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Douglas algebras on multiply connected domains

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In this paper we consider Douglas algebras related to the algebras H^{∞} of bounded analytic functions on multiply connected domains. Our main result is: Every closed subalgebra $\mathscr{B}$ of L^{∞} containing H^{∞} is generated by H^{∞} and the complex conjugates of single-valued interpolating Blaschke products which are invertible in $\mathscr{B}$. The result is also true for the algebras H^{∞} on finite bordered Riemann surfaces.

1. Introduction. Let Ω be a bounded connected open subset of the plane whose boundary Γ consists of $N+1$ non-intersecting, analytic Jordan curves. We denote by $H^{\infty}=H^{\infty}(\Omega)$ the algebra of bounded analytic functions on Ω. Denote by $L^{\infty}=L^{\infty}(\Gamma)$ the Banach algebra of essentially bounded, measurable functions on Γ with respect to the measure defined by arc length. By the correspondence between each function in H^{∞} and its nontangential boundary values, we also consider H^{∞} as a closed subalgebra of L^{∞} ([5: Chap. 4, Theorem 4. 4]).

In what follows, we always denote by $\mathscr{B}$ a closed subalgebra of L^{∞} which contains H^{∞}. A function $f\in H^{\infty}$ is called inner if $|f|=1$ a. e. on Γ. We call $\mathscr{B}$ a Douglas algebra over H^{∞} if $\mathscr{B}$ is generated by H^{∞} and the complex conjugates of inner functions that are invertible in $\mathscr{B}$. In the case that Ω is the open unit disk, Chang and Marshall showed that every $\mathscr{B}$ is a Douglas algebra ([4], [8: Chap. IX] and [10]). Our purpose of this note is to show the following result in the above situation.

MAIN THEOREM. *Let $\mathscr{B}$ be a closed subalgebra of L^{∞} containing H^{∞}. Then $\mathscr{B}$ is a Douglas algebra. More precisely, $\mathscr{B}$ is generated by H^{∞} and the complex conjugates of single-valued interpolating Blaschke products which are invertible in $\mathscr{B}$.*

The same result is also true for finite bordered Riemann surfaces. See § 5.

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2. The algebra $\mathbf{H^{\infty}+C}$. Let $\Gamma_0,\dots,\Gamma_n$ be the components of the boundary of Ω, where we let Γ_0 be the boundary of the unbounded connected component of $\boldsymbol{C}\backslash\Omega$. Let $C=C(\Gamma)$ and $C_k=C(\Gamma_k)$, $0\leqq k\leqq N$, denote the algebras of continuous functions on Γ and Γ_k, respectively. Define $H^\infty + C$ to be the set$\{f+g : f \in H^\infty,\ g \in C\}$. Then $H^\infty + C$ in a closed subalgebra of L^∞ ([1] and [14]).

Let $\mathscr{U}_k$ be the connected component of $\boldsymbol{C} \cup \{\infty\} \setminus \Gamma_k$ containing Ω for each k. Denote the open unit disk by D. There is a one-to-one conformal mapping Ψ_k of $\mathscr{U}_k$ onto D; note that Ψ_k extends to be analytic and conformal on a neighborhood of $\mathscr{U}_k \cup \Gamma_k$. Let $M(H^\infty)$ and $M(H^\infty(\mathscr{U}_k))$ denote the maximal ideal spaces of H^∞ and $H^\infty(\mathscr{U}_k)$, $0 \leqq k \leqq N$, respectively. Define the mappings

$$\hat{Z} : M(H^\infty) \rightarrow \bar{\Omega} \quad \text{by } \hat{Z}(\varphi) = \varphi(z)$$

and

$$\hat{\Psi}_k : M(H^\infty(\mathscr{U}_k)) \rightarrow \bar{D} \quad \text{by } \hat{\Psi}_k(\varphi) = \varphi(\Psi_k).$$

Let $h^\infty(\Omega)$ be the space of bounded harmonic functions on Ω. Let $S(H^\infty)$ be the Shilov boundary of H^∞. Then $M(L^\infty) = S(H^\infty)$ ([5: Chap. 6, Theorem 5. 2] and [9]).

