy problem (5. 19) give rise to singularities of the solution u (a fact which is not a priori obvious). Precisely we have the following theorem.

THEOREM 5. 4. *Let the hypotheses of Theorem* 5.2 *be satisfied and denote* $\lambda_j(t, x, \xi)$, $j=1, \dots, m$ *the hyperbolic roots of* P.

Let x_0 *be a point of* M *such that*:

i) $WF(f)\cap\pi^{-1}((0, x_0))=\phi$ ($\pi: T^*(\boldsymbol{R}\times M)\backslash 0\to\boldsymbol{R}\times M$ *being the canonical projection*).

ii) *for some* ξ_0, $(x_0, \xi_0)\in\bigcup_{j=0}^{m-k-1} WF(g_j)$.

Then there exists a $j\in\{1, \dots, m\}$ *for which*:

(5.38) $$(0, x_0, \lambda_j(0, x_0, \xi_0), \xi_0)\in WF(u),$$

where $u \in\mathscr{D}'_r(\boldsymbol{R}\times M)$ *is the solution of the Cauchy problem* (5.19).

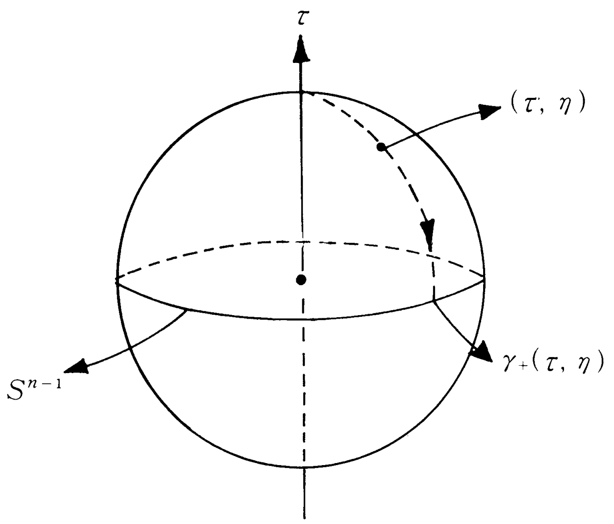
PROOF. The proof is based on the following fact. If $u \in C^\infty(\boldsymbol{R}_t; \mathscr{D}'(M))$ and if for some $x_0 \in M$ we have $(0, x_0; \pm 1, 0) \in N^*M\setminus WF(u)$, then $(x_0, \xi) \in\partial WF(u)$ iff $(0, x_0, \tau, \xi) \in WF(u)$ for some $\tau\in\boldsymbol{R}$.

Suppose this fact already proved and suppose that $(0, x_0; \lambda_j(0, x_0, \xi_0), \xi_0) \notin WF(u)$ for every j. From Duistermaat-Hörmander [11] it follows that $\{0, x_0, \tau, \xi_0)|\tau\in\boldsymbol{R}\}\cap WF(u)=\phi$. On the other hand, by hypothesis i) and the Corollary we know that $\pi^{-1}(0, x_0)\cap WF(u)\cap N^*M=\phi$ so that we should have $(x_0, \xi_0)\notin\partial WF(u)$. Since $\partial WF(u)\supset\bigcup_{j=0}^{m-k-1} WF(g_j)$ we get a contradiction.

To conclude we only have to prove the above mentioned property of $\partial WF(u)$.

Precisely, we have to prove that if $(0, x_0, \tau, \xi)\notin WF(u)$ for all $\tau\in\boldsymbol{R}$ then $(x_0, \xi)\notin\partial WF(u)$.

With no loss of generality, suppose $|\xi|=1$ and denote by $\gamma_\pm$ the projections along the meridians of the upper and lower semisphere in $\boldsymbol{R}^{n+1}$ (with the northern and southern pole cut-off) onto the equator S^{n-1} given by $\tau=0$.



Our hypothesis can be rewritten as $WF(u)\cap\{(0, x_0, \tau, \eta)|(\tau, \eta)\in[\gamma_+^{-1}(\xi)\cup\gamma_-^{-1}(\xi)\cup(1, 0)\cup(-1, 0)]\}=\phi$.

