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Some properties of interior G-algebras

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1. Introduction

Let G be a finite group and $R[G]$ be a group ring over a commutative ring R with the identity element 1_R. This group ring is one of the main objects in the representation theory of finite groups, and it is interesting to study two sided ideals of $R[G]$, for example, Jacobson radical is an important object in modular representation theory. In the case that R is a field of characteristic zero, $R[G]$ is a semi-simple ring by Maschke's theorem, and Wedderburn's theorem implies that any two sided ideal of $R[G]$ is a direct product of minimal two sided ideals of $R[G]$ with multiplicity one. But in the case that R is a field of characteristic p $(p>0)$ or, more generally, R is a p-adic complete discrete valuation ring with a maximal ideal (π), it is a complicated problem to determine two sided ideals of $R[G]$. Throughout this paper, the commutative ring R is a splitting field of characteristic p for G or a complete discrete valuation ring satisfying the residue field $k=R/(\pi)$ is a splitting field of characteristic p for G, and all ideals of $R[G]$ means that R-free two sided ideals of $R[G]$.

For an ideal I of $R[G]$ we set A_I the quotient ring $R[G]/I$. Then let ρ_I be the canonical map of $R[G]$ to A_I, then the map ρ_I is an epimorphism as R-algebra. So for a finite R-rank R-algebra A (we call simply it an R-algebra) and an epimorphism ρ of $R[G]$ to A the pair (A, ρ) is the dual case of two sided ideals of $R[G]$, and we call the pair (A, ρ) an epimorphic interior G-algebra. See [9]. In order to study ideals I of $R[G]$ or interior G-algebras (A, ρ) the R-module I or A becomes an $R[G\times G]$-module as following, respectively;

$(g, h)x=gxh^{-1}$, where x in an element of I and (g, h) is an element of the direct product $G\times G$.

$(g, h)a=\rho(g)a\rho(h^{-1})$, where a is an element of A and (g, h) is an element of $G\times G$.

By corollary 2 in [9], for an epimorphic interior G-algebra (A, ρ) the $R[G\times G]$-module A is indecomposable if and only if the center $Z(A)$ of the R-algebra A is a local ring. In this case, we call (A, ρ) an epimorphic local interior G-algebra.

On the other hand, in [8], J. A. Green studied Brauer's definition of defect groups of blocks in block theory, and defined the concept of G-algebras and its defect groups. After, M. Broué and L. Puig defined interior G-algebras, which is a special case of G-algebras, and its defect groups. See [3], [8] and [14]. Our epimorphic interior G-algebras are naturally interior G-algebras, and so we can define defect groups for an epimorphic local interior G-algebra.

As stated above, a defect group is an invariant of an epimorphic local interior G-algebra, but this group is too large subgroup of G. See section 2. In this paper, we define another invariant p-subgroup, which is a normal subgroup of a defect group, for an epimorphic interior G-algebra. We shall study some properties of this p-subgroup.

In section 2, we shall define epimorphic local interior G-algebras and defect groups, and present some properties of them. In section 3, we shall define a normal subgroup $(vtx_{G\times G}A)_1$ of a defect group of an epimorphic local interior G-algebra and give its elementary properties. In section 4, we shall give the relation the subgroup $(vtx_{G\times G}A)_1$ and a factor group of G. In section 5, we shall give another proof of Michler's theorem for embedding of group ring of defect group in block theory. In section 6, we shall present some finiteness condition of epimorphic local interior G-algebra using $(vtx_{G\times G}A)_1$. The theory of epimorphic local interior G-algebra is simple (but essential) in the case that G is a p-group. So we shall give some examples and facts in this case, in this case, in section 7.

Notation. Let A, B and C be sets and f and g maps of A to B and of B to C, respectively. We denote by $g\circ f$ the composition of f and g. For a subset D of A we denote by $f|_D$ the restriction. In this paper, all $R[G]$-modules are R-free left $R[G]$-modules with finite R-ranks. All R-algebras are assumed to have the identity element and be R-free modules with finite ranks. Whenever H is a subgroup of G, V is an $R[G]$-module and W is an $R[H]$-module, the $R[H]$-module $V\downarrow_H$ is a restricted module and the $R[G]$-module $W\uparrow^G$ is an induced module.

2. Epimorphic interior G-algebras

We give definitions of an interior G-algebra and an epimorphic interior G-algebra.

Definition 2.1. *For R-algebra A with the identity element 1_A and R-algebra homomorphism ρ of $R[G]$ to A such that $\rho(1)=1_A$, the pair (A,*

ρ) is called an interior G-algebra. Furthermore if R-algebra homomorphism ρ is epimorphic, we call (A, ρ) an epimorphic interior G-algebra.

For an interior G-algebra (A, ρ) the R-algebra A becomes a $R[G\times G]$ -module as stated in section 1.

