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KMO-Langevin equation and Fluctuation-Dissipation Theorem (II)

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§1. Introduction

Let $R=(R_{pq})_{1\leq p,q\leq d}$ be an $M(d\,;\boldsymbol{C})$-valued continuous and non-negative definite function on $\boldsymbol{R}$, that is, for any $u={}^t(u_1, u_2,\cdots, u_d)\in\boldsymbol{C}^d$, $(Ru, u)_{\boldsymbol{C}^d}=\sum_{p,q=1}^{d}\bar{u}_p u_q R_{pq}$ is a $\boldsymbol{C}$-valued continuous and non-negative definite function on $\boldsymbol{R}$, where d is a fixed integer through this paper and $M(d\,;\boldsymbol{C})$ denotes the set of $d\times d$-matrices over $\boldsymbol{C}$. We define an $M(d\,;\boldsymbol{C})$-valued holomorhic function $[R]$ on $\boldsymbol{C}^+$ by

$$(1.1)\qquad [R](\zeta)\equiv\frac{1}{2\pi}\int_0^\infty e^{i\zeta t}R(t)\,dt.$$

Furthermore, let $A=(A(t)\,;\,t\in\boldsymbol{R})$ be any stationary curve in a Hilbert space $\mathscr{H}^d$ with R as its covariance matrix:

$$((A_p(t),\ A_q(s))_{\mathscr{H}})_{1\leq p,q\leq d}=R(t-s),$$

where $A(t)={}^t(A_1(t),\ A_2(t),\cdots,\ A_d(t))$.

In [11], we have treated the case where $d=1$ and R satisfies the following conditions:

(1.2) $R(0)\neq 0$

(1.3) there exists a null set Λ_R in $\boldsymbol{R}-\{0\}$ such that $\lim_{\eta\downarrow 0}[R](\xi+i\eta)$ exists for any $\xi\in\boldsymbol{R}-\Lambda_R$

(1.4) there exist positive constants c and m such that $|[R](\zeta)|\geq c(1+|\zeta|^m)^{-1}$ for any $\zeta\in\boldsymbol{C}^+$.

In the first half of [11], we have obtained a complete structure of the function $[R]$ by introducing a second KMO-Langevin data (Theorems 4.1 and 4.2 in [11]). In the last half of [11], by using a spectral representation of the stationary curve A, we have introduced a Kubo noise from the point of view of Kubo's linear response theory in statistical physics. And we have applied the complete form of the function $[R]$ to derive an equation of motion describing the time evolution of the stationary curve A with the Kubo noise as the random force which we called a second KMO-Langevin equation (Theorem 7.1 in [11]). We have found that such a second KMO-Langevin equation gives a natural generalization of Stokes-Boussinesq-Langevin equation which describes the time evolution of Brownian motion with Alder-Wainwright effect ([12]). Furthermore we have shown that Kubo's fluctuation-dissipation theorem holds on the basis of the second KMO-Langevin equation with both a mathematical structure and a physical meaning (Theorems 8.1, 8.2 and 8.4 in [11]).

In the investigation of [11] stated above, it was fundamental to obtain a complete structure of the function $[R]$. For this reason, we shall in this paper obtain a complete form of the function $[R]$ for the d-dimensional case in (1.1) under the following more general conditions (1.5) and (1.6) than conditions (1.2), (1.3) and (1.4) assumed in case $d=1$:

(1.5) $\qquad R(0)\in GL(d\,;\boldsymbol{C})$

and

(1.6) $\qquad D\equiv\lim_{\varepsilon\downarrow 0}\int_0^{\infty}e^{-\varepsilon t}R(t)\,dt$ exists and $D\in GL(d\,;\boldsymbol{C})$,

where $GL(d\,;\boldsymbol{C})$ denotes the set of $d\times d$-regular matrices over $\boldsymbol{C}$.

We call the function $[R](2\pi R(0))^{-1}$ an $M(d\,;\boldsymbol{C})$-valued **mobility function** associated with R. We define two sets $\mathscr{R}$ and $\mathscr{L}$ by

$\mathscr{R}\equiv\{R\,;R$ is an $M(d\,;\boldsymbol{C})$-valued continuous and non-negative definite function on $\boldsymbol{R}$ satisfying conditions (1.5) and (1.6)$\}$

and

$\mathscr{L}\equiv\{(\alpha,\beta,\varkappa)\,;$

(i) $\alpha\in GL(d\,;\boldsymbol{C})$, $\alpha^*=\alpha$ and α is positive definite

(ii) $\beta\in GL(d\,;\boldsymbol{C})$

(iii) $\varkappa\equiv(\varkappa_{pq})_{1\le p,\,q\le d}\,;$

(a) for each $p,\ q\in\{1,2,\cdots,d\}$ $\varkappa_{pq}$ is a $\boldsymbol{C}$-valued Borel signed measure on $\boldsymbol{R}$

(b) for any $u={}^t(u_1,u_2,\cdots,u_d)\in\boldsymbol{C}^d$

$$\varkappa_u\equiv(\varkappa u,u)_{\boldsymbol{C}^d}=\sum_{p,\,q=1}^{d}\bar{u}_p u_q\varkappa_{pq}\ \text{is a Borel measure on } \boldsymbol{R}$$

(c) $\int_{\boldsymbol{R}}\frac{1}{1+\lambda^2}\varkappa_u(d\lambda)<\infty \qquad$ for any $u\in\boldsymbol{C}^d$

(d) $\varkappa^*=\varkappa$

(e) $\lim_{\varepsilon\downarrow 0}\int_{\boldsymbol{R}}\frac{1}{\pi}\frac{\varepsilon}{\lambda^2+\varepsilon^2}\varkappa(d\lambda)=\frac{1}{\sqrt{2\pi}}(\beta\cdot\alpha+(\beta\cdot\alpha)^*)$

(f) $\lim_{\eta\downarrow 0}(\lim_{\varepsilon\downarrow 0}\int_{\boldsymbol{R}}(\frac{1}{\lambda-i\eta}-\frac{1}{\lambda-i\varepsilon})\varkappa(d\lambda))=0$

(iv) for any $\zeta\in\boldsymbol{C}^+$

$$Z(\zeta)\equiv\beta-i\zeta-i\zeta\lim_{\varepsilon\downarrow 0}\int_{\boldsymbol{R}}\frac{1}{(\lambda-\zeta-i\varepsilon)(\lambda-i\varepsilon)}\varkappa(d\lambda)\cdot(\sqrt{2\pi}\alpha)^{-1} \in GL(d;\boldsymbol{C}) \tag{1.7}$$

(v) $\sup_{\eta>0}\int_{\boldsymbol{R}}\text{trace }(Z(\xi+i\eta)^{-1}\cdot\alpha+(Z(\xi+i\eta)^{-1}\cdot\alpha)^*)d\xi<\infty\}$.

In § 2 we shall review Mori's theory of generalized Brownian motion ([6]), which can be applied to the case where the stationary curve A is differentiable. We shall in § 3 proceed a sort of renormalization to the results in § 2 in order that we can treat the class $\mathscr{R}$. By using the result in § 3 through an approximation procedure, we shall in § 4 and § 5 show the following main theorems in this paper.

THEOREM 5.1 *There exists a bijective mapping Φ from $\mathscr{R}$ onto $\mathscr{L}$ such that*

$$[R](\zeta)=Z(\zeta)^{-1}\cdot\frac{\alpha}{\sqrt{2\pi}} \qquad \text{for any } \zeta\in\boldsymbol{C}^+,$$

where $R\in\mathscr{R}$ and $(\alpha,\beta,\varkappa)=\Phi(R)\in\mathscr{L}$.

Furthermore, we shall give a formula by which the triple $(\alpha,\beta,\varkappa)=\Phi(R)$ can be represented in terms of R.

THEOREM 5.2 *Let R and $(\alpha,\beta,\varkappa)$ be any element of $\mathscr{R}$ and $\mathscr{L}$, respectively, such that $\Phi(R)=(\alpha,\beta,\varkappa)$. Then*

(i) $\alpha=\frac{R(0)}{\sqrt{2\pi}}$

(ii) $\beta=R(0)\cdot D^{-1}$

(iii) $\varkappa(d\xi)=\frac{1}{(2\pi)^2}\lim_{\eta\downarrow 0}R(0)\cdot([R](\xi+i\eta)^{-1}+([R](\xi+i\eta)^{-1})^*)\cdot R(0)d\xi$ *in $\mathscr{D}'(\boldsymbol{R})$.*

Let R and $(\alpha,\beta,\varkappa)$ be any element of $\mathscr{R}$ and $\mathscr{L}$, respectively, such that $\Phi(R)=(\alpha,\beta,\varkappa)$. Then we call the triple $(\alpha,\beta,\varkappa)$ **a second KMO-Langevin data** associated with R.

REMARK 1.1 Let $(\alpha,\beta,\varkappa)\in\mathscr{L}$ be a second KMO-Langevin data associated with $R\in\mathscr{R}$. We note that, for $d=1$, the triple $(\alpha,\beta,\varkappa\cdot(\sqrt{2\pi}\alpha)^{-1})$ corresponds to the second KMO-Langevin data associated with R given in Definition 4.1 of [11].

Under the general framework carried out in § 5, we shall in § 6 characterize a set $\Phi(\mathscr{R}_{sm})$ of second KMO-Langevin data associated with elements of $\mathscr{H}_{sm}$ which were treated in § 3 (Theorem 6. 1):

$$\mathscr{R}_{sm}\equiv\{R\in\mathscr{R}\,;\int_{\boldsymbol{R}}\lambda^2(\Delta(d\lambda)u,u)_{C^d}<\infty \qquad \text{for any } u\in\boldsymbol{C}^d,$$

where Δ is a spectral measure matrix of $R\}$.
Furthermore, by combining Theorem 6. 1 with Theorems 5. 1 and 5. 2, we shall give a continued fraction expansion of the function $[R]$ for $R\in\mathscr{R}_{sm}$ (Theorems 6. 2 and 6. 3).