We can find a neighborhood $U\times\omega$ of (x_0, ξ) in $M\times S^{n-1}$, a $\delta>0$ and a neighborhood $\omega_\pm$ of $(\pm 1, 0)$ in S^n, such that

(5.39) $$WF(u)\cap\{(t, x, \tau, \eta)\in T^*(\boldsymbol{R}\times M)\backslash 0\,|\,|t|<\delta,\ x\in U,\ (\tau, \eta)\in[\gamma_+^{-1}(\omega)\cup\gamma_-^{-1}(\omega)\cup\omega_+\cup\omega_-]\}=\phi.$$

Take a symbol $\chi(t, x, \tau, \xi)\in S^0_{1,0}$ whose support is contained into $]-\delta, \delta[\times U\times(\omega_+\cup\omega_-)$ and such that $\chi(t, x, D_t, D_x)u\in C^\infty$.

Let $B(x, D_x)\in OPS^0_{1,0}(M)$ be a proper operator elliptic near (x_0, ξ), with symbol supported in $U\times\omega$. Now, for $|t|<\delta$ we have $WF(B(1-\chi))u\subset\{(t, x, \tau, \eta)\,|\,|t|<\delta, x\in U, (\tau, \eta)\in\gamma_+^{-1}(\omega)\cup\gamma_-^{-1}(\omega)\}$ so that $B(1-\chi)\ u\in C^\infty$ for $|t|<\delta$. Since $B\chi u\in C^\infty$, we conclude that $Bu=B\chi u+B(1-\chi)u\in C^\infty(]-\delta, \delta[\times M)$ and hence $(x_0, \xi)\notin\partial WF(u)$.

REMARK 5.2. It is worth noting that as a consequence of (5.38) and of Theorem 3.1 [6] the singularities of the Cauchy data g_j, $0\le j\le m-k-1$, propagate at least along a half bicharacteristic of $\tau-\lambda_i(t, x, \xi)$, for some $i=1, \dots, m$.

What we propose to do in the sequel is to give sufficient conditions which ensure propagation along a whole bicharacteristic of $\tau-\lambda_i(t, x, \xi)$ and (or) branching of singularities along at least two bicharacteristics related to two different factors $\tau-\lambda_i(t, x, \xi)$.

As the proof deeply relies on Theorem 3.1 of [6], we prepare some notation.

Suppose we are given a differential operator $P\in F^m_{m-k}(\boldsymbol{R}\times M)$. For convenience, we assume P given by (1.1) with ∂_t replaced by $D_t=\frac{1}{\sqrt{-1}}\partial_t$ so that the roots of the equation $\sigma_m(P_m)(t, x, \tau, \xi)$ are given by $\tau=\lambda_j(t, x, \xi)$, $j=1, \dots, m$.

For every $(x, \xi)\in S^*M$ we put $\rho_j(x, \xi)=(0, x, \lambda_j(0, x, \xi), \xi)$ and denote by $\gamma_j(x, \xi)$ the bicharacteristic of $\tau-\lambda_j(t, x, \xi)$ issued from $\rho_j(x, \xi)$. We also put $\gamma_j^{\pm}(x, \xi)=\gamma_j(x, \xi)\cap\{\pm t>0\}$. We now define the microlocal polynomials:

(5.40) $$I_j(x, \xi\,;\zeta)=(D_\tau\sigma_m(P_m))(\rho_j(x, \xi))\zeta+\sigma_{m-1}(P_{m-1})(\rho_j(x, \xi)),\quad j=1, \dots, m,\ \zeta\in\boldsymbol{C}.$$

We have the following result on branching of singularities.