Let (A, ρ) be an interior G-algebra and H be a subgroup of G. A subring A^H of A consists of the fixed points of A under the conjugate action of H. Whenever H' is a subgroup of G such that $H\leq H'$, the trace map $Tr_H^{H'}: A^H \longrightarrow A^{H'}$ is defined by $a\longrightarrow \sum\rho(g)a\rho(g^{-1})$, where g runs over a complete set of representatives in H' of H'/H. We denote by $A_H^{H'}$ the image $Tr_H^{H'}(A^H)$. Then $A_H^{H'}$ is a two sided ideal in $A^{H'}$. If A^G is a local ring, we call the interior G-algebra (A, ρ) a local interior G-algebra. For a local interior G-algebra (A, ρ) a defect qroup of (A, ρ) is the minimal subgroup of the family of subgroups H satisfying $A_H^G=A^G$. The defect group is determined uniquely up to conjugation in G and is a p-subgroup of G.

DEFINITION 2.2. *An epimorphic interior G-algebra (A, ρ) is an epimorphic local interior G-algebra if the subring $A^G=Z(A)$ is local.*

LEMMA 2.3. ([9] *Corollary* 2) *An epimorphic interior G-algebra (A, ρ) is an epimorphic local interior G-algĕbra if and only if an $R[G\times G]$ -module A is an indecomposable $R[G\times G]$-module.*

Let $End_R(V)$ be the R-endomorphism ring of V and ρ_V the representation of $R[G]$ introduced by V. Then the pair $(End_R(V), \rho_V)$ is an interior G-algebra. If the $R[G]$-module V is indecomposable if and only if the corresponding interior G-algebra $(End_R(V), \rho_V)$ is local. The $R[G]$ -module V is H-projective for a subgroup H of G, it the $R[G]$-module V is isomorphic to the direct summand of the $R[G]$-module $V\downarrow_H\uparrow^G$. By Higman's criteria for relative projectivity, V is H-projective if and only if $End_R(V)^G$ equals to $End_R(V)_H^G$. For an indecomposable $R[G]$-module V the minimal subgroup of the family of H satisfying V is H-projective is called the vertex of V, and denoted by $vtx_G V$. The defect group of the above local interior G-algebra $(End_R(V), \rho_V)$ equals the vertex $vtx_G V$.

We present some examples of epimorphic local interior G-algebras.

EXAMPLE 2.4. *In case that $R=k$, for a simple $k[G]$-module V, its k-endomorphism ring $End_k(V)$ and the representation ρ_V of $k[G]$ introduced by V the pair $(End_k(V), \rho_V)$ is an epimorphic local interior G-algebra. Then the defect group D of this interior G-algebra is the vertex of the simple module V.*

EXAMKLE 2.5. *Let e be an central primitive idempotent of $R[G]$, and $B=R[G]e$ be the corresponding block of $R[G]$. Then (B, ρ_B) becomes an epimorphic local interior G-algebra by the R-algebra homomorphism $\rho_B: x \longrightarrow xe$ for the element x of $R[G]$. We call this interior G-algebra a block interior G-algebra. The defect group of this interior G-algebra equals the defect group of B as classical case.*

We define morphisms between interior G-algebras.

DEFINITION 2.6. *Let (A, ρ) and (A', ρ') be interior G-algebras. The R-algebra homomorphism φ such that $\varphi(1_A)=1_A$, is a morphism of (A, ρ) to (A', ρ'), if $\rho'=\varphi\circ\rho$ is hold. If the above homomorphism is an isomorphism, then we call (A, ρ) and (A', ρ') is isomorphic by φ.*

A local interior G-algebra (A, ρ) belongs to a block $B=R[G]e$, if there exists a morphism of the block interior G-algebra to the interior G-algebra (A, ρ). This means that the image $\rho(B)$ is non-zero.

Let (A, ρ) be an epimorphic local interior G-algebra belonging a block B and $I=\{x\in R[G]\,; \rho(x)=0\}$ the kernel of ρ. Then we obtain an epimorphic interior G-algebra $(R[G]/I, \rho_I)$, where ρ_I is the canonical R-algebra homomorphism of $R[G]$ to $R[G]/I$, which is isomorphic to the interior G-algebra (A, ρ). Furthermore the interior G-algebra $(B/I\cap B, (\rho_I)|_B)$, where $(\rho_I)|_B$ is the restriction, is isomorphic to (A, ρ). Conversely let I be a two sided ideal of a block algebra B. Then the interior G-algebra $(B/I, \rho_I)$, where ρ_I is the restriction the canonical R-algebra homomorphism to B, is an epimorphic interior G-algebra. For a primitive idempotent f of the subring $(B/I)^G$ we set a R-algebra epimorphism ρ_f of $R[G]$ to the R-algebra $(B/I)f$ defined by $x\longmapsto\rho_I(x)f$, and have an epimorphic local interior G-algebra $((B/I)_f, \rho_f)$ belonging to the block B.