Finally, we shall in § 7 characterize a set $\Phi(\mathscr{R}_{ds})$ of second KMO-Langevin data associated with elements of $\mathscr{R}_{ds}$ which have spectral density matrices (Theorem 7. 1):

$$\mathscr{R}_{ds}\equiv\{R\in\mathscr{R}\,;\ R \text{ has a spectral density matrix}\}.$$

Simultaneously, the intersection $\Phi(\mathscr{R}_{sm})\cap\Phi(\mathscr{R}_{ds})=\Phi(\mathscr{R}_{sm}\cap\mathscr{R}_{ds})$ will be characterized (Theorem 7. 2).

In a forthcoming paper, we shall consider a stationary curve A in a Hilbert space $\mathscr{H}^d$ whose covariance matrix R belongs to the set $\mathscr{R}_{ds}$. And we shall derive a differential-integral equation which R satisfies and then an equation of motion describing the time evolution of the stationary curve A by introducing a Kubo noise as a random force, which will be called a second-KMO-Langevin equation. Furthermore, we shall find that Kubo's fluctuation-dissipation theorem holds on the basis of the second-KMO-Langevin equation, which gives a justification of the nomenclature of the $M(d\,;\boldsymbol{C})$-valued mobility function $[R](2\pi R(0))^{-1}$ and the second KMO-Langevin data $(\alpha,\beta,\varkappa)$.

§ 2. Mori's theory of generalized Brownian motion

Let $\mathscr{H}$ be a Hilbert space with an inner product $(\cdot,*)_{\mathscr{H}}$ and L a self-adjoint operator on $\mathscr{H}$. We denote by $(U(t)\,;t\in\boldsymbol{R})$ a strongly continuous one-parameter group of unitary operators on $\mathscr{H}$ whose infinitesimal generator is equal to iL:

$$(2.1) \qquad U(t)=e^{itL}.$$

Furthermore we are given d vectors $A_p\in\mathscr{H}$ $(1\leq p\leq d)$, where d is a fixed positive integer. Then we define a stationary curve $A=(A(t)\,;t\in\boldsymbol{R})$ in $\mathscr{H}^d$ by

$$(2.2)\qquad A(t)\equiv\begin{pmatrix}A_1(t)\\A_2(t)\\\vdots\\A_d(t)\end{pmatrix}=\begin{pmatrix}U(t)A_1\\U(t)A_2\\\vdots\\U(t)A_d\end{pmatrix}$$

and then a covariance matrix R_A on $\boldsymbol{R}$ by

$$(2.3)\qquad R_A(t)=((A_p(t),\ A_q(0))_{\mathscr{H}})_{1\le p,q\le d}.$$

In this section we suppose the following conditions:

$$(2.4)\qquad \{A_p;1\le p\le d\}\text{ is linearly independent in }\mathscr{H}$$

and

$$(2.5)\qquad \{A_p;1\le p\le d\}\subset\mathscr{D}(L).$$

We note that condition (2.4) is equivalent to the following condition:

$$(2.6)\qquad R_A(0)\in GL(d,\boldsymbol{C}).$$

It follows from (2.1), (2.2) and (2.5) that $A(t)$ satisfies the following equation:

$$(2.7)\qquad \dot{A}(t)=\begin{pmatrix}iLA_1(t)\\iLA_2(t)\\\vdots\\iLA_d(t)\end{pmatrix}\qquad\text{for any }t\in\boldsymbol{R},$$

where $\dot{A}(t)\equiv\frac{d}{dt}A(t)$.

We shall remind of Mori's theory of generalized Brownian motion ([6]). The object of Mori's theory is to rewrite a sort of partial differential equation (2.7) into an ordinary differential-integral equation, which is divided into a dissipative part and a fluctuating part from the point of view of a fundamental principle in statistical physics. And these parts are related each other through Mori's fluctuation-dissipation theorem.

Let $\mathscr{H}_0$ be the closed subspace generated by $\{A_p;1\le p\le d\}$ and $\mathscr{H}_1$ be the orthogonal complementary subspace of $\mathscr{H}_0$ in $\mathscr{H}$. We denote by P_0 the projection operator on $\mathscr{H}_0$. Then we define a linear operator L_1 in $\mathscr{H}_1$ by

$$(2.8)\qquad \begin{cases}\mathscr{D}(L_1)\equiv\mathscr{H}_1\cap\mathscr{D}(L)\\ L_1u\equiv(I-P_0)Lu\ \ (u\in\mathscr{D}(L_1)).\end{cases}$$

The fundamental point in the so-called projection method which is used in Mori's theory lies in the following

LEMMA 2.1 ([6]) *L_1 is a self-adjoint operator on $\mathscr{H}_1$.*

We define an element ω of $M(d;\boldsymbol{C})$ by

$$\omega=i^{-1}\dot{R}_A(0)\cdot R_A(0)^{-1}. \tag{2.9}$$

Furthermore, by virtue of Lemma 2.1, we can define a stationary curve $I_M=(I_M(t);t\in\boldsymbol{R})$ *in* $\mathscr{H}_1^d$ by

$$I_M(t)\equiv\begin{pmatrix} I_{M,1}(t)\\ I_{M,2}(t)\\ \vdots\\ I_{M,d}(t)\end{pmatrix}=\begin{pmatrix} V(t)(I-P_0)\dot{A}_1(0)\\ V(t)(I-P_0)\dot{A}_2(0)\\ \vdots\\ V(t)(I-P_0)\dot{A}_d(0)\end{pmatrix}, \tag{2.10}$$

where $(V(t);t\in\boldsymbol{R})$ is a strongly continuous one-parameter group of unitary operators on $\mathscr{H}_1$ whose infinitesimal generator is equal to iL_1:

$$V(t)=e^{itL_1}. \tag{2.11}$$

We define an $M(d;\boldsymbol{C})$-valued function ϕ_M on $\boldsymbol{R}$ by

$$\phi_M(t)=((I_{M,p}(t),\ I_{M,q}(0))_{\mathscr{H}_1})_{1\le p,q\le d}\cdot R_A(0)^{-1}. \tag{2.12}$$

Now we can state Mori's theory of generalized Brownian motion. Concerning the covariance matrix R_A, we have

THEOREM 2.1 ([6])

(i) *For any $t\in\boldsymbol{R}$*

$$\dot{R}_A(t)=i\omega\cdot R_A(t)-\int_0^t\phi_M(t-s)\cdot R_A(s)\,ds$$

(ii) *For any $\xi\in\boldsymbol{C}^+$*

$$\int_0^\infty e^{i\xi t}R_A(t)\,dt=(-i\omega-i\xi+\int_0^\infty e^{i\xi t}\phi_M(t)\,dt)^{-1}\cdot R_A(0).$$

Next, an equation of motion which describes the time evolution of $(A(t);t\in\boldsymbol{R})$ is given in

THEOREM 2.2 ([6]) *For any $t\in\boldsymbol{R}$*

$$\dot{A}(t)=i\omega\cdot A(t)-\int_0^t\phi_M(t-s)\cdot A(s)\,ds+I_M(t).$$

Furthermore, by (2.9), (2.10), (2.11) and (2.12), we have the following Mori's fluctuation-dissipation theorem.

THEOREM 2.3 ([6])

(i) $\omega = i^{-1}\dot{R}_A(0) \cdot R_A(0)^{-1}$

(ii) *For any* $t \in \boldsymbol{R}$ *and* $p, q \in \{1, 2, \cdots, d\}$

$(A_p(0), I_{M,q}(t))_{\mathscr{H}} = 0$

(iii) *For any* $s, t \in \boldsymbol{R}$

$((I_{M,p}(t), I_{M,q}(s))_{\mathscr{H}})_{1 \le p, q \le d} = \phi_M(t-s) \cdot R_A(0).$

DEFINITION 2.1 (i) An equation in Theorem 2.2 is said to be *Mori's memory kernel equation.*

(ii) We call ω, ϕ_M and I_M in Mori's memory kernel equation *frequency matrix, memory kernel matrix* and *Mori noise,* respectively.

Immediately from (2.7), Theorems 2.3 (i) and 2.3 (ii), we have

PROPOSITION 2.1

(i) $(\omega \cdot R_A(0))^* = \omega \cdot R_A(0)$

(ii) $\phi_M(\cdot) \cdot R_A(0)$ *is an* $M(d; \boldsymbol{C})$*-valued continuous and non-negative definite function on* $\boldsymbol{R}$.

§ 3. A renormalization of Mori's theory

In this section we shall treat the same situation as § 2 and call it a smooth case. By (2.3) and Proposition 2.1 (ii), we can apply Bochner's theorem to see that there exist two bounded Borel spectral measure matrices Δ and κ on $\boldsymbol{R}$ such that

$$(3.1) \qquad R_A(t) = \int_{\boldsymbol{R}} e^{-it\lambda} \Delta(d\lambda)$$

and

$$(3.2) \qquad \phi_M(t) \cdot R_A(0) = \int_{\boldsymbol{R}} \mathrm{e}^{-it\lambda} \kappa(\mathrm{d}\lambda).$$

We define two $M(d; \boldsymbol{C})$-valued holomorphic functions $[R_A]$ and $[\phi_M]$ on $\boldsymbol{R}$ by

$$(3.3) \qquad [R_A](\zeta) \equiv \frac{1}{2\pi} \int_0^\infty e^{i\zeta t} R_A(t)\, dt = \frac{1}{2\pi i} \int_0^\infty \frac{1}{\lambda - \zeta} \Delta(d\lambda)$$

and

$$(3.4) \qquad [\phi_M](\zeta) \equiv \frac{1}{2\pi} \int_0^\infty e^{i\zeta t} \phi_M(t)\, dt = \frac{1}{2\pi i} \int_{\boldsymbol{R}} \frac{1}{\lambda - \zeta} \kappa(d\lambda) \cdot R_A(0)^{-1}.$$

Besides conditions (2.4) and (2.5), we shall put the following condition (3.5) on R_A:

(3.5) $\quad D\equiv\lim_{\varepsilon\downarrow 0} 2\pi[R_A](i\varepsilon)$ exists and $D\in GL(d\,;\boldsymbol{C})$.

Then we define an element β of $GL(d\,;\boldsymbol{C})$ by

(3.6) $\quad \beta\equiv R_A(0)\cdot D^{-1}$.

Now, we see from Theorem 2.1 (ii), (3.5) and (3.6) that

LEMMA 3.1

(i) $[\phi_M](0+i0)\equiv\lim_{\varepsilon\downarrow 0}[\phi_M](i\varepsilon)$ *exists*

(ii) $-i\omega+2\pi[\phi_M](0+i0)=\beta$.