THEOREM 5.5. *Let $P(t, x, D_t, D_x)\in F^m_{m-k}(\boldsymbol{R}\times M)$ and let $u\in\mathscr{D}'_r(\boldsymbol{R}\times M)$ be such that $Pu=f\in C^\infty(\boldsymbol{R}\times M)$ and $\partial_t^j u|_{t=0}=g_j\in\mathscr{D}'(M)$, $j=0, \dots, m-k-1$.*

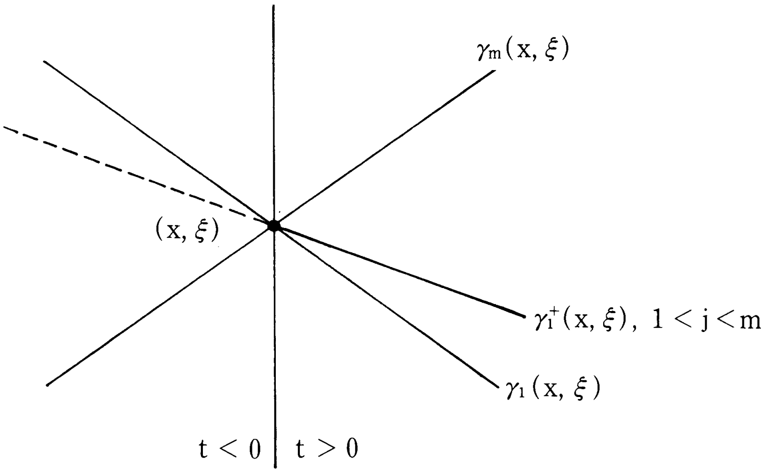
Let $(x, \xi)\in S^*M$ and suppose $\lambda_1(0, x, \xi)<\lambda_2(0, x, \xi)<\dots<\lambda_m(0, x, \xi)$. Then:

1. *Suppose that $\rho_1(x, \xi)\in WF(u)$ (resp. $\rho_m(x, \xi)\in WF(u)$) and $I_1(x, \xi\,;\zeta)\ne0$, $\forall\zeta\in\boldsymbol{Z}, \zeta\ge-(k-1)$ (resp. $I_m(x, \xi\,;\zeta)\ne0$, $\forall\zeta\in\boldsymbol{Z}$, $\zeta\ge-(k-1)$).*

Furthermore, suppose that for every j, $1<j<m$, either $\gamma_j^+(x, \xi)\cap WF(u)=\phi$ or $\gamma_j^-(x, \xi)\cap WF(u)=\phi$.

Then we have:

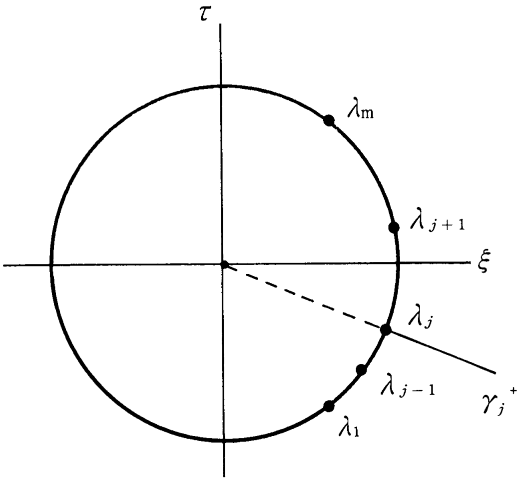
(5.41) $$\gamma_1(x, \xi)\cup\gamma_m(x, \xi)\subset WF(u).$$

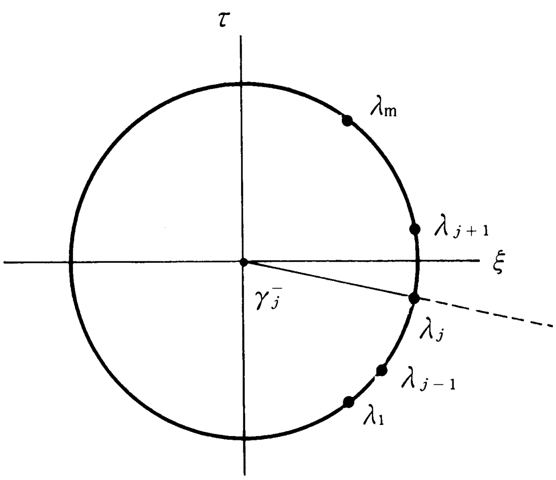


2. *In the case $m>2$, and $k=1$, let $\rho_j(x, \xi)\in WF(u)$ for some j, $1<j<m$.*