PROPOSITION 2. 1. (1) *For $\varphi \in M(H^\infty)$ with $\hat{Z}(\varphi) \in \Gamma$, there exists a unique representing measure μ_φ on $S(H^\infty)$.* (2) *Each function $u \in h^\infty(\Omega)$ has a unique continuous extension $\hat{u}$ to $M(H^\infty)$ such that for $\varphi \in M(H^\infty) \setminus \Omega$,*

$$\hat{u}(\varphi) = \int_{S(H^\infty)} u \, d\mu\varphi, \quad u \in h^\infty(\Omega).$$

(3) *Denote the nontangential limit of u by $u^* \in L^\infty(\Gamma)$. The Gelfand transform $\hat{u}^*$ of $\hat{u}^*$ coincides with $\hat{u}$ on $S(H^\infty)$ and the values of $\hat{u}$ on $\hat{Z}^{-1}(\Gamma_k)$ depend only on $u^*|_{\Gamma_k}$.*

PROOF. (1) For $\hat{Z}(\varphi) = \lambda \in \Gamma$, $\hat{Z}^{-1}(\lambda)$ is a peak set for H^∞. So the restriction of H^∞ to $\hat{Z}^{-1}(\lambda)$ is isomorphic to a fiber algebra of $H^\infty(D)$ ([7]). Thus $\varphi \in \hat{Z}^{-1}(\lambda)$ has a unique representing measure on $S(H^\infty)$.

(2) Let M_{rep} be the set of all representing measures for H^∞ on $S(H^\infty)$. Then M_{rep} is weak-$*$ compact in the space of all regular Borel measures on $S(H^\infty)$. Moreover let M^*_{rep} be the set of all $\mu \in M_{\text{rep}}$ such that

$$\int f \, d\mu = f\left(\int Z \, d\mu\right) \text{ for all } f \in h(\bar{\Omega}).$$

Here $h(\bar{\Omega})$ is the space of functions continuous on $\bar{\Omega}$ and harmonic on Ω. Then M^*_{rep} also is weak-$*$ compact. For $z \in \Omega$, let ω_z be the harmonic measure and $\varphi_z \in M(H^\infty)$ be the point evaluation. Then, $\omega_z \in M^*_{\text{rep}}$ and ω_z is a unique representing measure for φ_z contained in M^*_{rep}. Together with part

(1), M^*_{rep} contains a unique representing measure μ_φ for every $\varphi \in M(H^\infty)$. Thus M^*_{rep} is homeomorphic to $M(H^\infty)$ for the weak-$*$ topology. Now if we define for $u \in h^\infty(\Omega)$

$$\hat{u}(\varphi) = \int \hat{u}^* \, d\mu_\varphi, \; \mu_\varphi \in M^*_{\mathrm{rep}},$$

then $\hat{u}$ is continuous on $M(H^\infty)$ and the extension of u. Uniqueness of the extension of u follows from the corona theorem ([5: Chap. 6, Theorem 6.3] and [11]).

(3) For $\varphi \in S(H^\infty)$, μ_φ is the Dirac measure δ_φ. So $\hat{u} = \hat{u}^*$ on $S(H^\infty)$. Since $\hat{u}^*$ is identified with $(u^* | \Gamma_k)$ on $\hat{Z}^{-1}(\Gamma_k)$, the values $\hat{u}(\hat{Z}^{-1}(\Gamma_k))$ are determined by $u^* | \Gamma_k$.

NOTE. There is another proof for (2). Namely, one may define the set M^*_{rep} as the set of logmodular measures for H^∞ in the proof (cf. [6: Chap. IV, Corollary 7.6]).