Next we define for each $\varepsilon>0$ an $M(d\,;\boldsymbol{C})$-valued function γ_ε on $\boldsymbol{R}$ by

(3.7) $$\gamma_\varepsilon(t)\equiv-\chi_{(0,\infty)}(t)\int_t^\infty e^{-\varepsilon s}\phi_M(s)\,ds.$$

A direct calculation yields

LEMMA 3.2

(i) *For any* $\varepsilon>0$ *and* $p,\ q\in\{1,2,\cdots,d\}$
$(\gamma_\varepsilon)_{pq}\in L^1(\boldsymbol{R})$

(ii) *For any* $\varepsilon>0$ *and* $\zeta\in\boldsymbol{C}^+$
$$(-i\zeta)\int_0^\infty e^{i\zeta t}\gamma_\varepsilon(t)\,dt=2\pi([\phi_M](\zeta+i\varepsilon)-[\phi_M](i\varepsilon))$$

(iii) *For any* $\varepsilon>0$ *and* $\zeta\in\boldsymbol{C}^+$
$$\int_0^\infty e^{i\zeta t}\gamma_\varepsilon(t)\,dt=\int_{\boldsymbol{R}}\frac{1}{(\lambda-\zeta-i\varepsilon)(\lambda-i\varepsilon)}\varkappa(d\lambda)\cdot R_A(0)^{-1}.$$

Furthermore we define for each $\varepsilon>0$ two $M(d\,;\boldsymbol{C})$-valued functions K_ε and L_ε on $\boldsymbol{C}^+\cup\boldsymbol{R}$ by

(3.8) $$K_\varepsilon(\zeta)\equiv\int_{\boldsymbol{R}}\frac{1}{(\lambda-\zeta-i\varepsilon)(\lambda-i\varepsilon)}\chi(d\lambda)\cdot R_A(0)^{-1}$$

and

(3.9) $\quad L_\varepsilon(\zeta)\equiv(-i\zeta)K_\varepsilon(\zeta)$.

It then follows from Lemmas 2.1 (i), 3.2 (ii) and 3.2 (iii) that

LEMMA 3.3

(i) *For each* $\varepsilon>0$

(a) K_ε *and* L_ε *are holomorphic on* $\boldsymbol{C}^+$

(b) K_ε *and* L_ε *are continuous on* $\boldsymbol{C}^+\cup\boldsymbol{R}$

(ii) *For any* $\zeta\in\boldsymbol{C}^+$

(a) $K(\zeta)\equiv\lim_{\varepsilon\downarrow 0} K_\varepsilon(\zeta)$ *exists*

(b) $L(\zeta)\equiv\lim_{\varepsilon\downarrow 0} L_\varepsilon(\zeta)$ *exists and* $L(\zeta)=(-i\zeta)K(\zeta)$.

Immediately from (3.8), (3.9) and Lemma 3.3 (ii) (b), we have

LEMMA 3.4

(i) *For any* $\varepsilon>0$ *and* $\zeta\in C^+$

$$L_\varepsilon(\zeta)=\frac{1}{i}\int_R\left(\frac{1}{\lambda-\zeta-i\varepsilon}-\frac{1}{\lambda-i\varepsilon}\right)\kappa(d\lambda)\cdot R_A(0)^{-1}$$

(ii) *For any* $\varepsilon>0$, $\zeta\in C^+$ *and* $\eta_0>0$

$$L_\varepsilon(\zeta)=L_\varepsilon(i\eta_0)+\frac{1}{i}\int_R\frac{\zeta-i\eta_0}{(\lambda-\zeta-i\varepsilon)(\lambda-i(\eta_0+\varepsilon))}\kappa(d\lambda)\cdot R_A(0)^{-1}$$

(iii) *For any* $\zeta\in C^+$ *and* $\eta_0>0$

$$L(\zeta)=L(i\eta_0)+\frac{1}{i}\int_R\frac{\zeta-i\eta_0}{(\lambda-\zeta)(\lambda-i\eta_0)}\kappa(d\lambda)\cdot R_A(0)^{-1}$$

(iv) *K and L are holomorphic on* C^+.

By using β in (3.6), we see from Lemmas 3.1 (ii), 3.2 (ii), 3.2 (iii) and 3.3 (ii) that Theorem 2.1 can be renormalized into the following

THEOREM 3.1 *For any* $\zeta\in C^+$

$$[R_A](\zeta)=(\beta-i\zeta+L(\zeta))^{-1}\cdot\frac{R_A(0)}{2\pi}.$$

For future use, we shall investigate some properties of the spectral measure matrix κ in (3.2).

LEMMA 3.5

(i) $\lim_{\eta\downarrow 0} L(i\eta)=\lim_{\eta\downarrow 0}\eta K(i\eta)=0$

(ii) *For any* $\zeta=\xi+i\eta\in C^+$

$$\frac{1}{\pi}\int_R\frac{\eta}{(\lambda-\xi)^2+\eta^2}\kappa(d\lambda)=[\phi_M](\zeta)\cdot R_A(0)+([\phi_M](\zeta)\cdot R_A(0))^*$$

(iii) $\lim_{\eta\downarrow 0}\frac{1}{\pi}\int_R\frac{\eta}{\lambda^2+\eta^2}\kappa(d\lambda)=\frac{1}{2\pi}(\beta\cdot R_A(0)+(\beta\cdot R_A(0))^*)$

(iv) $\kappa(\{0\})=0$.

RROOF (i) follows from (3.5), (3.6) and Lemma 3.3 (ii) (b). For the proof of (ii), let any $\zeta=\xi+i\eta\in C^+$ be fixed. Since κ is a bounded spectral measure matrix, we see from (3.2) that

$$\frac{1}{\pi}\int_R\frac{\eta}{(\lambda-\xi)^2+\eta^2}\kappa(d\lambda)$$

$$=\frac{1}{2\pi}\int_R\left(\int_0^\infty e^{-i(\lambda-\xi-i\eta)t}dt+\int_0^\infty e^{i(\lambda-\xi+i\eta)t}dt\right)\kappa(d\lambda)$$

$$=\frac{1}{2\pi}\left(\int_0^\infty e^{i(\xi+i\eta)t}\phi_M(t)\cdot R_A(0)\,dt+\int_0^\infty e^{i(-\xi+i\eta)t}(\phi_M(t)\cdot R_A(0))^*\,dt\right)$$
$$=[\phi_M](\xi)\cdot R_A(0)+([\phi_M](\xi)\cdot R_A(0))^*,$$

which gives (ii). By Lemma 3.1 (i),

$$(3.10)\qquad [\phi_M](0+i0)\cdot R_A(0)=\frac{1}{2\pi}(\beta\cdot R_A(0)+i\,\omega\cdot R_A(0)).$$

Therefore, combining (3.10) with Proposition 2.1 (i), we find that (iii) follows from (ii). In particular, (iii) implies (iv). (Q.E.D.)

§ 4. A mobility function $[R_A](2\pi R(0))^{-1}$ of a stationary curve A

In this section we shall consider a stationary curve $A=(A(t)\,;t\in\boldsymbol{R})$ in (2.2) with covariance matrix R_A in (2.3) satisfying conditions (2.6) and (3.5) only. The difference between § 3 and § 4 is that we do not suppose condition (2.5) in § 2. For this reason we define for each $n\in\boldsymbol{N}$ and $p\in\{1, 2, \cdots, d\}$ a vector $A_{n,p}$ in $\mathscr{H}$ by

$$(4.1)\qquad A_{n,p}\equiv n\int_0^\infty e^{-nt}U(t)A_p(0)\,dt$$

and a vector A_n in $\mathscr{H}^d$ by

$$(4.2)\qquad A_n\equiv\begin{pmatrix}A_{n,1}\\ A_{n,2}\\ \vdots\\ A_{n,d}\end{pmatrix}.$$

(**Step** 1) From a general theory of semi-groups of linear operators, we have

LEMMA 4.1
(i) *For each* $n\in\boldsymbol{N}$
$\{A_{n,p}\,;1\le p\le d\}$ *is linearly independent*
(ii) *For each* $n\in\boldsymbol{N}$
$\{A_{n,p}\,;1\le p\le d\}\subset\mathscr{D}(L)$
(iii) $\lim_{n\to\infty} A_n=A(0)$ *in* $\mathscr{H}^d$.

For each $n\in\boldsymbol{N}$ we define a covariance matrix R_n on $\boldsymbol{R}$ and an $M(d\,;\boldsymbol{C})$-valued holomorphic function $[R_n]$ on $\boldsymbol{C}^+$ by

(4.3) $$R_n(t)\equiv((U(t)A_{n,p}, A_{n,q})_{\mathscr{H}})_{1\le p,q\le d}$$

and

(4.4) $$[R_n](\xi)\equiv\frac{1}{2\pi}\int_0^{\infty}e^{i\xi t}R_n(t)\,dt.$$

Immediately from Lemma 4.1 (iii), we have

LEMMA 4.2

(i) $\lim_{n\to\infty} R_n(t)=R_A(t)$ *for any* $t\in\boldsymbol{R}$

(ii) $\lim_{n\to\infty}[R_n](\xi)=[R_A](\xi)$ *for any* $\xi\in\boldsymbol{C}^+$.

(**Step** 2) Similarly as Lemma 4.3 in [11] for one-dimensional case, we have

LEMMA 4.3 *For each* $n\in\boldsymbol{N}$ *and any* $\xi\in\boldsymbol{C}^+$ *with* Im $\xi\neq n$,

$$[R_n](\xi)=\frac{n}{2\pi(n-i\xi)}\{\frac{1}{2}\int_{\boldsymbol{R}}e^{-n|t|}R(t)+\frac{n}{n+i\xi}(2\pi[R_A](\xi)-\int_0^{\infty}e^{-nt}R(t)dt)\}.$$

By condition (3.5) and Lemma 4.3, we have

LEMMA 4.4

(i) *For each* $n\in\boldsymbol{N}$

$$D_n\equiv 2\pi\lim_{\varepsilon\downarrow 0}[R_n](i\varepsilon) \text{ exists}$$

(ii) $\lim_{n\to\infty} D_n=D$.