Let $id_{R[G]}$ be the identity map. Then the pair $(R[G], id_{R[G]})$ is an interior G-algebra, and $R[G]$ becomes a $R[G\times G]$-module. Since this $R[G\times G]$-module is a permutation module the dual module $R[G]^*$ is isomorphic to $R[G]$. Therefore for an epimorpic interior G-algebra (A, ρ) the $R[G\times G]$-module A is isomorphic to some two sided ideal I^* of $R[G]$. Therefore as $R[G\times G]$-module, the discussion of epimorphic interior G-algebras is equivalent to that of two sided ideals of $R[G]$.

Let (A, ρ) be an epimorphic interior G-algebra, $\bar{A}=A/(\pi)$ and $\bar{\rho}$ a k-algebra epimorphism of $k[G]$ to A introduced by ρ. Since (A, ρ) is an epimorphic interior G-algebra the subring A^G is the center of A, and $(\bar{A}, \bar{\rho})$ is local if and only if so is (A, ρ). Furthermore the defect qroup of $(\bar{A}, \bar{\rho})$

equals the defect group of (A, ρ). See [7] Ch. 1 lemma 17, 4.

LEMMA 2.7. *Let (A, ρ) and (A', ρ') be local interior G-algebras, and assume that there exists a morphism of (A, ρ) to (A', ρ'). Then the defect group of (A, ρ) contains the defect group of (A', ρ') up to G-conjugate.*

PROOF. Let φ be a morphism of (A, ρ) to (A', ρ'). Since the image $\varphi(A_H^G)$ is contained in A'^G_H for a subgroup of G the lemma is immediately from the definition.

LEMMA 2.8. *Let (A, ρ) be an epimorphic local interior G-algebra belonging to a block B with defect group D. Then there exists a simple $k[G]$ -module V such that the defect group of (A, ρ) contains the vertex of V and is contained in D up to G-conjugate.*

PROOF. We may prove this in the case $R=k$ equals a field of characteristic p. Since (A, ρ) belongs to B the defect group of (A, ρ) is contained in D by lemma 2.7. Since (A, ρ) be an epimorphic local interior G-algebra there exist a two sided ideal I of B such that (A, ρ) is isomorphic to $(B/I, \rho_I)$. On the other hand, for the maximal ideal I' containing I the canonical homomorphism of B/I to B/I' introduces a morphism of (A, ρ) to $(B/I', \rho_{I'})$. By Wedderburn's structure theorem, there exists a simple k $[G]$-module V such that the interior G-algebra $(End_k(V), \rho_V)$ is isomorphic to the interior G-algebra $(B/I', \rho_{I'})$. Thus we obtain a morphism of (A, ρ) to $(End_k(V), \rho_V)$. Therefore the defect group of (A, ρ) contains the defect group of $(End_k(V), \rho_V)$, which equals the vertex of V.

THEOREM 2.9. *Let B be a block of G with defect group D. Then all epimorphic local interior G-algebras belonging to B have defect group D if and noly if all simple $k[G]$-modules belonging to B have vertex D. In particular, if the defect group D of the block B is an abelian group, then all epimorphic local interior G-algebras have defect groups D.*

PROOF The first statement of theorem is immediately from lemma 2.8. The second statement is implies from Knörr's theorem in [10].

COROLLARY 2.10. *Let (A, ρ) be an epimorphic local interior G-algebra and P a normal p-subgroup of G. Then P is contained in the defect group of (A, ρ).*

PROOF. Since vertices of any simple $k[G]$-modules contains P lemma 2.8. implies corollary.

3. The invariant $(\mathbf{vtx}_{G\times G}\mathbf{A})_1$.

In this section, we define a invariant $(vtx_{G\times G}A)_1$ of an epimorphic interior G-algebra (A, ρ), which is a normal subgroup of a defect group of (A, ρ).

For an epimorphic local interior G-algebra (A, ρ) lemma 2.2 implies that the $R[G\times G]$-module A is an indecomposable $R[G\times G]$-module. The following theorem is concerned with a vertex of the $R[G\times G]$-module A.

THEOREM 3.1. ([9] Theorem) *Let (A, ρ) be an eqimorphic local interior G-algebra with defect group D. Then the vertex $vtx_{G\times G}A$ contains $D^\Delta=\{(g, g)\in G\times G : g\in D\}$ and is contained in $D\times D$ up to $G\times G$-conjugate.*

By this theorem, there occurs the vertex $vtx_{G\times G}A$ contains D^Δ and is contained in $D\times D$, where D is a defect group of (A, ρ). From this fact, we can determine a normal subgroup of D as following. Let D be a subgroup of G and L a subgroup of $G\times G$ such that L contains D^Δ and is contained in $D\times D$. For the group L we define a subgroup L_1 by $L_1=\{g\in G : (g, 1)\in L\}$. Then L_1 is a normal subgroup of D. Then we have the following lemma.