PROPOSITION 2.2. *For a fixed point $z_0 \in \Omega$, Let $\omega = \omega_{z_0}$ be the harmonic measure on Γ. Suppose that $\mathscr{B}$ is a closed subalgebra of L^∞ such that $\mathscr{B} \supset H^\infty$ and ω is multiplicative on $\mathscr{B}$. Then $\mathscr{B} = H^\infty$.*

For the proof see [6: Chap. IV, Theorem 7.7].

Since $(Z\text{-}z_0)^\wedge$ does not vanish on $M(H^\infty) \setminus \Omega$, if $\mathscr{B} \supsetneq H^\infty$, then $Z\text{-}z_0$ is invertible in $\mathscr{B}$. From Runge's theorem, Propositions 2.1 (1) and 2.2, we have the following.

COROLLARY 2.3. *Let $\mathscr{B}$ be a closed subalgebra of L^∞ properly containing H^∞. Then $\mathscr{B}$ contains $H^\infty + C$ and $M(\mathscr{B})$ is identified with a compact subset of $M(H^\infty) \setminus \Omega$.*

REMARK. Let F be the Ahlfors function for Ω ([2] and [5: Chap. 5]). Then F is an inner function on Γ and continuous on $\bar{\Omega}$. It follows from the corollary that $H^\infty + C$ is generated by H^∞ and $\bar{F}$. So $H^\infty + C$ is a Douglas algebra.

3. Lemmas. In this section we show several lemmas.

LEMMA 3.1.

$$H^\infty + C = \bigoplus_{k=0}^{N} (H^\infty(\mathscr{U}_k) + C_k)$$

Namely, for $f \in H^\infty + C$, we have $f = \chi_{\Gamma_0} f_0 + \ldots + \chi_\Gamma f_N$, where χ_{Γ_k} is the characteristic function of Γ_k, $f_k \in H^\infty(\mathscr{U}_k) + C_k$, $0 \leqq k \leqq N$, and $\|f\| =$ $\max(\|f_0\|, \ldots, \|f_N\|)$.

PROOF. Since $\chi_{\Gamma_k}\in H^\infty+C$, it is sufficient to prove $\chi_{\Gamma_k}(H^\infty+C)=H^\infty(\mathscr{U}_k)+C_k$ for each k. Let $f=g+u$, where $f\in H^\infty$ and $u\in C$. Write

$$g(z)=\sum_{k=0}^{N} g_k(z), \text{ where } g_k(z)=\frac{1}{2\pi i}\int_{\Gamma_k}\frac{g(\zeta)}{\zeta-z}d\zeta.$$

Then $g_k\in H^\infty(\mathscr{U}_k)$ and $g_k\in C_j(j\neq k)$. So $\chi_{\Gamma_k}f\in H^\infty(\mathscr{U}_k)+C_k$. Conversely it is clear that $H^\infty(\mathscr{U}_k)+C_k\subset\chi_{\Gamma_k}(H^\infty+C)$.

LEMMA 3.2. *If $\mathscr{B}$ is a closed subalgebra of L^∞ properly containing H^∞, then*

(1) *$\mathscr{B}_k=\chi_{\Gamma_k}\mathscr{B}$ is the closed subalgebra of $L^\infty(\Gamma_k)$ containing $H^\infty(\mathscr{U}_k)+C_k$ for each k.*

(2) *$M(\mathscr{B})=\bigcup_{k=0}^{N} M(\mathscr{B}_k)$, where $M(\mathscr{B}_j)\cap M(\mathscr{B}_k)=\phi$ for $j\neq k$.*

(3) *$\mathscr{B}=\mathscr{B}_0\oplus\ldots\oplus\mathscr{B}_N$, a direct sum.*

By proposition 2.1, the proof is immediate.