On the other hand, it is easy to see

LEMMA 4.5

$$\lim_{\eta\to\infty}\eta[R_A](i\eta)=\frac{R_A(0)}{2\pi}.$$

Therefore, by using conditions (2.6) and (3.5) again, we find from Lemmas 4.4 (ii) and 4.5 that

LEMMA 4.6 *There exist* $T_0>0$, $\varepsilon_0>0$ *and* $n_0\in\boldsymbol{N}$ *such that*

(i) $[R_A](i\eta)\in GL(d;\boldsymbol{C})$ *for any* $\eta\in(0,2\varepsilon_0]\cup[T_0,\infty)$

(ii) $D_n\in GL(d;\boldsymbol{C})$ *for any* $n\in\boldsymbol{N}\cap(n_0,\infty)$.

We define elements α, α_n, β and $\beta_n(n\geq n_0)$ of $GL(d;\boldsymbol{C})$ by

(4.5) $$\alpha\equiv\frac{R_A(0)}{\sqrt{2\pi}} \text{ and } \alpha_n\equiv\frac{R_n(0)}{\sqrt{2\pi}}$$

and

(4.6) $\beta \equiv R_A(0)D^{-1}$ and $\beta_n \equiv R_n(0)D_n^{-1}$.

It then follows from Lemmas 4.2 (i) and 4.4 (i) that

LEMMA 4.7
(i) $\alpha^* = \alpha$ *and* $\alpha_n^* = \alpha_n$ $(n \geq n_0)$
(ii) $\lim_{n\to\infty} \alpha_n = \alpha$
(iii) $\lim_{n\to\infty} \beta_n = \beta$.

(**Step** 3) By virtue of Lemmas 4.1 (i), 4.4 (i) and (4.6) (ii), we can apply Theorem 2.1 (ii), Proposition 2.1, (3.2), Lemmas 3.1 and 3.5 (iv) to stationary curves A_n $(n \geq n_0)$ in order to find that

LEMMA 4.8 *There exist for each* $n \geq n_0$ *a frequency matrix* ω_n, *a memory kernel matrix* ϕ_n *and a bounded Borel spectral measure matrix* $\varkappa_n$ *such that*
(i) $(\omega_n \cdot R_n(0))^* = \omega_n \cdot R_n(0)$
(ii) $\phi_n(t) R_n(0) = \int_{\boldsymbol{R}} e^{-it\lambda} \varkappa_n(d\lambda)$ $\quad (t \in \boldsymbol{R})$
(iii) $\varkappa_n(\{0\}) = 0$
(iv) $\varkappa_n^* = \varkappa_n$
(v) *for any* $\zeta \in \boldsymbol{C}^+$
$$-i\omega_n - i\zeta + 2\pi[\phi_n](\zeta) = \frac{\alpha_n}{\sqrt{2\pi}} \cdot [R_n](\zeta)^{-1}$$
(vi) $-i\omega_n + 2\pi[\phi_n](0+i0) = \beta_n$
(vii) *for any* $\zeta \in \boldsymbol{C}^+$
$$\beta_n - i\zeta + 2\pi([\phi_n](\zeta) - [\phi_n](0+i0)) = \frac{\alpha_n}{\sqrt{2\pi}} \cdot [R_n](\zeta)^{-1}.$$

(**Step** 4) We define for each $n \geq n_0$ a bounded Borel symmetric spectral measure matrix $\varkappa_n^{(s)}$ on $\boldsymbol{R}$ by

(4.7) $\varkappa_n^{(s)}(d\lambda) \equiv \frac{1}{2}(\varkappa_n(d\lambda) + \varkappa_n(-d\lambda))$.

LEMMA 4.9
(i) *For any* $t \in \boldsymbol{R}$
$$\frac{1}{2}\{\phi_n(t) \cdot R_n(0) + (\phi_n(t) \cdot R_n(0))^*\} = \int_{\boldsymbol{R}} e^{-it\lambda} \varkappa_n^{(s)}(d\lambda)$$
(ii) *For any* $\zeta \in \boldsymbol{C}^+$
$$\pi\{[\phi_n](\zeta) \cdot R_n(0) + ([\phi_n](\zeta) \cdot R_n(0))^*\}$$

$$=(-i\xi)\int_{\boldsymbol{R}}\frac{1}{(\lambda-\xi)(\lambda+\xi)}\kappa_n^{(s)}(d\lambda)$$

(iii) *For any* $\eta>0$

$$\pi\{[\phi_n](i\eta)\cdot R_n(0)+([\phi_n](i\eta)\cdot R_n(0))^*\}=\int_{\boldsymbol{R}}\frac{\eta}{\lambda^2+\eta^2}\kappa_n^{(s)}(d\lambda)$$

(iv) *For any* $\eta>0$

$$\begin{aligned}&2\pi\{[\phi_n](i\eta)\cdot R_n(0)+([\phi_n](i\eta)\cdot R_n(0))^*\}\\&\quad=\alpha_n\cdot([R_n](i\eta))^{-1}\cdot\alpha_n+(\alpha_n\cdot([R_n](i\eta))^{-1}\cdot\alpha_n)^*-2\eta\sqrt{2\pi}\ \alpha_n.\end{aligned}$$

PROOF (i) follows from Lemmas 4.8 (ii) and 4.8 (iv). By (i), we see that

$$\begin{aligned}\text{the left hand side of (ii)}&=\frac{1}{i}\int_{\boldsymbol{R}}\frac{1}{\lambda-\xi}\kappa_n^{(s)}(d\lambda)\\&=\frac{1}{2i}\left(\int_{\boldsymbol{R}}\frac{1}{\lambda-\xi}\kappa_n^{(s)}(d\lambda)+\int_{\boldsymbol{R}}\frac{1}{\lambda-\xi}\kappa_n^{(s)}(-d\lambda)\right)\\&=\text{the right hand side of (ii).}\end{aligned}$$

In particular, (iii) follows immediately from (ii). By operating both hand sides of (iii) for $\xi=i\eta$ to $R_n(0)$ and then noting (4.5), we have

$$-i\omega_n\cdot R_n(0)+\eta\sqrt{2\pi}\ \alpha_n+2\pi[\phi_n](i\eta)\cdot R_n(0)=\alpha_n\cdot[R_n](i\eta)^{-1}\cdot\alpha_n. \tag{4.8}$$

By taking the adjoint of both hands sides of (4.8) and then noting Lemmas 4.7 (i) and 4.8 (i), we have

$$i\omega_n\cdot R_n(0)+\eta\sqrt{2\pi}\ \alpha_n+2\pi([\phi_n](i\eta)\cdot R_n(0))^*=(\alpha_n\cdot[R_n](i\eta)^{-1}\alpha_n)^*. \tag{4.9}$$

Therefore, by summing up (4.8) and (4.9), we obtain (iv). (Q. E. D.)

(**Step** 5) Let any $u={}^t(u_1, u_2, \cdots, u_d)\in\boldsymbol{C}^d$ be fixed. We define for each $n\geq n_0$ a bounded Borel measure $\tilde{\kappa}_{u,n}^{(s)}(d\lambda)$ on $[-\infty,\infty]$ by

$$\tilde{\kappa}_{u,n}^{(s)}(d\lambda)\equiv\frac{1}{\varepsilon_0^2+\lambda^2}\sum_{p,q=1}^{d}\bar{u}_p u_q(\kappa_n^{(s)})_{pq}((-\infty,\infty)\cap d\lambda). \tag{4.10}$$

LEMMA 4.10

(i) $\sup_{n\geq n_0}\tilde{\kappa}_{u,n}^{(s)}([-\infty,\infty])<\infty$

(ii) *There exist a subsequence* $(n_k\,;k\in\boldsymbol{N})(\lim_{k\to\infty}n_k=\infty)$ *and a bounded Borel measure* $\tilde{\kappa}_u$ *on* $[-\infty,\infty]$ *such that*

$$w-\lim_{k\to\infty}\tilde{\kappa}_{u,n_k}^{(s)}=\tilde{\kappa}_u \text{ on } [-\infty,\infty].$$

PROOF By Lemmas 4.9 (iii) and (iv), we have

$$(4.11)\quad 2\varepsilon_0\tilde{\kappa}^{(s)}_{u,n}([-\infty,\infty])=((\alpha_n\cdot[R_n](i\varepsilon_0)^{-1}\cdot\alpha_n+(\alpha_n\cdot[R_n](i\varepsilon_0)^{-1}\cdot\alpha_n)^* -2\varepsilon_0\sqrt{2\pi}\ \alpha_n)u,\ u)_{C^d}.$$

Therefore, by Lemmas 4.2 (ii), 4.6 (i) and 4.7 (ii), we find that (4.11) implies (i). Since for any $c>0$ $\{m$; Borel measure on $[-\infty,\infty]$ with $m([-\infty,\infty])\leq c\}$ is relative compact, (ii) follows immediately from (i).

(Q. E. D.)