LEMMA 3.2. *For a subgroup L of $G\times G$ such that L contains D^Δ and is contained in $D\times D$, the correspondence $L\longrightarrow L_1$ of the family of subgroups of $G\times G$ which contain D^Δ and is contained in $D\times D$ to the family of normal subgroups of D is bijective.*

PROOF. Let Q be a normal subgroup of D. We define a subgroup $Q_{-1}=\{(gh, g) : g\in Q, h\in D\}$ of $G\times G$. Then we easily check $(L_1)_{-1}=L$ and $(Q_{-1})_1=Q$, and obtain lemma.

The next theorem give another invariant p-subgroup of epimorphic interior G-algebras, and this is a main object in this paper.

THEOREM 3.3. *Let (A, ρ) be an epimorphic local interior G-algebra with fixed defect group D. Then there exists vertex $vtx_{G\times G}A$ which contains D^Δ and is contained in $D\times D$. Furthermore $(vtx_{G\times G}A)_1$ is a normal subgroup of D and is unique up to $N_G(D)$-conjugate.*

PROOF. By the definition of vertex, there exists $vtx_{G\times G}A$ containing D^Δ. By Theorem 3.1, there exists the elements g and h of G such that $vtx_{G\times G}A$ is contained in $D^g\times D^h$, and so D^Δ is contained in $D^g\times D^h$. Therefore $D=D^g$ and $D=D^h$. This fact and lemma 3.2 implies theorem.

The determined normal subgroup $(vtx_{G\times G}A)_1$ is seen an interesting object in order to study epimorphic local interior G-algebras or two sided ideals of a block algebra.

EXAMPLE 3.4. *For an epimorphic local interior G-algebra* $(End_k(V), \rho_V)$ *with defect group D in example* 2.4 *the subgroup* $(vtx_{G\times G}(End_k(V)))_1$ *equals D. For an epimorphic local interior G-algebra* (B, ρ_B) *with defect group D in example* 2.5 *the subgroup* $(vtx_{G\times G}B)_1$ *is* $\langle 1\rangle$.

The following theorem, in [9], give a characterization in the case that $(vtx_{G\times G}A)_1$ is $\langle 1\rangle$.

THEOREM 3.5. ([9] theorem) *Let* (A, ρ) *be an epimorphic local interior G-algebra with defect group D belonging to a block B. Then the following are equivalent.*

(1) *The normal subgroup* $(vtx_{G\times G}A)_1$ *of D equals* $\langle 1\rangle$.

(2) *The restriction* $A\downarrow_{G\times\langle 1\rangle}$ *is a projective* $R[G\times\langle 1\rangle]$*-module.*

(3) *The restriction* $\rho|_B : B\longrightarrow A$ *is an R-algebra isomorphism.*

Furthermome in this case, D equals the defect group of the block B.

PROOF. Since $(vtx_{G\times G}A)_1$ equals $\langle 1\rangle$ is equivalent that $vtx_{G\times G}A$ equals D^{Δ} the Mackey decomposition theorem implies (1) to (2). By lemma 4 in [9], we have (2) to (3). (3) to (1) from Theorem 8.9 ch. 3 [7].

This theorem gives similar result to R-free two sided ideals of $R[G]$, which is a dual case of local interior G-algebra. Then we obtain the following corollary.

COROLLARY 3.6. ([9] Corollary 5) *Let I be an R-free indecomposable two sided ideal of* $R[G]$. *If the vertex* $vtx_{G\times G}I$ *is contained in G, then I is a block ideal of* $R[G]$.

Next we study the relation between $(vtx_{G\times G}A)_1$ and some group determined by an epimorphic interior G-algebra (A, ρ). For (A, ρ) the R-algebra A becomes a left $R[G]$-module through ρ, and for an indecomposable direct summand V of this left $R[G]$-module A we have a vertex $vtx_G V$. On the other hand, K is a kernel of the left $R[G]$-module A. Then we have the following.

PROPOSITION 3.7. *Let* (A, ρ) *be an epimorphic interior G-algebra, and V and K as above. If P is a p-sylow subgroup of K, then* $(vtx_{G\times G}A)_1$ *contains* $vtx_G V$ *up to G-conjugate and P is contained in* $vtx_G V$ *up to G-conjugate. In particular P is contained in* $(vtx_{G\times G}A)_1$ *up to G-conjugate.*

PROOF. We can see left $R[G]$-module A the restriction $A\downarrow_{G\times\langle 1\rangle}$, where A is a $R[G\times G]$-module, and V is an indecomposable direct summand of $A\downarrow_{G\times\langle 1\rangle}$. The first statement from this. Let S be a p-Sylow subgroup of G containing P and W an indecomposable direct summand of $V\downarrow_S$ satisfying $vtx_G V$ equals $vtx_S W$. Because K is a kernel of $A\downarrow_{G\times\langle 1\rangle}$, the subgroup P is contained in the kernel of W. This implies that P is contained in $vtx_S W = vtx_G V$.