Then:

α) If $I_j(x, \xi; \zeta)\neq 0$, $\forall\zeta\in \boldsymbol{Z}$, $\zeta\leq -1$, then either $\gamma_j^+(x, \xi)$ or $\gamma_j^-(x, \xi)$ is contained in $WF(u)$. Moreover, if only one half of the bicharacteristic $\gamma_j(x, \xi)$ is included in $WF(u)$ then both $\rho_{j-1}(x, \xi)$ and $\rho_{j+1}(x, \xi)$ belong to $WF(u)$.





β) If $I_j(x, \xi; \zeta)\neq 0$, $\forall\zeta\in\boldsymbol{Z}$, then either $\gamma_j^+(x, \xi)$ or $\gamma_j^-(x, \xi)$ is inchluded in $WF(u)$ and either $\rho_{j-1}(x, \xi)$ or $\rho_{j+1}(x, \xi)$ belongs to $WF(u)$.

Proof of point 1. To be definite suppose $\rho_1(x, \xi)\in WF(u)$. By Theorem 5.4 we know that $(0, x, \pm 1, 0)\notin WF(u)$ so that by well known results on propagation we have $(0, x, \tau, \xi)\notin WF(u)$ for all $\tau<\lambda_1(0, x, \xi)$

and all $\tau > \lambda_m(0, x, \xi)$. By the hypothesis on $I_1(x, \xi; \zeta)$ and the fact that $\rho_1(x, \xi) \in WF(u)$, applying Theorem 3.1 of [6], we obtain that $\rho_2(x, \xi) \in WF(u)$. Since either $\gamma_2^+(x, \xi) \cap WF(u) = \phi$ or $\gamma_2^-(x, \xi) \cap WF(u) = \phi$, by the same Theorem we get $\rho_3(x, \xi) \in WF(u)$. Proceeding in this way we obtain that $\rho_m(x, \xi) \in WF(u)$ so that (5.34) follows by the same quoted Theorem.

To prove point 2. we simply apply Theorem 3.1 of [6] (with $r = k = 1$).

REMARK 5.3. In Theorem 5.5 the hypothesis $f \in C^\infty(\boldsymbol{R} \times M)$ can be replaced, for a fixed $x \in M$, by the condition $WF(f) \cap \pi^{-1}(0, x) = \phi$, $\pi : T^*(\boldsymbol{R} \times M) \backslash 0 \to \boldsymbol{R} \times M$ being the projection onto the base.

Statements 1. and 2. hold provided we replace everywhere the bicharacteristics by small arcs of them.

Examples

1. The Euler-Poisson-Darboux operator.

Let $P = t(D_t^2 - \sum_{j=1}^n D_{x_j}^2) + \alpha(t, x) D_t + \sum_{j=1}^n \beta_j(t, x) D_{x_j} + \gamma(t, x)$, $t \in \boldsymbol{R}$, $x \in \boldsymbol{R}^n$, with α, β_j and γ smooth functions.

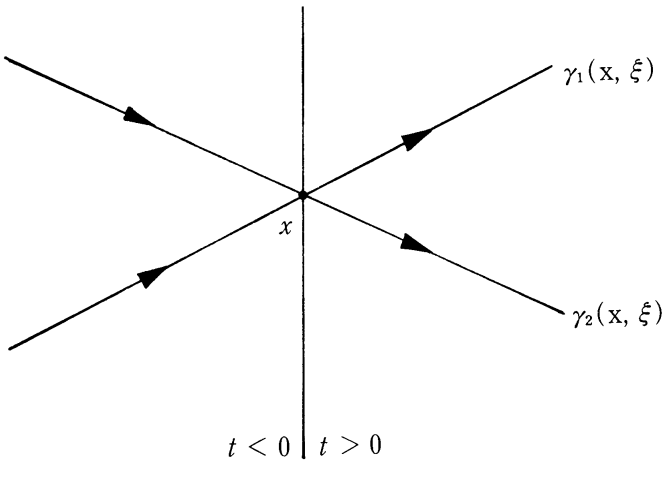
Suppose that $u \in \mathscr{D}'_T(\boldsymbol{R} \times \boldsymbol{R}^n)$ satisfies the Cauchy problem $Pu = 0$, $u|_{t=0} = g \in \mathscr{D}'(\boldsymbol{R}^n)$.