LEMMA 4.11 $\tilde{\kappa}_u(\{-\infty, 0, +\infty\})=0$

PROOF For each $\eta>0$ we define a bounded continuous function $g(\cdot\,;\eta)$ on $[-\infty,\infty]$ by

$$(4.12)\quad \begin{cases} g(\lambda\,;\eta)\equiv\eta\cdot\dfrac{\lambda^2+\varepsilon_0^2}{\lambda^2+\eta^2} & \text{for } \lambda\in\boldsymbol{R} \\ g(\pm\infty\,;\eta)\equiv\eta. \end{cases}$$

By (4.10) and Lemma 4.10 (ii), we have, for any $\eta>0$,

$$(4.13)\quad \int_{[-\infty,\infty]} g(\lambda\,;\eta)\tilde{\kappa}_u(d\lambda)=\lim_{k\to\infty}\sum_{p,q=1}^{d}\bar{u}_p u_q\int_{\boldsymbol{R}}\frac{\eta}{\lambda^2+\eta^2}(\kappa^{(s)}_{n_k})_{pq}(d\lambda).$$

Therefore, by Lemmas 4.2 (ii), 4.6 (i), 4.7 (ii), 4.9 (iii) and 4.9 (iv), we see from (4.12) and (4.13) that for any $\eta\in(0,2\varepsilon_0]\cup[T_0,\infty)$

$$(4.14)\quad \eta\tilde{\kappa}_u(\{-\infty,+\infty\})+\frac{1}{\eta}\tilde{\kappa}_u(\{0\})+\eta\int_{\boldsymbol{R}-\{0\}}\frac{\lambda^2+\varepsilon_0^2}{\lambda^2+\eta^2}\tilde{\kappa}_u(d\lambda)$$
$$=\frac{1}{2}((\alpha\cdot[R_A](i\eta)^{-1}\cdot\alpha+(\alpha\cdot[R_A](i\eta)^{-1}\cdot\alpha)^* -2\eta\sqrt{2\pi}\alpha)u,\ u)_{C^d}.$$

By dividing both hand sides of (4.14) by η, for any $\eta\in(0,2\varepsilon_0]\cup[T_0,\infty)$,

$$(4.15)\quad \tilde{\kappa}_u(\{-\infty,\infty\})+\frac{1}{\eta^2}\tilde{\kappa}_u(\{0\})+\int_{\boldsymbol{R}-\{0\}}\frac{\lambda^2+\varepsilon_0^2}{\lambda^2+\eta^2}\tilde{\kappa}_u(d\lambda)$$
$$=\frac{1}{2}((\alpha\cdot(\eta[R_A](i\eta))^{-1}\cdot\alpha+(\alpha\cdot(\eta[R_A](i\eta))^{-1}\cdot\alpha)^* -2\sqrt{2\pi}\ \alpha)u,\ u)_{C^d}.$$

Therefore, by letting tend η to infinity in (4.15), we see from Lemma 4.5, (4.5) and Lemma 4.7 (i) that $\tilde{\kappa}_u(\{-\infty,\infty\})=0$. On the other hand, by multiplying both hand sides of (4.14) by η, we have

$$(4.16)\quad \tilde{\kappa}_u(\{0\})\leq\frac{\eta}{2}(\alpha\cdot[R_A](i\eta)^{-1}\cdot\alpha+(\alpha\cdot[R_A](i\eta)^{-1}\cdot\alpha)^* -2\eta\sqrt{2\pi}\ \alpha)u,\ u)_{C^d}.$$

Therefore, by letting η tend to zero in (4.16), we see from condition (3.5) and Lemma 4.6 (i) that $\tilde{\kappa}_u(\{0\})=0$. Thus we have completed the proof of Lemma 4.11. (Q.E.D.)

(**Step** 6) Let any $u={}^t(u_1, u_2, \cdots, u_d)\in \boldsymbol{C}^d$ be fixed. We define for each $n\geq n_0$ two bounded Borel measures $\tilde{\kappa}^{(+)}_{u,n}$ and $\tilde{\kappa}^{(-)}_{u,n}$ on $[0, +\infty]$ by

$$(4.17)\qquad \tilde{\kappa}^{(+)}_{u,n}(B)\equiv\sum_{p,q=1}^{d}\bar{u}_p u_q\int_{B\cap[0,\infty]}\frac{1}{\varepsilon_0^2+\lambda^2}(\kappa_n)_{pq}(d\lambda)$$

and

$$(4.18)\qquad \tilde{\kappa}^{(-)}_{u,n}(B)\equiv\sum_{p,q=1}^{d}\bar{u}_p u_q\int_{B\cap[0,\infty]}\frac{1}{\varepsilon_0^2+\lambda^2}(\kappa_n)_{pq}(-d\lambda)$$

for any Borel set B in $[0,\infty]$.

LEMMA 4.12

(i) $\frac{1}{2}(\tilde{\kappa}^{(+)}_{u,n}+\tilde{\kappa}^{(-)}_{u,n})\leq\tilde{\kappa}^{(s)}_{u,n}$

(ii) *There exist a subsequence* $(m_k\,;k\in\boldsymbol{N})$ *of* $(n_k\,;k\in\boldsymbol{N})$ *in Lemma 4.10* (ii) $(\lim_{k\to\infty} m_k=\infty)$, *two bounded Borel measures* $\tilde{\kappa}^{(+)}_u$ *and* $\tilde{\kappa}^{(-)}_u$ *on* $[0,\infty]$ *such that*

(a) $w-\lim_{k\to\infty}\tilde{\kappa}^{(+)}_{u,m_k}=\tilde{\kappa}^{(+)}_u$

(b) $w-\lim_{k\to\infty}\tilde{\kappa}^{(-)}_{u,m_k}=\tilde{\kappa}^{(-)}_u$.

PROOF (i) follows from (4.7), (4.10), (4.17) and (4.18). By (i) and Lemma 4.10 (i), we have (ii), similarly as Lemma 4.10 (ii). (Q.E.D.)

LEMMA 4.13

(i) $\tilde{\kappa}^{(+)}_u(\{0,\infty\})=0$

(ii) $\tilde{\kappa}^{(-)}_u(\{0,\infty\})=0$.

PROOF Let f be any non-negative element of $C([-\infty,\infty])$. By Lemmas 4.10 (ii), 4.12 (i) and 4.12 (ii), we have

$$(4.19)\qquad \int_{[-\infty,\infty]}fd\tilde{\kappa}_u=\lim_{k\to\infty}\int_{[-\infty,\infty]}fd\tilde{\kappa}^{(s)}_{u,m_k}$$
$$\geq\lim_{k\to\infty}\int_{[0,\infty]}fd\tilde{\kappa}^{(s)}_{u,m_k}$$

$$\geq \frac{1}{2} \varliminf_{k\to\infty} \left(\int_{[0,\infty]} f d\tilde{\varkappa}^{(+)}_{u, m_k} + \int_{[0,\infty]} f d\tilde{\varkappa}^{(-)}_{u, m_k} \right)$$
$$= \frac{1}{2} \left(\int_{[0,\infty]} f d\tilde{\varkappa}^{(+)}_{u} + \int_{[0,\infty]} f d\tilde{\varkappa}^{(-)}_{u} \right).$$

Therefore, by taking a sequence $(f_n; n\in \boldsymbol{N})$ in $C([-\infty,\infty])$ such that $0\leq f_n \leq 1$ and $\lim_{n\to\infty} f_n = \chi_{\{0,\infty\}}$, we find from (4.19) that

$$\frac{1}{2}(\tilde{\varkappa}^{(+)}_{u}(\{0,\infty\}) + \tilde{\varkappa}^{(-)}_{u}(\{0,\infty\}) \leq \tilde{\varkappa}_{u}(\{0,\infty\}),$$

which completes the proof of Lemma 4.13, by noting Lemma 4.11.(Q.E.D.)

(**Step** 7) Let any $u={}^t(u_1, u_2, \cdots, u_d)\in \boldsymbol{C}^d$ be fixed. We define bounded Borel measures $\tilde{\varkappa}_{u,n}(n\geq n_0)$, $\tilde{\tilde{\varkappa}}_u$ on $[-\infty,\infty]$ and a Borel measure $\varkappa_u$ on $\boldsymbol{R}$ by

$$(4.20) \qquad \tilde{\varkappa}_{u,n}(B) \equiv \sum_{p,q=1}^{d} \bar{u}_p u_q \int_{B\cap \boldsymbol{R}} \frac{1}{\varepsilon_0^2+\lambda^2} (\varkappa_n)_{pq}(d\lambda)$$

$$(4.21) \qquad \tilde{\tilde{\varkappa}}_u(B) \equiv \tilde{\varkappa}^{(+)}_u(B\cap[0,\infty]) + \tilde{\varkappa}^{(-)}_u((-B)\cap[0,\infty])$$

for any Borel set B in $[-\infty,\infty]$, and

$$(4.22) \qquad \varkappa_u(d\lambda) \equiv (\varepsilon_0^2+\lambda^2)\tilde{\tilde{\varkappa}}_u(\boldsymbol{R}\cap d\lambda).$$

LEMMA 4.14

(i) $\tilde{\varkappa}_{u,n}(\{0\})=0$

(ii) $\tilde{\tilde{\varkappa}}_u(\{-\infty, 0, \infty\})=0$

(iii) $w-\lim_{k\to\infty} \tilde{\varkappa}_{u,m_k} = \tilde{\tilde{\varkappa}}_u$

(iv) $\varkappa_u(\{0\})=0$

(v) $\int_{\boldsymbol{R}} \frac{1}{1+\lambda^2} \varkappa_u(d\lambda) < \infty.$

PROOF (i) follows from Lemma 4.8 (iii) and (4.20). (ii) follows from Lemma 4.13 and (4.21). For the proof of (iii), let f be any element of $C([-\infty,\infty])$. By noting (i), we see from (4.17), (4.18) and (4.20) that

$$(4.23) \qquad \int_{[-\infty,\infty]} f d\tilde{\varkappa}_{u,n} = \int_{[-\infty,0]} f d\tilde{\varkappa}_{u,n} + \int_{[0,\infty]} f d\tilde{\varkappa}_{u,n}$$
$$= \int_{[0,\infty]} f(-\lambda) \tilde{\varkappa}^{(-)}_{u,n}(d\lambda) + \int_{[0,\infty]} f(\lambda) \tilde{\varkappa}^{(+)}_{u,n}(d\lambda).$$

Therefore, by letting n tend to infinity along the sequence $(m_k; k\in \boldsymbol{N})$ in

(4.23), we find from Lemma 4.10 (ii), (4.21) and Lemma 4.14 (ii) that

$$\begin{aligned}\lim_{k\to\infty}\int_{[-\infty,\infty]} fd\tilde{\varkappa}_{u,m_k} &= \int_{[0,\infty]} f(-\lambda)\tilde{\varkappa}_u^{(-)}(d\lambda)+\int_{[0,\infty]} f(\lambda)\tilde{\varkappa}_u^{(+)}(d\lambda) \\ &= \int_{[-\infty,0]} f(\lambda)\tilde{\tilde{\varkappa}}_u(d\lambda)+\int_{[0,\infty]} f(\lambda)\tilde{\tilde{\varkappa}}_u(d\lambda) \\ &= \int_{[-\infty,\infty]} f(\lambda)\tilde{\tilde{\varkappa}}_u(d\lambda),\end{aligned}$$

which gives (iii). (iv) and (v) follow immediately from (4.22) and Lemma 4.14 (ii). (Q.E.D.)