The above normal subgroup K appears in the theory of dimension subgroups in integral representation theory.

4. Reduction by normal subgroup.

In this section, we discuss the case of a reduction by a normal subgroup of G. Let N be a normal subgroup of G and G^0 a factor group G/N and for a subgroup of G containing N we use the same notation. Let (A, ρ^0) be an epimorphic interior G^0-algebra. Then we have an epimorphic interior G-algebra (A, ρ), where ρ is the composition of ρ^0 and the homomorphism of $R[G]$ to $R[G^0]$ introduced by the canonical map $G \longrightarrow G^0$. Similarly any $R[G^0]$-module becomes an $R[G]$-module. For an epimorphic local interior G^0-algebra (A, ρ^0) the introduced interior G-algebra (A, ρ) is an epimorphic local interior G-algebra. We determine the defect group of (A, ρ) and the normal subgroup $(\mathrm{vtx}_{G\times G}A)_1$.

LEMMA 4.1. *Let N be a normal subgroup of G whose p-Sylow subgroup S and H be a subgroup of G containing N such that the factor group H/N is p-group. If $QN=H$ for a p-subgroup Q of H containing S, then Q is a p-Sylow subgroup of H.*

PROOF. Let P be a p-Sylow subgroup of H containing Q. Since $H/N \simeq QN/N \simeq Q/N\cap Q$ is a p-group the index $|Q:N\cap Q|$ is equal to the index $|P:S|$. On the other hand, the p-subgroup $N\cap Q$ contains S and is contained in N, and so $N\cap Q$ equals S. Thus the order of P equals that of Q. Therefore P equals Q.

LEMMA 4.2. *Let N be a normal subgroup of G and K a subgroup of G containing N. Let (A, ρ^0) is an interior G^0-algebra. Then for the introduced interior G-algebra (A, ρ), $A^G = A^G_K$ if and only if $A^{G^0} = (A_{k^0})^{G^0}$.*

PROOF. Immediately from the definition of A^G_K.

LEMMA 4.3. *Let N be a normal subgroup of G and H a subgroup of*

G containing N. Whenever (A, ρ^0) is a local interior G^0-algebra with defect group H^0, a defect qroup of the introduced interior G-algebra (A, ρ) equals a p-Sylow subgroup of H. Similarly whenever V is an indecomposable $R[G^0]$-module with vertex H^0, the vertex of the introduced $R[G]$-module V equals a p-Sylow subgroup of H.

PROOF. Let P be an p-Sylow subgroup of H. Since the defect group of (A, ρ^0) is H^0 lemma 4.2 and the definition of defect group implies that the defect group of (A, ρ) is contained in H. Therefore the defect group of (A, ρ) is contained in P up to G-conjugate. Next we assume that the defect group of (A, ρ) is a proper subgroup of P. Let Q be the defect group of (A, ρ) and S a p-Sylow subgroup of N. Since S is contained in N and for an element g of N the element $\rho(g)$ equals 1_A the p-subgroup S is contained in Q, and so we may assume that Q contains S and is contained in P. By assumption, the p-subgroup Q is a proper subgroup of P, lemma 4.1 implies that QN is a proper subgroup of H. Therefore the factor group QN/N is a proper subgroup of H. Since the defect group Q of (A, ρ) is contained in QN the defect group of the interior G^0-algebra (A, ρ^0) is contained in QN/N by lemma 4.2. This is contradiction to the definition H^0 and the defect group of (A, ρ) equals P. Since $(End_R(V), \rho_V)$ is a local interior G^0-algebra the second statement is proved by the first statement.

THEOREM 4.4. *Let N be a normal subgroup of G and $K \leq H$ subgroups of G containing N. Whenever (A, ρ^0) is an epimorphic local interior G^0-algebra which has defect group H^0 and the normal subgroup $(vtx_{G^0 \times G^0}(A))_1 = K^0$ of H^0, then the defect group of the introduced interior G-algebra (A, ρ) equals a p-Sylow subgroup of H and the normal subgroup $(vtx_{G \times G}A)_1$ equals a p-Sylow subgroup of K.*

PROOF. The first statement is immediately from lemma 4.3. Before proving the second statement, we determine the vertex $vtx_{G \times G}A$. Let the vertex $vtx_{G^0 \times G^0}(A)$ be $L/N \times N$, where L is a subgroup of $G \times G$ containing $N \times N$. By the second statement of lemma 4.3, the vertex $vtx_{G \times G}A$ is a p-Sylow subgroup of L. Let P be a p-Sylow subgroup of H and Q a p-Sylow subgroup of K. Since K is a normal subgroup of H we may assume that Q is a normal subgroup of P. Therefore the set $\{(gh, h) : g \in Q,\ h \in P\} = Q_{-1}$ becomes a p-subgroup of L of order $|Q| \cdot |P|$. On the other hand, since $(L/N \times N)_1 = K^0$ the order of L equals $|H| \cdot |K|$, and Q_{-1} is a p-Sylow subgroup of L. Thus by lemma 4.2, $(vtx_{G \times G}A)_1$ equals Q.

