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Stably solitary foliations

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§ 1. Introduction and the statement of results.

Let ϕ be the 1-dimensional foliation of S^3 determined by the Hopf fibration $p: S^3 \rightarrow S^2$. As is easily verified, ϕ admits no transverse foliations of codimension one. But there exists a small perturbation ϕ' of ϕ which is transverse to the Reeb foliation of S^3. Thus it is in general unstable under the small perturbation that a foliation admits no transverse foliations.

For codimension one foliations of 3-manifolds, there are various investigations of the transverse foliations ([9], [5], [6], [8] and [7]). In [8], Tamura showed that for every closed oriented 3-manifold, there exists a codimension one foliation which admits transverse 2-plane fields and admits no transverse foliations of codimension one. We say such a foliation *solitary*. In this paper we consider the foliations which satisfy the solitarity stably under the perturbation. More precisely we prove

THEOREM 1. *Let M be a closed oriented 3-manifold and let* $\mathrm{Fol}^2(M)$ *denote the space of codimension one C^2 foliations of M with the C^2-topology (see* [2] *and* [1]). *Then there exist a foliation $\mathscr{F}$ and a neighborhood $\mathfrak{N}$ of $\mathscr{F}$ in* $\mathrm{Fol}^2(M)$ *such that every element $\mathscr{F}' \in \mathfrak{N}$ is solitary.*

REMARK 1. For every closed oriented 3-manifold and for every homotopy class τ, Yano ([11]) constructed a 1-dimensional foliation consisting in τ which admits no transverse foliations stably under the perturbation.

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§ 2. Proof of Theorem 1.

Theorem 1 is proved from the following two theorems.

THEOREM 2 (Tamura [8]). *Let $\mathscr{F}^{(n)}$ be the codimension one C^∞ foliation of the solid torus $S^1 \times D^2$ as below. Then for $n \neq 0$, $\mathscr{F}^{(n)}$ is solitary.*

THEOREM 3. *Let $\mathscr{F}^{(n)}$ be the foliation of $S^1 \times D^2$ as below and let $K = (S^1 \times D^2)'$ be the compact codimension 0 submanifold of $S^1 \times D^2$ as below. Then there exists a neighborhood $\mathfrak{N}$ of $\mathscr{F}^{(n)}$ in* $\mathrm{Fol}^2(S^1 \times D^2)$ *such that for every element $\mathscr{F}' \in \mathfrak{N}$, there is a compact codimension 0 saturated submanifold*

K' such that the restriction $\mathscr{F}'|K'$ is C^0 isotopic to $\mathscr{F}|K$.

THE CONSTRUCTION OF $\mathscr{F}^{(n)}$.

Let $h_n: S^1\times D^2\to S^1\times D^2$ be the C^∞ diffeomorphism defined by

$$h_n(e^{2\pi ix},\ re^{2\pi iy})=(e^{2\pi ix},\ re^{2\pi i(y+nx)}),\ 0\leq x\leq 1,\ 0\leq y\leq 1,\ 0\leq r\leq 1.$$