(**Step** 8) We define a finite subset $E=\{e_p, e_{pq}, \tilde{e}_{pq}\,;1\le p<q\le d\}$ of $\boldsymbol{C}^d$ by

(4.24) $e_p\equiv{}^t(0,\cdots,0,\underset{\widehat{p}}{1},0,\cdots,0)$

(4.25) $e_{pq}\equiv e_p+e_q$

and

(4.26) $\tilde{e}_{pq}\equiv e_p+ie_q.$

Furthermore we choose simultaneously the subsequence $(n_k\,;k\in\boldsymbol{N})$ in Lemma 4.10 (ii) and the subsequence $(m_k\,;k\in\boldsymbol{N})$ in Lemma 4.12 (ii) for elements of E. Then we define $\boldsymbol{C}$-valued signed Borel measures $\varkappa_{pp}$, $\varkappa_{pq}$ and $\varkappa_{qp}$ $(1\le p<q\le d)$ on $\boldsymbol{R}$ by

(4.27) $\varkappa_{pp}\equiv\varkappa_{e_p}$

(4.28) $\varkappa_{pq}\equiv\frac{1}{2}\{\varkappa_{e_{pq}}-\varkappa_{e_p}-\varkappa_{e_q}-i(\varkappa_{\tilde{e}_{pq}}-\varkappa_{e_p}-\varkappa_{e_q})\}$

and

(4.29) $\varkappa_{qp}\equiv\bar{\varkappa}_{pq}$

And we define $\varkappa$ by

(4.30) $\varkappa\equiv(\varkappa_{pq})_{1\le p,q\le d}.$

LEMMA 4.15 *For any element f of $C([-\infty,\infty])$*

$$\lim_{k\to\infty}\int_{\boldsymbol{R}}\frac{f(\lambda)}{\lambda^2+\varepsilon_0^2}\varkappa_{m_k}(d\lambda)=\int_{\boldsymbol{R}}\frac{f(\lambda)}{\lambda^2+\varepsilon_0^2}\varkappa(d\lambda).$$

PROOF From the manner of choosing a subsequence $(m_k\,;k\in\boldsymbol{N})$, we see from (4.20), Lemmas 4.14 (ii) and 4.14 (iii) that for any $v={}^t(v_1,v_2,\cdots,v_d)$ of E

$$\lim_{k\to\infty}\sum_{p,q=1}^{d}\bar{v}_p v_q\int_{\boldsymbol{R}}\frac{f(\lambda)}{\lambda^2+\varepsilon_0^2}(\varkappa_{m_k})_{pq}(d\lambda)=\int_{\boldsymbol{R}}f(\lambda)\tilde{\tilde{\varkappa}}_v(d\lambda).$$

In particular, we have, for $1\leq p<q\leq d$,

$$(4.31)\quad \lim_{k\to\infty}\int_{\boldsymbol{R}}\frac{f(\lambda)}{\lambda^2+\varepsilon_0^2}(\varkappa_{m_k})_{pp}(d\lambda)=\int_{\boldsymbol{R}}f(\lambda)\tilde{\tilde{\varkappa}}_{e_p}(d\lambda)$$

$$(4.32)\quad \lim_{k\to\infty}\int_{\boldsymbol{R}}\frac{f(\lambda)}{\lambda^2+\varepsilon_0^2}\{(\varkappa_{m_k})_{pp}+(\varkappa_{m_k})_{qq}+(\varkappa_{m_k})_{pq}+(\varkappa_{m_k})_{qp}\}(d\lambda)$$
$$=\int_{\boldsymbol{R}}f(\lambda)\tilde{\tilde{\varkappa}}_{e_{pq}}(d\lambda)$$

and

$$(4.33)\quad \lim_{k\to\infty}\int_{\boldsymbol{R}}\frac{f(\lambda)}{\lambda^2+\varepsilon_0^2}\{(\varkappa_{m_k})_{pp}+(\varkappa_{m_k})_{qq}+i(\varkappa_{m_k})_{pq}-i(\varkappa_{m_k})_{qp}\}(d\lambda)$$
$$=\int_{\boldsymbol{R}}f(\lambda)\tilde{\tilde{\varkappa}}_{\tilde{e}_{pq}}(d\lambda).$$

It then follows from (4. 31), (4. 32) and (4. 33) that for $1\leq p<q\leq d$

$$(4.34)\quad \lim_{k\to\infty}\int_{\boldsymbol{R}}\frac{f(\lambda)}{\lambda^2+\varepsilon_0^2}(\varkappa_{m_k})_{pq}(d\lambda)$$
$$=\int_{\boldsymbol{R}}f(\lambda)\frac{1}{2}\{\tilde{\tilde{\varkappa}}_{e_{pq}}-\tilde{\tilde{\varkappa}}_{e_p}-\tilde{\tilde{\varkappa}}_{e_q}-i(\tilde{\tilde{\varkappa}}_{\tilde{e}_{pq}}-\tilde{\tilde{\varkappa}}_{e_p}-\tilde{\tilde{\varkappa}}_{e_q})\}(d\lambda)$$

$$(4.35)\quad \lim_{k\to\infty}\int_{\boldsymbol{R}}\frac{f(\lambda)}{\lambda^2+\varepsilon_0^2}(\varkappa_{m_k})_{qp}(d\lambda)$$
$$=\int_{\boldsymbol{R}}f(\lambda)\frac{1}{2}\{\tilde{\tilde{\varkappa}}_{e_{pq}}-\tilde{\tilde{\varkappa}}_{e_p}-\tilde{\tilde{\varkappa}}_{e_q}+i(\tilde{\tilde{\varkappa}}_{\tilde{e}_{pq}}-\tilde{\tilde{\varkappa}}_{e_p}-\tilde{\tilde{\varkappa}}_{e_q})\}(d\lambda).$$

Therefore, we can see from (4. 22), (4. 27), (4. 28), (4. 29), (4. 31), (4.34) and (4. 35) that Lemma 4. 15 holds. (Q. E. D.)

LEMMA 4. 16 *For any* $u={}^t(u_1, u_2, \cdots, u_d)$ *of* $\boldsymbol{C}^d$

$$\sum_{p,q=1}^{d}\bar{u}_p u_q\varkappa_{pq}=\varkappa_u.$$

PROOF By choosing the subsequence $(m_k\,;k\in\boldsymbol{N})$ in Lemma 4. 12 (ii) for u from the subsequence $(m_k\,;k\in\boldsymbol{N})$ in Lemma 4. 15, we find from (4. 20), (4. 22), Lemmas 4. 14 (iii) and 4. 15 that for any $f\in C([-\infty,\infty])$

$$\int_{\boldsymbol{R}}\frac{f(\lambda)}{\lambda^2+\varepsilon_0^2}\Big(\sum_{p,q=1}^{d}\bar{u}_p u_q\varkappa_{pq}\Big)(d\lambda)=\int_{\boldsymbol{R}}\frac{f(\lambda)}{\lambda^2+\varepsilon_0^2}\varkappa_u(d\lambda),$$

which completes the proof of Lemma 4. 14. (Q. E. D).

By Lemmas 4. 8 (iii), 4. 8 (iv) and 4. 15, we obtain

LEMMA 4.17
(i) $\varkappa(\{0\})=0$
(ii) $\varkappa^*=\varkappa$.

(**Step** 9) We define for each $n\geq n_0$ and $\varepsilon>0$ an $M(d;\boldsymbol{C})$-valued holomorphic function $L_{n,\varepsilon}$ *on* $\boldsymbol{C}^+\cup\boldsymbol{R}$ by

$$(4.36)\qquad L_{n,\varepsilon}(\zeta)\equiv(-i\zeta)\int_{\boldsymbol{R}}\frac{1}{(\lambda-\zeta-i\varepsilon)(\lambda-i\varepsilon)}\varkappa_n(d\lambda)\cdot(\sqrt{2\pi}\ \alpha_n)^{-1}.$$

Immediately from Lemma 4.8 (ii), we have

LEMMA 4.18 *For any* $n\geq n_0$, $\varepsilon>0$ *and* $\zeta\in\boldsymbol{C}^+$
$L_{n,\varepsilon}(\zeta)=2\pi([\phi_n](\zeta+i\varepsilon)-[\phi_n](i\varepsilon))$.

Now we shall show the following key lemma.

LEMMA 4.19 *For any* $\eta>0$ *with* $0<\eta\leq\varepsilon_0$

$$\lim_{n\to\infty}(\lim_{\varepsilon\downarrow 0}L_{n,\varepsilon}(i\eta))=\lim_{\varepsilon\downarrow 0}(\lim_{n\to\infty}L_{n,\varepsilon}(i\eta)).$$

PROOF By Lemmas 4.8 (vii) and 4.18, we see that for any $n\geq n_0$

$$(4.37)\qquad \lim_{\varepsilon\downarrow 0}L_{n,\varepsilon}(i\eta)=\sqrt{2\pi}\ \alpha_n\cdot[R_n](i\eta)^{-1}-\beta_n-\eta I$$

and so by Lemmas 4.2 (ii), 4.6 (i), 4.7 (ii) and 4.7 (iii),

$$(4.38)\qquad \lim_{n+\infty}(\lim_{\varepsilon\downarrow 0}L_{n,\varepsilon}(i\eta))=\sqrt{2\pi}\ \alpha\cdot[R](i\eta)^{-1}-\beta-\eta I.$$

On the other hand, by using Lemmas 4.8 (vii) and 4.18 again, we see that for any $n\geq n_0$ and $\varepsilon>0$

$$\begin{aligned}(4.39)\qquad L_{n,\varepsilon}(i\eta)&=\{(\beta_n-i(i(\eta+\varepsilon))+2\pi([\phi_n](i(\eta+\varepsilon))-[\phi_n](0+i0)))\\&\qquad-(\beta_n-i(i\varepsilon)+2\pi([\phi_n](i\varepsilon)-[\phi_n](0+i0))\}-\eta I\\&=\sqrt{2\pi}\ \alpha_n\cdot([R_n](i(\eta+\varepsilon))^{-1}-[R_n](i\varepsilon)^{-1})-\eta I.\end{aligned}$$

Therefore, by using Lemmas 4.2 (ii), 4.6 (i) and 4.7 (ii) again and letting n tend to infinity in (4.39) for any fixed $\varepsilon>0$

$$(4.40)\qquad \lim_{n\to\infty}L_{n,\varepsilon}(i\eta)=\sqrt{2\pi}\ \alpha\cdot([R](i(\eta+\varepsilon))^{-1}-[R](i\varepsilon)^{-1})-\eta I.$$

Furthermore, by letting ε tend to zero in (4.40), we see from (4.5) and (4.6) that

(4.41) $\lim_{\varepsilon \downarrow 0}(\lim_{n+\infty} L_{n,\varepsilon}(i\eta)) = \sqrt{2\pi}\ \alpha \cdot [R](i\eta)^{-1} - \beta - \eta I.$

Thus, by (4.38) and (4.41), we have lemma 4.19. (Q.E.D.)