We apply this theorem to the kernel of a block of G. Let B be a block of G. The kernel of B is the intersection of all indecomposable $R[G]$-modules belonging to B. It is well-known that the kernel of B is a normal p'-subgroup of G. But in [13], H. Pahling proves the kernel of B is equal to the set $\{g \in G : \rho_B(g)=1\}$, and We prove that $\{g \in G : \rho_B(g)=1\}$ becomes a p'-group using theorem 3. 5.

COROLLARY 4. 5. *Let B be a block of G. Then the kernel of B is a normal p'-subgroup.*

PROOF. Let (B, ρ_B) be the corresponding block interior G-algebra. We may prove that $\{g \in G : \rho_B(g)=1\}=N$ is a p'-subgroup of G. Let G^0 be a factor group G/N. For a element gN of G^0 we define the element $\rho^0(gN)=\rho_B(g)$ in B. Then ρ^0 is an R-algebra epimorphism of $R[G^0]$ to B, and obtain an epimorphic local interior G^0-algebra (B, ρ^0). It is easy to check that the interior G-algebra introduced from (B, ρ^0) is isomorphic to the interior G-algebra (B, ρ_B). Since the vertex $vtx_{G\times G}B$ equal D by theorem 3. 5, theorem 4. 4 implies that N is p'-subgroup.

5. Embedding of defect groups.

In [11], G. Michler prove the following theorem using character theory.

THEOREM 5. 1. *Let $B=R[G]e$ be a block with defect group D, and (B, ρ_B) the corresponding block interior G-algebra. Then the restriction $(\rho_B)|_{R[D]} : R[D] \longrightarrow B$ is a monomorphism.*

In this section, we prove this theorem using interior G-algebra theory, in particular, we remark that the fact $(vtx_{G\times G}B)_1=\langle 1\rangle$ is essential to prove this.

To prove theorem we present the following lemma.

LEMMA 5. 2. *Let V be an indecomposable $R[G]$-module which has vertex D and source W. Then for a subgroup of G containing D there exists an indecomposable direct summand U of the restriction $V\downarrow_H$ such that vertex of U equals D and source of U is W.*

PROOF. Since H contains D the $R[G]$-module V is H-projective, and V is a direct summand of $V\downarrow_H\uparrow^G$. So there exists direct summand U such that V is a direct summand of $U\uparrow^G$. Since $vtx_G V$ equals D the subgroup D is contained in $vtx_H U$ up to H-conjugate. On the other hand, V is a direct summand of $W\uparrow^G$, and Mackey decomposition theorem implies that U is a direct summand of $W^{g}\downarrow_{(gDg^{-1}\cap H)}\uparrow^{H}$ for an element g of G. Therefore the vertex $vtx_{H}U$ is contained in $gDg^{-1}\cap H$, and $vtx_{H}U$ is contained in D up to G-conjugate. Thus $vtx_{H}U$ equals D up to G-conjugate.

Next we prove that source of U is W. Since W is a direct summand of $U\downarrow_{D}$ and $vtx_{D}W$ is D the definition of source implies that W is a source of U.

PROOF OF THEOREM 5.1. First we present a general definition concerned with the trace map. Let V be an $R[D\times D]$-module, H an subgroup of $D\times D$ and V^{H} the R-submdule of V consists from the fixed elements by the a ction of H. The R-linear map $Tr_{D}{}^{D\times D}$; $V^{D}\longrightarrow V^{D\times D}$ defined by $v\longrightarrow(d, 1)v$, where d runs over the element of D.

By example 3.4, $vtx_{G\times G}B$ equals D^{Δ}. Let W be a source of A. W is an $R[D^{\Delta}]$-module. Lemma 5.2 implies that there exists an indecomposable direct summand U of $A\downarrow_{D\times D}$ such that $vtx_{D\times D}U$ equals D^{Δ} and the source of U equals W. Since U is an indecomposable direct summand of $W\uparrow^{D\times D}$ Green's theorem for indecomposability of induced module implies that U is isomorphic to $W^{D\times D}$. The $R[D^{\Delta}]$-module have the trivial module R_{D} and the induced module $R_{D}\uparrow^{D\times D}$ is isomorphic to $R[D\times D]$-module $R[D]$. Therefore the $R[D\times D]$-module U have a submodule which is isomorphic to $R[D]$.

Since $Tr_{D}{}^{D\times D}(R[D]^{D})$ is non zero $Tr_{D}{}^{D\times D}(B^{D})$ is non zero.