Let $\{a_n\}$ be the points in Ω with no limit point in Ω. Let $G(z\,;a_n)$ be the Green's function for Ω with pole at a_n. If $\{a_n\}$ satisfies that $\sum_{n=1}^{\infty} G(z\,;a_n)<\infty$ for each $z\in\Omega$, we define

$$B(z)=\exp\left[-\sum_{n=1}^{\infty} G(z\,;a_n)-i^*\left(\sum_{n=1}^{\infty} G(z\,;a_n)\right)\right]$$

where $^*u$ denotes a harmonic conjugate of a real harmonic function u. We call B the Blaschke product on Ω for $\{a_n\}$. Note that B may not be single-valued ([5: Chap. 7]).

For a closed subalgebra $\mathscr{B}$, let $\mathscr{B}^{-1}$ be the set of invertible elements of $\mathscr{B}$.

LEMMA 3.3. *Let $\mathscr{B}$ be a closed subalgebra of L^∞ and $\mathscr{B}\supsetneqq H^\infty$. Suppose that B is a single-valued Blaschke product on Ω with its zeros $\{a_n\}$ accumulating only at points on Γ_k for some k, and that b is the Blaschke product on $\mathscr{U}_k$ with the same zeros $\{a_n\}$. Then $B\in\mathscr{B}^{-1}$ if and only if $b\in\mathscr{B}_k^{-1}$.*

PROOF. Firstly we note that $b^{-1}B\in(H^\infty)^{-1}$. In fact, by the assumption, b has the form

$$b=\prod_n w_n\circ\Psi_k, \text{ where } w_n=-\frac{|a_n'|}{a_n'}\frac{z-a_n'}{1-\bar{a}_n'z}, \quad a_n'=\Psi_k(a_n).$$

We obtain the following factorization of $w_n\circ\Psi_k$ as a function in H^∞:

$$w_n\circ\Psi_k=W_n\exp(v_n+i^*v_n),$$

where W_n is a Blaschke product on Ω and $v_n=\log|w_n\circ\Psi_k/W_n|$. Then $B=\prod_n W_n$. Since v_n are negative harmonic functions on Ω and $\sum_n v_n(a_1)=\log|(b/B)(a_1)|>-\infty$, $\sum v_n$ converges to a harmonic function v uniformly on every compact subset of Ω, by Harnack's theorem. Also, note that v_n is continuous on $\bar{\Omega}$ and vanishes on Γ_k. By the reflection principle and the maximum principle, we see that $\sum v_n$ converges to v uniformly on a neighborhood of Γ_k. On the other hand, the same argument implies that $\sum G(z\,;a_n)$ is harmonic on a neighborhood of Γ_j $(j\neq k)$. So $v=\log|b|-\log|B|$ is harmonic on a neighborhood of $\bar{\Omega}$. Consequently $b=B\exp(v+i^*v)$ and $b^{-1}B$ has no zeros and is analytic on Γ. Thus $b^{-1}B\in(H^\infty)^{-1}$.

Suppose $b\in\mathscr{B}_k^{-1}$. Since b is analytic on Γ_j $(j\neq k)$ and has no zeros there, $b\in\mathscr{B}_j^{-1}$ $(j\neq k)$. So $b\in\mathscr{B}^{-1}$ and $B\in\mathscr{B}^{-1}$. Conversely, $b=bB^{-1}$. $B\in\mathscr{B}^{-1}$. Thus $b\in\mathscr{B}_k^{-1}$ by Lemma 3.2.

We recall the notion of interpolating Blaschke products. A sequence $\{a_n\}_{n=1}^\infty$ in Ω is called interpolating if for any $\{w_n\}\in l^\infty$, there is a function $f\in H^\infty$ with $f(a_n)=w_n$. If $\{a_n\}=S_0\cup\ldots\cup S_N$ where $S_j\cap S_k=\phi$ for $j\neq k$ and all limit points of S_k lie on Γ_k, $0\leqq k\leqq N$, then $\{a_n\}$ is an interpolating sequence if and only if S_k is an interpolating sequence for $H^\infty(\mathscr{U}_k)$, $k=0,\ldots,N$ ([11]). A Blaschke product is called an interpolating Blaschke product if its zeros are all simple and form an interpolating sequemce.