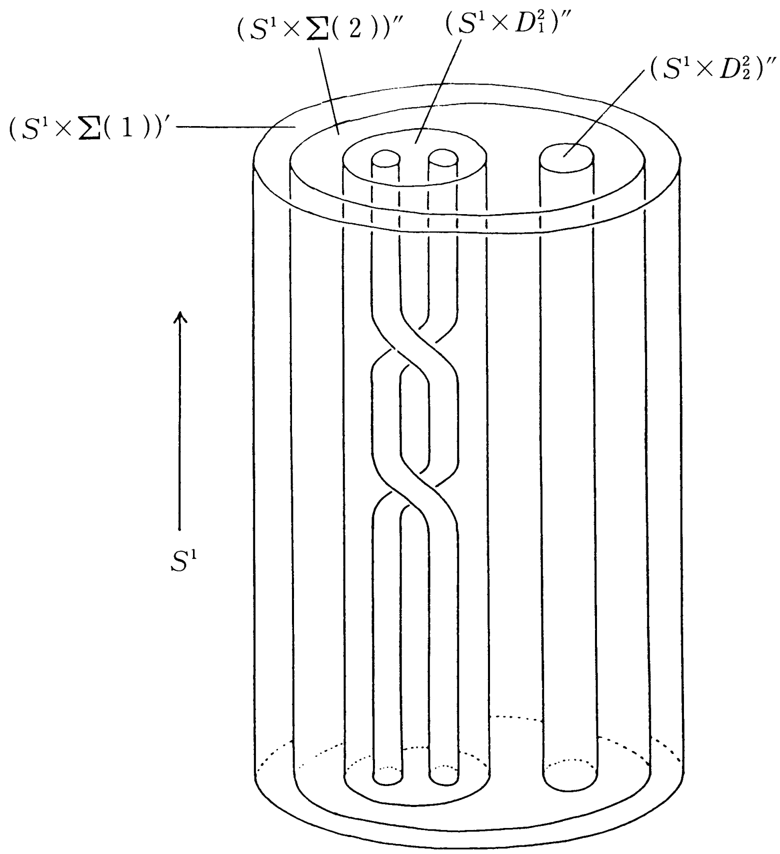
Let $\Sigma(k)=D^2-\bigcup_{i=1}^{k}\text{Int}\ D_i^2$ be the k-punctured 2-disc.

We have a decomposition of the solid torus as follows:

$$S^1\times D^2=h_n(S^1\times\Sigma(2))\cup h_n(S^1\times D_1^2)\cup h_n(S^1\times D_2^2).$$

Let $\mathscr{F}_{\pi,k}$ be the C^∞ foliation of $S^1\times\Sigma(k)$ obtained by the turbulization of the product foliation $\{\{t\}\times\Sigma(k)\,;\,t\in S^1\}$ along the boundary $S^1\times\partial\Sigma(k)$. Let $\mathscr{F}_0^{(n)}=\{h_n(L)\,;\,L\in\mathscr{F}_{\pi,2}\}$. Then the union of $\mathscr{F}_0^{(n)}$ and two Reeb components $\mathscr{F}_R$ of $h_n(S^1\times D_1^2)$ and $h_n(S^1\times D_2^2)$ determines a codimension one C^∞ foliation of $S^1\times D^2$, which is denoted by $\mathscr{F}_1^{(n)}$. Consider the decomposition of $S^1\times D^2$:

$$S^1\times D^2=(S^1\times\Sigma(1))'\cup(S^1\times D^2)',\ \text{where}$$
$$(S^1\times D^2)'=(S^1\times\Sigma(2))''\cup(S^1\times D_1^2)''\cup(S^1\times D_2^2)''\text{(see Figure 1).}$$



$\mathscr{F}^{(n)}$ for $n=1$

Figure 1.

Let $\mathscr{F}^{(n)}$ be the foliation of $S^1\times D^2$ consisting of foliations $\mathscr{F}_{\pi,1}$ of $(S^1\times\Sigma(1))'$, $\mathscr{F}_{\pi,2}$ of $(S^1\times\Sigma(2))''$, $\mathscr{F}_1^{(n)}$ of $(S^1\times D_1^2)''$ as above, and the Reeb component $\mathscr{F}_R$ of $(S^1\times D_2^2)''$. Furthermore we can impose the following conditions:

(a) Each compact leaf of $\mathscr{F}^{(n)}$ has an expanding holonomy along the leaf loop which is homotopic to the core circle $S^1\times\{0\}$ of $S^1\times D^2$.

(b) For the compact leaf $\partial(S^1\times D^2)$, the above holonomy map is C^∞-tangent to the identity.

(c) For the other compact leaves, the above holonomy maps are hyperbolic.

PROOF OF TEOREM 1 FROM THEOREMS 2 AND 3.

For each closed oriented 3-manifold M, take a transversely oriented C^∞ foliation $\mathscr{F}_0$ of codimension one with a Reeb component and with a transverse 2-plane field, and replace the Reeb component by the foliation $\mathscr{F}^{(n)}$ of Theorem 2. Since the perturbation keeps the topological type of $\mathscr{F}^{(n)}|K$ by Theorem 3, the perturbed foliation $\mathscr{F}'$ is also solitary again by Theorem 2.