We assume that the restriction $\rho\mid_{R[D]}$; $R[D]\longrightarrow B$ is not injective. Then the ideal $I=\{x\in R[D]\,;\rho(x)\}=0$ is non zero. We denote t by the element $\sum d$, where d runs over the elements of D. Since $R[D]$ has an unique minimal ideal (t) the element t is contained in I. Therefore the image $\rho(t)$ is zero in A. For an element x in B^{D} $Tr_{D}{}^{D\times D}(x)=\rho(t)x=0$, and $Tr_{D}{}^{D\times D}(B^{D})$ equals 0. This is contradiction.

6. The case of $(\mathbf{vtx}_{G\times G}\mathbf{A})_{1}$ a cyclic group.

In section 7, we shall see it occurs that the group algebra $R[G]$ has infinite may isomorphism classes of interior G-algebra. In this section, we discuss the finite case using $(vtx_{G\times G}A)_{1}$. This fact impresses us that the subgroup $(vtx_{G\times G}A)_{1}$ of an epimorphic interior G-algebra is similar to the vertex of an indecomposable $R[G]$-module.

Throughout this section D is a p-subgroup of G and Q is a normal subgroup of D. Let $\mathfrak{A}(D, Q)$ be a family of isomorphism classes of epimorphic local interior G-algebra (A, ρ) satisfying that the defect group of (A, ρ) is D and the subgroup $(vtx_{G\times G}A)_{1}$ is Q up to $N_{G}(D)$-conjugate.

Then we have the following theorem.

THEOREM 6.1. *Let D, Q and $\mathfrak{A}(D, Q)$ be as above. If the subgroup Q is a cyclic p-group, then the family of the isomorphism classes of R-algebra A satisfying the interior G-algebra (A, ρ) is in $\mathfrak{A}(D, Q)$ has finite many elements.*

PROOF. By proposition 3.7, an indecomposable direct summand of the left $R[G]$-module A is Q-projective. Since Q is a cyclic group there exists finite many isomorphism classes of these indecomposable direst summands. On the other hand, the R-rank of A is smaller or equal than the order of G. Hence for the epimorphic local interior G-algebra (A, ρ) in $\mathfrak{A}(D, Q)$ the isomorphism classes of the left $R[G]$-modules A are finite.

For two epimorphic local interior G-algebras (A, ρ) and (A', ρ') we may prove whenever the left $R[G]$-modules A and A' are isomorphic, then the R-algebras A and A' are isomorphic. Of course, the R-algebra A is a left A-module as regular module. Since ρ and ρ' are epimorphisms we obtain the following isomorphisms.

$$End_G(A) \simeq End_A(A) \simeq A$$

and

$$End_G(A') \simeq A'.$$

By assumption, A and A' are isomorphic as left $R[G]$-module, and the endomorphism rings $End_G(A)$ and $End_{A'}(A')$ are isomorphic as R-algebra. This implies the theorem.

7. Examples (p-groups)

In this section, we give some properties of an interior P-algebra, where P is a p-group. This is a special case of interior G-algebras, but give us implicit facts in general case. In particular, it is remarkable that this case is similar to the principal block interior G-algebra.

LEMMA 7.1. *Let P be a p-group and (A, ρ) an epimorphic interior P-algebra. Then the following assertions are hold.*

(1) *The epimorphic interior P-algebra (A, ρ) is local.*

(2) *The defect group of (A, ρ) is P.*

(3) *The vertex $vtx_{P\times P}A$ contains P^{Δ} and is contained in $P\times P$.*

PROOF. (1). Since the algebra A is isomorphic to a quotient algebra of $R[P]$ and the group ring $R[P]$ is local the algebra $A^P=$(the center of A) is local. (2). By theorem 2.9, the defect group of (A, ρ) contains the vertex of the trivial $R[P]$-module, and this implies (2). (3). This is from theorem

3.1 and (2).

By lemma 7.1, the subgroup $(vtx_{P\times P}A)_1$ is a normal subgroup of P. We list some examples of this subgroup.

EXAMPLE 7.2. *Let $P=\langle g\rangle$ be a cyclic p-group of order p^n and $R=k$. Then the group ring $k[P]$ is uniserial and for $1\leq l\leq p^n$ there is an unique interior*
P-algebra (A_1, ρ_1) such that $dim_k A_l=l$. The set $(A_l, \rho_l): l=1, 2,\cdots, p^n$ is the set of all epimorphic interior P-algebras. Let P_l be the subgroup of P with index $|P:P_l|$ is p^l.

$$(vtx_{P\times P}A_l)_1=\begin{cases} P_1 & \text{if } l=p^n. \\ P_2 & \text{if } p^{n-1} \text{ exactly divides } l. \\ P_3 & \text{if } p^{n-2} \text{ exactly divides } l. \\ \vdots & \qquad\vdots \\ P_{n-1} & \text{if } p \text{ exactly divides } l. \\ P & \text{otherwise.} \end{cases}$$