Now, we shall show the following key lemma which may have an independent interest.

LEMMA 3.4. *If B is a Blaschke product on Ω with simple zerons, then there exists a single-vaued Blaschke product on Ω with simple zeros such that its zeros coincide with B's but a finite number.*

If one admits that the single-valued Blaschke product may have multiple zeros, the lemma follows from [12]. To prove the lemma in the present form, we shall need te following version of Widom [13].

LEMMA 3.5. *There is a compact subset K of Ω such that for any real number $c_1,\ldots,c_N$ there are mutually distinct finite points a_n in K such that*

$$(3.1)\qquad \sum_n\int_{\Gamma_k}{}^*dG(\zeta\,;a_n)\equiv c_k \pmod{2\pi}$$

for $k=1,\ldots,N$.

PROOF OF LEMMA 3.4. We may assume that B has no zeros in K. Then, by Lemma 3.5, there exists a finite Blaschke product B_1 scuh that the zeros of B_1 are simple and lying in the set K and such that BB_1 is single-valued on Ω. Clearly, BB_1 has the desired property.

PROOF of LEMMA 3.5. Let u_k, $k=1,\dots,N$, be the harmonic function on Ω whose boundary values are 1 on Γ_k and 0 on Γ_j, $j\neq k$. By the Green's formula,

$$u_k(z)=\frac{1}{2\pi}\int_{\Gamma_k}{}^*dG(\zeta\,;\,z),\ z\in\Omega.$$

Now we use the method used in [13; Lemma 6]. Namely, set $(a_1,\dots,a_z)\in\Omega^N$. Then

$$(3.\ 2)\qquad \frac{1}{2\pi}\int_{\Gamma_k}{}^*d\Big(\sum_{n=1}^{N}G(\zeta\,;\,a_n)\Big)=\sum_{n=1}^{N}u_k(a_n)$$

which is the period of the function ${}^*(\sum_n G(\zeta\,;\mathrm{a}_n))$ around Γ_k. Now, the Jacobi matrix of the mapping

$$(x_1,\ y_1,\ \dots,\ x_N,\ y_N)\rightarrow\Big(\sum_{n=1}^{N}u_k(a_n)\Big)_{k=1,\dots,N},\ a_n=x_n+iy_n$$

is given by

$$(3.3)\qquad \Big(\frac{\partial u_k}{\partial \mathrm{x}_n}(\mathrm{a}_n),\ \frac{\partial u_k}{\partial \mathrm{y}_n}(\mathrm{a}_n)\Big)_{k,\ n=1,\dots,N}.$$

Since u_k are linearly independent, the rank of matrix is N for $(\mathrm{a}_1,\dots,a_N)\in\Omega^N$ except for nowhere dense closed subset of Ω^N. Now let us take a single-valued finite Blaschke product φ on Ω. For example, one may consider the Ahlfors function for Ω. Note that φ is a ν-sheet mapping of Ω ontoD except for a finite number of branch points. Thus we can find suitable points $a_1^1,\dots,a_N^1$ in Ω so that $w_n=\varphi=(a_n^1)$ are mutually distinct, $\varphi^{-1}(w_n)=\{a_n^1,\dots,a_n^\nu\}$ consists of ν distinct points and the Jacobi matrix (3. 3) at $(a_1^1,\dots,a_N^1)$ has rank N. Clearly, $b^0(z)=\prod_{n=1}^{N}\{(\varphi(z)-w_n)/(1-\bar{w}_n\varphi(z))\}$ is a single-valued Blaschke product with ν times N distinct zeros. We consider the periods of differentials

$$(3.4)\qquad \sum_{n=1}^{\mathrm{N}}{}^*dG(\zeta\,;\,a_n)+\sum_{l=2}^{\nu}\sum_{n=1}^{N}{}^*dG(\zeta\,;\,a_n^l).$$