(**Step** 10) We define for each $\varepsilon > 0$ two $M(d\,;\boldsymbol{C})$-valued functions K_ε and L_ε on $\boldsymbol{C}^+ \cup \boldsymbol{R}$ by

(4.42) $$K_\varepsilon(\zeta) \equiv \int_{\boldsymbol{R}} \frac{1}{(\lambda - \zeta - i\varepsilon)(\lambda - i\varepsilon)} \varkappa(d\lambda) \cdot (\sqrt{2\pi}\ \alpha)^{-1}$$

and

(4.43) $$L_\varepsilon(\zeta) \equiv (-i\zeta) K_\varepsilon(\zeta).$$

By Lemmas 4.2 (i) and 4.15, we have

Lemma 4.20 *For any* $\varepsilon > 0$ *and* $\zeta \in \boldsymbol{C}^+$

$$\lim_{k \to \infty} L_{m_k,\varepsilon}(\zeta) = L_\varepsilon(\zeta).$$

Similarly as Lemmas 3.4 (i) and 3.4 (ii), it follows from (4.42) and (4.43) that

Lemma 4.21

(i) *For any* $\varepsilon > 0$ *and* $\zeta \in \boldsymbol{C}^+$

$$L_\varepsilon(\zeta) = \frac{1}{i} \int_{\boldsymbol{R}} \left(\frac{1}{\lambda - \zeta - i\varepsilon} - \frac{1}{\lambda - i\varepsilon} \right) \varkappa(d\lambda) \cdot (\sqrt{2\pi}\ \alpha)^{-1}$$

(ii) *For any* $\varepsilon > 0$ *and* $\zeta \in \boldsymbol{C}^+$

$$L_\varepsilon(\zeta) = L_\varepsilon(i\varepsilon_0) + \frac{1}{i} \int_{\boldsymbol{R}} \frac{\zeta - i\varepsilon_0}{(\lambda - \zeta - i\varepsilon)(\lambda - i(\varepsilon_0 + \varepsilon))} \varkappa(d\lambda) \cdot (\sqrt{2\pi}\ \alpha)^{-1}.$$

By combining (4.41) with Lemmas 4.20 and 4.21 (ii), we can define an $M(d\,;\boldsymbol{C})$-valued function L on $\boldsymbol{C}^+$ by

(4.44) $$L(\zeta) \equiv \lim_{\varepsilon \downarrow 0} L_\varepsilon(\zeta).$$

By noting Lemmas 4.14 (v) and 4.16, we can show that

Lemma 4.22

(i) *For any* $\zeta \in \boldsymbol{C}^+$

$$L(\zeta) = L(i\varepsilon_0) + \frac{1}{i} \int_{\boldsymbol{R}} \frac{\zeta - i\varepsilon_0}{(\lambda - \zeta)(\lambda - i\varepsilon_0)} \varkappa(d\lambda) \cdot (\sqrt{2\pi}\ \alpha)^{-1}$$

(ii) *L is holomorphic on* $\boldsymbol{C}^+$.

Now we shall show the following main lemma.

Lemma 4.23 *For any* $\zeta \in \boldsymbol{C}^+$

(i) $\beta - i\xi + L(\xi) \in GL(d\,; \boldsymbol{C})$

(ii) $[R_A](\xi) = (\beta - i\xi + L(\xi))^{-1} \cdot \frac{\alpha}{\sqrt{2\pi}}$.

PROOF We define an $M(d\,;\boldsymbol{C})$-valued function F on $\boldsymbol{C}^+$ by

$$F(\xi) \equiv (\beta - i\xi + L(\xi)) \cdot [R_A](\xi). \tag{4.45}$$

It follows from Lemma 4.22 (ii) that F is holomorphic on $\boldsymbol{C}^+$. By Lemmas 4.2 (ii), 4.6 (i), 4.7 (ii), 4.7 (iii), 4.8 (vii), 4.18, 4.19, 4.20 and (4.44), we find that

$$F(i\eta) = \frac{\alpha}{\sqrt{2\pi}} \qquad \text{for any } \eta \in (0, \varepsilon_0].$$

Therefore, by the theorem of identity, we have Lemma 4.23. (Q. E. D.)

By using Lemma 4.23, we shall show

LEMMA 4.24 $\lim_{\eta \downarrow 0} L(i\eta) = 0$.

PROOF By substituting $\xi = i\eta\,(\eta > 0)$ into both hand sides of Lemma 4.23 (ii) and then letting η tend to zero, we see from condition (3.5), (4.5) and (4.6) that $\lim_{\eta \downarrow 0} L(i\eta)$ exists and then

$$(\beta + \lim_{\eta \downarrow 0} L(i\eta))^{-1} = \beta^{-1},$$

which gives Lemma 4.24. (Q. E. D.)

(**Step** 11) Finally, by collecting the results which we have investigated into Step 1 to Step 10, we shall show the following

THEOREM 4.1 *Let $A = (A(t)\,;\, t \in \boldsymbol{R})$ be any stationary curve in* (2.2) *with covariance matrix R_A in* (2.3) *satisfying conditions* (2.6) *and* (3.5). *Then there exists a unique triple $(\alpha, \beta, \varkappa)$ such that for any $\xi \in \boldsymbol{C}^+$*

$$[R_A](\xi) = (\beta - i\xi + L(\xi))^{-1} \cdot \frac{\alpha}{\sqrt{2\pi}}, \tag{4.46}$$

where

$$L(\xi) = (-i\xi) \lim_{\varepsilon \downarrow 0} \int_{\boldsymbol{R}} \frac{1}{(\lambda - \xi - i\varepsilon)(\lambda - i\varepsilon)} \varkappa(d\lambda) \cdot (\sqrt{2\pi}\ \alpha)^{-1}. \tag{4.47}$$

Here

(i) $\alpha \in GL(d ; \boldsymbol{C})$, $\alpha^* = \alpha$ *and* α *is positive definite*

(ii) $\beta \in GL(d ; \boldsymbol{C})$

(iii) $\varkappa = (\varkappa_{pq})_{1 \leq p, q \leq d}$;

(a) *for each* p, $q \in \{1, 2, \cdots, d\}$ $\varkappa_{pq}$ *is a* $\boldsymbol{C}$*-valued Borel signed measure on* $\boldsymbol{R}$

(b) *for any* $u = {}^t(u_1, u_2, \cdots, u_d) \in \boldsymbol{C}^d$

$$\varkappa_u \equiv \sum_{p, q=1}^{d} \bar{u}_p u_q \varkappa_{pq} \text{ is a Borel measure on } \boldsymbol{R}$$

(c) $\int_{\boldsymbol{R}} \frac{1}{1+\lambda^2} \varkappa_u(d\lambda) < \infty$ *for any* $u \in \boldsymbol{C}^d$

(d) $\varkappa^* = \varkappa$

(e) $\lim_{\varepsilon \downarrow 0} \int_{\boldsymbol{R}} \frac{1}{\pi} \frac{\varepsilon}{\lambda^2 + \varepsilon^2} \varkappa(d\lambda) = \frac{1}{\sqrt{2\pi}} (\beta \cdot \alpha + (\beta \cdot \alpha)^*)$

(f) $\lim_{\eta \downarrow 0} (\lim_{\varepsilon \downarrow 0} \int_{\boldsymbol{R}} (\frac{1}{\lambda - i\eta} - \frac{1}{\lambda - i\varepsilon}) \varkappa(d\lambda)) = 0$

(iv) $Z(\zeta) \equiv \beta - i\zeta + L(\zeta) \in GL(d ; \boldsymbol{C})$ *for any* $\zeta \in \boldsymbol{C}^+$

(v) $\sup_{\eta > 0} \int_{\boldsymbol{R}} \operatorname{trace} \, (Z(\xi + i\eta)^{-1} \cdot \alpha + (Z(\xi + i\eta)^{-1} \cdot \alpha)^*) d\xi < \infty.$

REMARK 4.1 It will be shown in the proof of Theorem 4.1 that, for any triple $(\alpha, \beta, \varkappa)$ satisfying conditions (i), (ii) and (iii) in Theorem 4.1, $L(\zeta)$ in (4.47) is always well-defined and $Z(\zeta)^{-1} \cdot \alpha + (Z(\zeta)^{-1} \cdot \alpha)^*$ is a non-negative definite matrix for any $\zeta \in \boldsymbol{C}^+$.