EXAMPLE 7.3. *Let $P=\langle g\rangle\times\langle h\rangle$ be a Klein's four group, $R=k$ a field of characteristic 2 such that $|k|>2$. Then all left (two sided) ideals of $k[P]$ are isomorphic to the $k[P]$-modules introduced by the following representations.*

$$2(\gamma): g\longmapsto\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ h\longmapsto\begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}.$$
$$2(\infty): g\longmapsto\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ h\longmapsto\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$
$$3: \quad g\longmapsto\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ h\longmapsto\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$
$$1: \quad g\longmapsto 1, \quad h\longmapsto 1.$$

where γ is an element of k. See [4]. According to the above representations, we set interior P-algebras $(A_{2(\gamma)}, \rho_{2(\gamma)})$, $(A_{2(\infty)}, \rho_{2(\infty)})$, (A_3, ρ_3) and (A_1, ρ_1) respectively. Then

$$(vtx_{P\times P}A_{2(\gamma)})_1=\begin{cases} P & \text{if } \gamma\neq 0, 1, \infty. \\ \langle g\rangle & \text{if } \gamma=\infty. \\ \langle h\rangle & \text{if } \gamma=0. \\ \langle gh\rangle & \text{if } \gamma=1. \end{cases}$$

$$(vtx_{P\times P}A_3)_1=P$$
$$(vtx_{P\times P}A_1)_1=P.$$

Example 7.2 and 7.3 implies the following proposition.

PROPOSITION 7.4. *Let P be a p-group and $R=k$. Then the number of the isomorphic classes of epimorphic interior P-algebras is finite if and only if the group P is a cyclic group.*

PROOF. If P is cyclic, by example 7.2, the number of the isomorphic classes of epimorphic interior P-algebras is finite.

Assume that P is not cyclic. Then there exists a normal subgroup Q of P such that the factor group P/Q is the elementary p-group of order p^2. So we may prove that the number of isomorphic classes of epimorphic interior P-algebras is infinite in the case that $P=\langle g\rangle\times\langle h\rangle$ is an elementary p-group of order p^2. But we obtain the representation

$$g\longrightarrow\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\quad h\longrightarrow\begin{pmatrix}1 & \gamma\\ 0 & 1\end{pmatrix},$$

where γ is the element of the unit of k, and have the corresponding epimorphic interior P-ablebra $(A_{2(\gamma)}, \rho_{2(\gamma)})$. Since $(A_{2(\gamma)}, \rho_{2(\gamma)})$ is isomorphic to $(A_{2(\gamma')}, \rho_{2(\gamma')})$ if and only if $\gamma=\gamma'$ proposition is hold.

The following proposition is concerned with the existence of two sided ideals of $R[P]$.

PROPOSITION 7.5. *Let Q be a normal subgroup of P. Then there exists an epimorphic interior P-algebra (A, ρ) such that $(vtx_{P\times P}A)_1$ equals Q.*

PROOF. By proposition 4.4, for the epimorphic interior P-algebra $(R[P/Q], \rho_Q)$, where ρ_Q is the R-algebra homomorphism introduced by the canonical homomorphism $P\longrightarrow P/Q$, the normal subgroup $(vtx_{P\times P}R[P/Q])_1$ equals Q. This implies proposition.

Finally, we give the relation between the epimorphic interior P-algebra with $(vtx_{P\times P}A)_1=Q$ and the epimorphic interior P-algebra $(R[P/Q], \rho_Q)$.

PROPOSITION 7.6. *Let (A, ρ) be as above. Then there exists an (unique) morphism of (A, ρ) onto $(R[P/Q], \rho_Q)$.*

PROOF. Since $(vtx_{P\times P}A)_1$ equals Q the vertex $vtx_{P\times P}A$ equals $L=\bar{Q}_{-1}$ and a source W. Let R_L be the trivial $R[L]$-module, then the $R[P\times P]$-module $R[P/Q]$ is isomorphic to $R_L\uparrow^{P\times P}$. Because P is p-group, there exists a surjective homomorphism of W to R_L, and so there exists a surjective homomorphism $W\uparrow^{P\times P}$ to $R_L\uparrow^{P\times P}$. The Green's theorem and the above remark implies that there exists a surjective homomorphism φ of the $R[P\times P]$-module A to the $R[P\times P]$-module $R[P/Q]$.

Obviously, the image $\varphi(1_A)$ is in the center $Z(R[P/Q])$. Since φ is epimorphism and an $R[P\times P]$-homomorphism we have

$\varphi(A)=\varphi(\rho(R[P])1_A)=R[P/Q]\varphi(1_A)=R[P/Q]$. Therefore $\varphi(1_A)=x$ is an unit of $R[P/Q]$. Now we define a map φ_0 of A to $R[P/Q]$ as following.

$$\varphi_0\,;\, a \longrightarrow x^{-1}\varphi(a).$$

Then $\varphi_0(1_A)$ equals 1 in $R[P/Q]$. It is easyly checked that φ_0 is a morphism of (A, ρ) onto $(R[P/Q], \rho_Q)$. The uniqueness is from the definition of morphism.

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