Note that for $\mathrm{a}_n=\mathrm{a}_n^1$, (3. 4) corresponds to the Blaschke product $b^0(z)$. The N-tuples of their periods are given by

$$\Big(\sum_{n=1}^{\mathrm{N}}u_k(a_n)+\sum_{l=2}^{\nu}\sum_{n=1}^{N}u_k(a_n^l)\Big)_{k=1,\dots,\mathrm{N}}.$$

This yields the same Jacobi matrix (3. 3) as a function of $a_1,\dots,a_N$. Since it has rank N at $(a_1^1, \dots, a_N^1)$, the set of periods of b^1, the Blaschke product with zeros $\{a_1, \dots, a_N, a_1^l, \dots, a_N^l : l=2, \dots, \nu\}$, forms an open neighborhood of the identity, when $a_1, \dots, a_N$ run through a small neighborhood U_1 of the set' $\{a_1^1, \dots, a_N^1\}$. From the construction, b^1 has different zeros. Set $K_1=\bar{U}_1 \cup \{a_n^l ; 1 \leqq n \leqq N, 2 \leqq l \leqq \nu\}$. Let U_2 be any relatively compact open subset of Ω with $K_1 \cap U_2 = \phi$. Then, the set $K=K_1 \cup \bar{U}_2$ has the required property. In fact, as in the proof of [13: Lemma 6], there exist a finite number of points ξ_m in U_2 such that

$$\sum_m \int_{\Gamma_k} {}^*dG(\xi ; \xi_m) \equiv c_k \pmod{2\pi}$$

for $k=1, \dots, N$, where ξ_m may not be distinct. Perturbing ξ_m slightly, we have distinct ξ'_m. Choosing $a_1, \dots, a_N \in U_1$ in an appropriate way and adding (3. 4) to $\sum\limits_m {}^*dG(\xi'_m)$, we obtain (3. 1) relabeling $\{a_1, \dots, a_N\} \cup \{a_j^l ; 2 \leqq j \leqq N, 1 \leqq l \leqq \nu\} \cup \{\xi'_n\}$ as $\{a_n\}$.

4. Proof of the main theorem. We may assume that $\mathscr{B}' \supsetneqq H^\infty + C$. The proof runs as follows. As defined in § 2, let Ψ_k be the Riemann mapping of $\mathscr{U}_k$ onto D for each k. For a fumction $f \in \mathscr{B}$, $f|\Gamma_k \in L^\infty(\Gamma_k)$. By Chang and Marshall's theorem ([4] and [10], [8: Chap. IX, Theorem 3. 1]), there exist an interpolating Blaschke product b_k and $g_k \in H^\infty(D)$ *with* $\overline{d_k \circ \Psi_k} \in \mathscr{B}|\Gamma_k$ such that

$$(4.1) \qquad \|f - \overline{b_k \circ \Psi_k} g_k \circ \Psi_k\|_{\Gamma_k} < \varepsilon$$

for any $\varepsilon > 0$. To simplify the argument we consider the case $k=0$. Let $\{a_n\}$ be the zeros of $b_0 \circ \Psi_0$. Including to $g_0 \circ \Psi_0$ the finite Blaschke part of $b_0 \circ \Psi_0$ whose zeros are not on Ω, We may assume that $\{a_n\}$ is an interpolating sequence on Ω and $g_0 \circ \Psi_0 \in H^\infty$. Now let B_0 be the Blaschke product for $\{a_n\}$ on Ω. By Lemma 3. 4, we take a single-valued interpolating Blaschke product $\tilde{B}_0$ which has the same zeros as B_0 except a finite number of zeros. By the same metod in the proof of Lemma 3. 3, we can write