In the rest of this section we prove Theorem 3. For it, we use the following theorem due to Hirsch([2])(For our purpose, minor modifications are carried out).

THEOREM 4(Theorem 1.1 of [2]).

Let M be a closed 3-manifold and let $\mathrm{Fol}^2(M)$ be the space of C^2 foliations of M of codimension one with the C^2 topology (defined in [2]). Let L be a toral leaf of a codimension one foliation $\mathscr{F}$ and let $\alpha\in\pi_1(L, x_0)$ for some base point $x_0\in L$. Assume the linear holonomy along α is non-trivial (and thus hyperbolic). Then there exists $\varepsilon_0>0$ with the following properties :

If $0<\varepsilon<\varepsilon_0$, then there exists a neighborhood $\mathfrak{N}\subset\mathrm{Fol}^2(M)$ of $\mathscr{F}$ such that for every $\mathscr{F}'\in\mathfrak{N}$, there are a compact leaf $L'\in\mathscr{F}'$ and a C^2-diffeomorphism $h: L\to L'$ satisfying $d(x, h(x))<\varepsilon$, where d is the induced distance by a Riemannian metric on M. Moreover L' is unique.

PROOF OF THEOREM 3.

STEP 1. We will prove the small perturbation keeps the topological type of the Reeb component $\mathscr{F}^{(n)}|(S^1\times D_2^2)''$. Let L denote the compact leaf $\partial(S^1\times D_2^2)''$ and take a small collar neighborhood $C: L\times[-\varepsilon, \varepsilon]\to S^1\times D^2$ of L such that $C(L\times\{0\})=L$, $C(L\times[0, \varepsilon])\subset(S^1\times D_2^2)''$, and $C(L\times\{t\})$ is transverse to $\mathscr{F}^{(n)}$ for each $t\in[-\varepsilon, \varepsilon]-\{0\}$.

Let $N=(S^1\times D_2^2)''-C(L\times[0, \varepsilon])$ and let $P: S^1\times D^2\to N$ be a product structure such that (i) for each $t\in S^1$, $P(\{t\}\times D^2)$ is a leaf of $\mathscr{F}^{(n)}|N$ and (ii) for each $x\in D^2$, $P(S^1\times\{x\})$ is transverse to $\mathscr{F}^{(n)}|N$.

Let $\mathscr{F}'$ be a small C^2 perturbation of $\mathscr{F}^{(n)}$. By Theorem 4, there exists a compact leaf L' of $\mathscr{F}'$ in $C(L\times[-\varepsilon, \varepsilon])$ near L. Let α' be a loop in L'

homotopic to the core circle of $S^1\times D^2$. Then the linear holonomy along α' is also non-trivial. Consider $\mathscr{F}'|N$. We may assume $C(L\times\{\varepsilon\})$ and the circles $P(S^1\times\{x\})$, $x\in D^2$, are transverse to $\mathscr{F}'$.

Then we see $\mathscr{F}'|N$ is a product foliation by 2-discs. From this $\mathscr{F}'|\partial N$ is a trivial foliation by circles. This implies the holonomy group of L' is of rank 1 and thus $\mathscr{F}'|T$ is a Reeb component with hyperbolic holonomy, where T denotes the closure of the connected component of $S^1\times D^2-L'$ containing N. A C^0 isotopy from T to $(S^1\times D_2^2)''$ is easily constructed. We see also $\mathscr{F}'|(S^1\times D_1^2)''$ has two Reeb components near $(h_n(S^1\times D_1^2))''$ and $(h_n(S^1\times D_2^2))''$.