The hyperbolic roots are $\lambda_1(\xi) = -|\xi|$, $\lambda_2(\xi) = |\xi|$ and $I_1(x, \xi; \zeta) = 2i\zeta - \alpha(0, x) - \langle \beta(0, x), \xi/|\xi| \rangle$, $I_2(x, \xi; \zeta) = -I_1(x, -\xi; \zeta)$. As a consequence of Theorem 5.5 if $\forall \zeta \in \boldsymbol{Z}_+ I_1(x, \xi; \zeta) \neq 0$ and $I_2(x, \xi; \zeta) \neq 0$ $\forall (x, \xi) \in WF(g)$, we obtain:

$$WF(u) \supset \bigcup_{(x, \xi) \in WF(g)} (\gamma_1(x, \xi) \cup \gamma_2(x, \xi)).$$

On the other hand, supposing in addition that $I_p(x; \zeta) = \zeta + i\alpha(0, x) \neq 0$, $\forall \zeta \in \boldsymbol{R}$ and $\forall x$, from Theorems 5.2, 5.4 we obtain:

$$WF(u)|_{\{t \neq 0\}} = \bigcup_{(x, \xi) \in WF(g)} (\gamma_1(x, \xi) \cup \gamma_2(x, \xi))|_{\{t \neq 0\}}$$

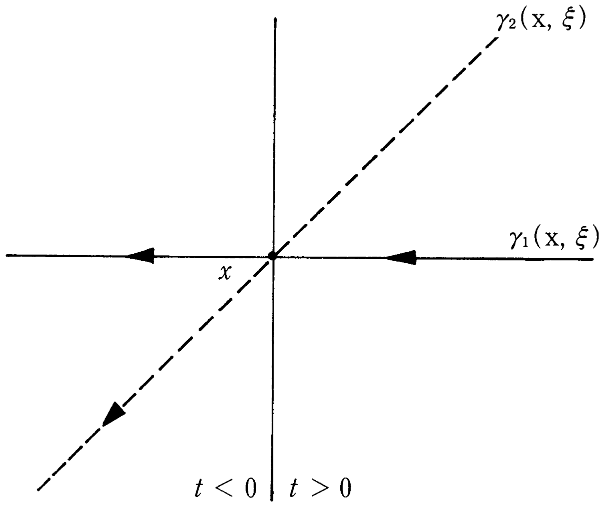


2. Let $P=tD_t(D_t-D_x)+\alpha(t, x)D_t$, $t\in\boldsymbol{R}$, $x\in\boldsymbol{R}$, with α a smooth function.

Suppose that $u\in\mathscr{D}'_r(\boldsymbol{R}\times\boldsymbol{R})$ satisfies the Cauchy problem $Pu=0$, $u|_{t=1}=g\in\mathscr{D}'(\boldsymbol{R})$. Supposing $I_p(x;\zeta)=\zeta+i\alpha(0, x)\neq 0$, $\forall\zeta\in\boldsymbol{Z}_+$ and $\forall x$, we have $u(t, x)=1_t\otimes g(x)$ so that:

$$WF(u)=\{(t, x, 0, \xi)\,|\,(x, \xi)\in WF(g)\}.$$

The hyperbolic roots are $\lambda_1(\xi)=0$, $\lambda_2(\xi)=\xi$ and $I_1(x, \xi;\zeta)\equiv 0$.



Acknowledgements. J. E. L. thanks the Italian Research Council for supporting his stay at the University of Bologna during the preparation of this work. C. P. has been partially supported by a NATO collaborative research grant, n. 215-81. A. B. and C. P. were supported by the C. N. R. gruppo G. N. A. F. A.

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