$$b_0 \circ \Psi_0 / \tilde{B}_0 = \exp(u_0 + i^* u_0) B'_0 / B''_0,$$

where $u_0 = \log|b_0 \circ \Psi_0 / B_0|$ and B'_0, B''_0 are finite Blaschke products. The left hand side is single-valued and the right does not vanish on Γ_0 and is continuous there. Therefore $b_0 \circ \Psi_0 / \tilde{B}_0 \in C(\Gamma_0)$, $|b_0 \circ \Psi_0 / \tilde{B}_0| = 1$ on Γ_0 and $(b_0 \circ \Psi_0 / \tilde{B}_0)^{-1} = \overline{(b_0 \circ \Psi_0 / \tilde{B}_0)} \in C(\Gamma_0)$. On the other hand, we have $\tilde{B}_0 \in \mathscr{B}^{-1}$ by the fact $(b_0 \circ \Psi_0)^{-1} \in \mathscr{B}|\Gamma_0$ and Lemma 3. 3. Consequently, it follows from (4. 1) that

$$\|f-\tilde{\tilde{B}}_0\,\tilde{g}_0\|_{\Gamma_0}=\|f-\overline{b_0\circ\Psi_0 g_0}\circ\Psi_0\|_{\Gamma_0}<\varepsilon$$

for any $\varepsilon>0$, where $\tilde{g}_0=\overline{(b_0\circ\Psi_0/\tilde{B}_0)}g\circ\Psi_0\in H^\infty(\mathscr{U}_0)+C(\Gamma_0)$.

We have the same work for any k, $1\leqq k\leqq N$. That is, for any function $f\in\mathscr{B}$,

$$\|f-(\mathrm{X}_{\Gamma_0}\tilde{\tilde{B}}_0\,\tilde{g}_0+\ \dots\ +\mathrm{X}_{\Gamma_N}\ \tilde{\tilde{B}}_N\ \tilde{g}_N)\|<\varepsilon,$$

$\mathrm{X}_{\Gamma_k}g_k\in H^\infty+C$ and $\tilde{B}_k$ is an interpolating Blaschke product on Ω with $B_k\in\mathscr{B}^{-1}$ for each k. Here by the remark in § 2, we finish the proof.

It is routine to see the next two results from the main theorem and Proposition 2.1(2) (cf. [8: Chap. IX]).

COROLLARY 4.1. Let $\mathscr{B}$ be a closed subalgebra of L^∞ containing H^∞. Then

$$M(\mathscr{B})=\{\varphi\in M(H^\infty): |\hat{q}(\varphi)|=1 \text{ for } q\in H^\infty_i \text{ inner, } q\in\mathscr{B}^{-1}\}.$$

COROLLARY 4.2 *Let $\mathscr{B}_1$ and $\mathscr{B}_2$ be closed subalgebras of L^∞ containing H^∞. Then, $\mathscr{B}_1=\mathscr{B}_2$ if and only if $M(\mathscr{B}_1)=M(\mathscr{B}_2)$. That is, every closed algebra between H^∞ and L^∞ is unipuely determined by its maximal ideal space.*

5. The case of finite bordered Riemann surfaces. We can extend the above results to the case that Ω is a finite bordered Riemann surface whose boundary Γ consists of disjoint analytic simple closed curves $\Gamma_0,\dots,\Gamma_N$. In this section we shall briefly sketch the proof.

There exists a Cauchy differential (elementary differential) $\omega(p.\,q)=f(z,q)dz$ in $z+z(p)$, on a neighborhood $\bar{\Omega}$ ([3: Chap. VI, § 6, Satz 44]). The function $f(z,q)$ of q and the differential $f(z,q)dz$ of z are analytic on $\bar{\Omega}$ except for $p=q$ and it has the form

$$f(z,q)=\frac{1}{z-\xi}+R(z,\xi),\ z=z(p),\ \xi=\xi(q)$$

in a neighborhood of $p=q$, where $R(z,\xi)$ is analytic in both z and ξ.