STEP 2. We will prove the perturbation keeps the topological type of $\mathscr{F}^{(n)}|(h_n(S^1\times\Sigma(2))''$. Let $\mathscr{F}'$ denote a small perturbation of $\mathscr{F}^{(n)}$. Let $L_1=\partial(h_n(S^1\times D_1^2))''$ and let $L_2=\partial(h_n(S^1\times D_2^2))''$. Let L_i' be the compact leaf of $\mathscr{F}'$ corresponding to $L_i(i=1, 2)$ and let T_i' be the solid torus bounded by L_i'. By a lemma of Kopell([2]), for each non-compact leaf L' of $\mathscr{F}'$ outside both T_1' and T_2' and spiraling to L_1' and L_2', the intersection of L' and a collar of L_i' is a circle. Thus by the bundle structure argument as in Step 1, we see L' is diffeomorphic to the open 2-punctured 2-disc. This shows Step 2.

By a similar argument as above, we see that for $\mathscr{F}'$ there exists a compact codimension 0 submanifold K' near $K=(S^1\times D^2)'$ such that $\mathscr{F}'|K'$ is C^0 isotopic to $\mathscr{F}^{(n)}|K$. This completes the proof of Theorem 3.

REMARK 2. For a transverse pair $(\mathscr{F}, \mathscr{G})$ of codimension one foliations of a 3-manifold, we say $(\mathscr{F}, \mathscr{G})$ *unraisable to a total foliation* if there exists no codimension one foliation $\mathscr{H}$ such that $(\mathscr{F}, \mathscr{G}, \mathscr{H})$ forms a total foliation. Every closed oriented 3-manifold admits such a pair(see [7]). For such a pair, we can consider the stability as in the case of the solitary foliations. For example, let M be the total space of the S^1-bundle over the 2-torus $T^2=\mathbb{R}^2/\mathbb{Z}^2$ with non-trivial Euler class, and let $\pi: M\rightarrow T^2$ be the projection. Let $(\bar{\mathscr{F}}, \bar{\mathscr{G}})$ be the transverse pair of codimension one foliations of T^2 such that each leaf of $\bar{\mathscr{F}}$ (resp. $\bar{\mathscr{G}}$) is a circle parallel to the x-axis (resp. the y-axis) of T^2. Then the induced pair $(\pi^*\bar{\mathscr{F}}\ \pi^*\bar{\mathscr{G}})$ is unraisable to a total foliation(by Milnor [4] and Wood [10]). We can verify this pair does not satisfy the stability in the sense above. We now pose the following

PROBLEM. Find a stable unraisable pair.

REFERENCES

[1] D. B. A. EPSTEIN, A topology for the space of foliations, Lecture Notes in Math. 597, Geometry and Topology, Rio de Janeiro 1976, Springer Verlag, 132–150.

[2] M. W. HIRSCH, Stability of compact leaves of foliations, Dynamical systems (edited

by M. Peixoto), Academic Press, New York, 1973, 135–153.

[3] N. KOPELL, Commuting diffeomorphisms, Global analysis, Proc. Symp. in pure Math., vol. 14, A. M. S., 1970, 165–184.

[4] J. MILNOR, On the existence of a connection with zero curvature, Comm. Math. Helv., 32 (1958), 215–223.

[5] T. NISHIMORI, Existence problem of transverse foliations for some foliated 3-manifolds, Tohoku Math. J., 34 (1982), 179–238.

[6] T. NISHIMORI, Foliations transverse to the turbulized foliations of punctured torus bundles over a circle, Hokkaido Math. J., 13(1984), 1–25.

[7] A. SATO, Every 3-manifold admits a transverse pair of codimension one foliations which cannot be raised to a total foliation, to appear in Foliations (edited by I. Tamura), Advanced Studies in Pure Mathematics, vol. 5, N.-Holland/ Kinokuniya, 1984.

[8] I. TAMURA, Dynamical Systems on Foliations and Existence Problem of Transverse Foliations, to appear in ibid..

[9] I. TAMURA and A. SATO, On transverse foliations, Publ. Math. I. H. E. S., 54 (1981), 5–35.

[10] J. WOOD, Bundles with totally disconnected structure group, Comm. Math. Helv., 46 (1971), 257–273.

[11] K. YANO, Non-singular Morse-Smale flows on 3-manifolds which admit transverse foliations, to appear in Foliations (edited by I. Tamura).

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