Now, Proposition 2.1(1) can be shown in the same manner ; to see part (2), we only have to consider analytic functions $Z_1,\dots,Z_\nu$ *on* $\bar{\Omega}$, instead of a single Z, such that $Z_1,\dots,Z_\nu$ separate the points of Ω ; part (3) is routine.

For $k=0,\dots,N$, choose an annulus A_k in Ω with $\partial A_k\supset\Gamma_k$ and let Ψ_k be a one-to-one analytic map of A_k onto $\{z: r_k<|z|<1\}$ such that $\Psi_k(\Gamma_k)=T=\{|z|=1\}$.

Let $A(\bar{\Omega})$ be the Banach algebra of functions continuous on $\bar{\Omega}$ and analytic on Ω. If μ is a measure on Γ orthogonal to $A(\bar{\Omega})$, μ is absolutely continuous with respect to the harmonic measure ω_z for a point $z\in\Omega$. For

$A(\bar{\Omega})|_{\Gamma}$ is a hypo-Dirichlet algebra on Γ and Ω is connected. Now we can show in the same way as [14] that $H^{\infty}(\Omega)+C(\Gamma)$ is a closed subalgebra of $L^{\infty}(\Gamma)$. Moreover, it follows that $\chi_{\Gamma_k}(H^{\infty}+C)\cong H^{\infty}(D)+C(T)$ via ψ_k. In fact, as we have seen, $\chi_{\Gamma_k}(H^{\infty}(A_k)+C(\Gamma_k))\cong H^{\infty}(D)+C(T)$ via ψ_k. Since $H^{\infty}|_{A_k}\subset H^{\infty}(A_k)$, $\chi_{\Gamma_k}(H^{\infty}+C)\subset\chi_{\Gamma_k}(H^{\infty}(A_k)+C(\Gamma_k))$. For $f\in H^{\infty}(A_k)$, define

$$f_1(z)=\frac{1}{2\pi i}\int_{\Gamma_k}f(\xi)\omega(\xi, z)$$

and

$$f_2(z)=\frac{1}{2\pi i}\int_{\{|\psi_k|=r_k\}}f(\xi)\omega(\xi, z).$$

Then $f_1\in H^{\infty}(\Omega)$ and $f_2\in C(\Gamma_k)$. So $f\in\chi_{\Gamma_k}(H^{\infty}+C)$. Consequently, $\chi_{\Gamma_k}(H^{\infty}+C)\cong\chi_{\Gamma_k}(H^{\infty}(A_k)+C(\Gamma_k))$.

So we can show the same results as in §§ 2 and 3 but Lemma 3. 5, which can be stated in the following form.

LEMMA 5. 1. *Let $C_1, \dots, C_{2g+N}$ be a homological base of Ω, where g is the genus of Ω and $C_1, \dots, C_{2g}$ are nondividing cycles. Then there is a compact subset K of Ω such that for any real number $c_1, \dots, c_{2g+N}$ there are mutually distinct finite points a_n in K such that*

$$\sum_n\int_{C_k}{}^*dG(\xi\,;\,a_n)\equiv c_k(\bmod 2\pi)$$

for $k=1, \dots, 2g+N$.

In the proof, we only have to use the reproducing kernel $\varphi(C_k)$ in the space of square integrable differentials on Ω in addition to the harmonic measure du_k. Then (3. 2) takes the following form

$$\frac{1}{2\pi}\int_{C_k}{}^*d(\sum_{n=1}^{N}G=\xi\,;\,a_n))=\sum_{n=1}^{N}\int_{a_0}^{a_n}\varphi(C_k)$$

where $a_0\in\Gamma_0$ and a_n lie in $\tilde{\Omega}=\Omega\setminus(\bigcup_{k=1}^{2g}c_k)$.

The remaining part of the proof can be shown in a similar manner as before, where one may use $H^{\infty}(A_k)$ as intermediate algebras between H^{∞} fnd $H^{\infty}(D)$ if need be. The detail will be omitted.

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