PROOF At first we shall show that the triple $(\alpha, \beta, \varkappa)$ in (4.5), (4.6) and (4.30) is our desired one. (i) and (ii) are clear. By Lemmas 4.14 (v), 4.16, 4.17 (ii) and 4.23, it suffices to prove (iii-e), (iii-f) and (v). By (4.22), Lemma 4.14 (iii), (4.20) and (4.7), we see that for any $\varepsilon \in (0, 2\varepsilon_0)$ and any $u = {}^t(u_1, u_2, \cdots, u_d) \in \boldsymbol{C}^d$

$$\begin{aligned} \int_{\boldsymbol{R}} \frac{\varepsilon}{\lambda^2 + \varepsilon^2} \varkappa_u(d\lambda) &= \sum_{p, q=1}^{d} \bar{u}_p u_q \lim_{k \to \infty} \int_{\boldsymbol{R}} \frac{\varepsilon}{\lambda^2 + \varepsilon^2} (\varkappa_{m_k})_{pq}(d\lambda) \\ &= \sum_{p, q=1}^{d} \bar{u}_p u_q \lim_{k \to \infty} \int_{\boldsymbol{R}} \frac{\varepsilon}{\lambda^2 + \varepsilon^2} (\varkappa_{m_k}^{(s)})_{pq}(d\lambda). \end{aligned}$$

Furthermore it follows from Lemmas 4.2, 4.6 (i), 4.9 (iii) and 4.9 (iv) that

$$\begin{aligned} \int_{\boldsymbol{R}} \frac{\varepsilon}{\lambda^2 + \varepsilon^2} \varkappa_u(d\lambda) = \frac{1}{2} ((\alpha \cdot [R_A](i\varepsilon)^{-1} \cdot \alpha + (\alpha \cdot [R_A](i\varepsilon)^{-1} \cdot \alpha)^* \\ - 2\varepsilon \sqrt{2\pi} \, \alpha) u, u)_{\boldsymbol{C}^d} \end{aligned}$$

and so by Lemma 4.16, for any $\varepsilon \in (0, 2\varepsilon_0)$,

$$(4.48)\quad \int_{\boldsymbol{R}} \frac{\varepsilon}{\lambda^2+\varepsilon^2}\varkappa(d\lambda)=\frac{1}{2}\{\alpha\cdot[R_A](i\varepsilon)^{-1}\cdot\alpha+(\alpha\cdot[R_A](i\varepsilon)^{-1}\cdot\alpha)^* -2\varepsilon\sqrt{2\pi}\ \alpha\}.$$

By letting ε tend to zero in (4.48), we see from condition (3.5), (4.5) and (4.6) that (iii-e) holds. By Lemma 4.21 (i), for any $\varepsilon>0$ and $\eta>0$,

$$(4.49)\quad L_\varepsilon(i\eta)=\int_{\boldsymbol{R}} \frac{\varepsilon}{(\lambda-i(\eta+\varepsilon))(\lambda-i\eta)}\varkappa(d\lambda) +\frac{1}{i}\int_{\boldsymbol{R}}\left(\frac{1}{\lambda-i\eta}-\frac{1}{\lambda-i\varepsilon}\right)\varkappa(d\lambda).$$

By Lemma 4.14 (v), we can apply Lebesgue's convergence theorem to the first term in the right hand side of (4.49) to obtain

$$(4.50)\quad L(i\eta)=\frac{1}{i}\lim_{\varepsilon\downarrow 0}\int_{\boldsymbol{R}}\left(\frac{1}{\lambda-i\eta}-\frac{1}{\lambda-i\varepsilon}\right)\varkappa(d\lambda)$$

for any $\eta>0$. Therefore, by Lemma 4.24, we have (iii-f). By (3.3), for any $\zeta=\xi+i\eta\in\boldsymbol{C}^+$,

$$(4.51)\quad \frac{1}{2}([R_A](\zeta)+[R_A](\zeta)^*)=\frac{1}{2\pi}\int_{\boldsymbol{R}}\frac{\eta}{(\lambda-\xi)^2+\eta^2}\Delta(d\lambda).$$

Hence, by Lemma 4.23,

$$(4.52)\quad Z(\zeta)^{-1}\cdot\alpha+(Z(\zeta)^{-1}\cdot\alpha)^*=\sqrt{\frac{2}{\pi}}\int_{\boldsymbol{R}}\frac{\eta}{(\lambda-\xi)^2+\eta^2}\Delta(d\lambda).$$

Since Δ is a spectral measure matrix of the covariance matrix R_A, it follows that for any $u={}^t(u_1, u_2, \cdots, u_d)\in\boldsymbol{C}^d$

(4.53) $\sum_{p,q=1}^{d} \bar{u}_p u_q \Delta_{pq}$ is a bounded Borel measure on $\boldsymbol{R}$.

Therefore, by (4.52) and (4.53), we find that for any $\zeta\in\boldsymbol{C}^+$

(4.54) $Z(\zeta)^{-1}\cdot\alpha+(Z(\zeta)^{-1}\cdot\alpha)^*$ is a non-negative definite matrix and for any $\eta>0$ and $u={}^t(u_1, u_2, \cdots, u_d)\in\boldsymbol{C}^d$

$$(4.55)\quad \int_{\boldsymbol{R}}((Z(\xi+i\eta)^{-1}\cdot\alpha+(Z(\xi+i\eta)^{-1}\cdot\alpha)^*)u, u)_{\boldsymbol{C}^d}d\xi =\sqrt{\frac{2}{\pi}}\sum_{p,q=1}^{d}\bar{u}_p u_q\Delta_{pq}(\boldsymbol{R}).$$

By using the same consideration as Lemma 4.15, we see from (4.54) and (4.55) that (v) holds.

Finally we shall show the uniqueness of the triple $(\alpha, \beta, \varkappa)$. Let $(\tilde{\alpha}, \tilde{\beta},$

$\tilde{\kappa}$) be another triple satisfying (4.46), (i), (ii), (iii), (iv) and (v). By Lemma 4.22 (i), for any $\eta>0$,

$$(4.56)\quad \eta[R_A](i\eta) = \{\frac{\tilde{\beta}}{\eta}+(\frac{\lim_{\varepsilon\downarrow 0}\tilde{L}_\varepsilon(i\varepsilon_0)}{\eta}+\int_R\frac{\eta-\varepsilon_0}{\eta(\lambda-i\eta)(\lambda-i\varepsilon_0)}\tilde{\kappa}(d\lambda))\cdot(\sqrt{2\pi}\ \tilde{\alpha})^{-1}\}^{-1}\cdot \cdot\frac{\tilde{\alpha}}{\sqrt{2\pi}},$$

where $\tilde{L}_\varepsilon(\zeta)=(-i\zeta)\int_R\frac{1}{(\lambda-\zeta-i\varepsilon)(\lambda-i\varepsilon)}\tilde{\kappa}(d\lambda).$

By letting η tend to infinity in (4.56), we see from Lemma 4.5 and (iii-c) that

$$(4.57)\quad \tilde{\alpha}=\frac{R_A(0)}{\sqrt{2\pi}}.$$

By noting (4.49) and (4.50), we find that (iii-f) is equivalent to

$$(4.58)\quad \lim_{\eta\downarrow 0}(\lim_{\varepsilon\downarrow 0}\tilde{L}_\varepsilon(i\eta))=0.$$

Therefore, it follows from condition (3.5), (4.46) and (4.57) that

$$(4.59)\quad \tilde{\beta}=R_A(0)\cdot D^{-1}.$$

Since for any $\zeta=\xi+i\eta\in C^+$

$$\tilde{L}(\zeta)=\lim_{\varepsilon\downarrow 0}\int_R\{(\frac{\eta+\varepsilon}{(\lambda-\xi)^2+(\eta+\varepsilon)^2}-\frac{\varepsilon}{\lambda^2+\varepsilon^2}) -i(\frac{\lambda-\xi}{(\lambda-\xi)^2+(\eta+\varepsilon)^2}-\frac{\lambda}{\lambda^2+\varepsilon^2})\}\tilde{\kappa}(d\lambda)\cdot(\sqrt{2\pi}\ \tilde{\alpha})^{-1},$$

we see from (i), (iii-c), (iii-d) and (iii-e) that

$$\begin{aligned}(4.60)\quad &\tilde{\alpha}^{-1}\cdot\tilde{Z}(\zeta)+(\tilde{\alpha}^{-1}\cdot\tilde{Z}(\zeta))^* \\ &=\tilde{\alpha}^{-1}\cdot\tilde{\beta}+(\tilde{\alpha}^{-1}\cdot\tilde{\beta})^*+2\eta\tilde{\alpha}^{-1} \\ &\qquad +2\tilde{\alpha}^{-1}\cdot\int_R\frac{\eta}{(\lambda-\xi)^2+\eta^2}\tilde{\kappa}(d\lambda)\cdot(\sqrt{2\pi}\ \tilde{\alpha})^{-1} \\ &\qquad -2\tilde{\alpha}^{-1}\cdot\sqrt{\frac{\pi}{2}}(\tilde{\beta}\cdot\tilde{\alpha}+(\tilde{\beta}\cdot\tilde{\alpha})^*)\cdot(\sqrt{2\pi}\ \tilde{\alpha})^{-1} \\ &=2\eta\tilde{\alpha}^{-1}+\sqrt{\frac{2}{\pi}}\tilde{\alpha}^{-1}\cdot\int_R\frac{\eta}{(\lambda-\xi)^2+\eta^2}\tilde{\kappa}(d\lambda)\cdot\tilde{\alpha}^{-1},\end{aligned}$$

where $\tilde{Z}(\zeta)=\tilde{\beta}-i\zeta+\tilde{L}(\zeta).$

By (4.46) and (4.59), we find that for any $\zeta=\xi+i\eta\in\boldsymbol{C}^+$

(4.61) $$\frac{1}{\pi}\int_{\boldsymbol{R}}\frac{\eta}{(\lambda-\xi)^2+\eta^2}\tilde{\varkappa}(d\lambda)$$
$$=\frac{1}{2\pi}\tilde{\alpha}([R_A](\xi+i\eta)^{-1}+([R_A](\xi+i\eta)^{-1})^*)\tilde{\alpha}-\sqrt{\frac{2}{\pi}}\eta\tilde{\alpha}.$$

By noting (iii-c), we can apply Lemma 4.27 in [11] to (4.6) to find that for any $f\in C_0(\boldsymbol{R})$

(4.62) $$\int_{\boldsymbol{R}}f(\lambda)\tilde{\varkappa}(d\lambda)$$
$$=\frac{1}{2\pi}\tilde{\alpha}\cdot\lim_{\eta\downarrow 0}\int_{\boldsymbol{R}}([R_A](\xi+i\eta)^{-1}+([R_A](\xi+i\eta)^{-1})^*)d\xi\cdot\tilde{\alpha}.$$

Thus, it follows from (4.57), (4.59) and (4.62) that the triple $(\tilde{\alpha},\tilde{\beta},\tilde{\varkappa})$ can be uniquely determined by the covariance matrix R_A. (Q. E. D.)

By taking account of Kubo's linear response theory in statistical physics ([2], [3], [4] and [5]), we shall give

DEFINITION 4.1 We call the function $[R_A](2\pi R(0))^{-1}$ an $M(d;\boldsymbol{C})$-valued *mobility function* of the stationary curve A.

§ 5. $\mathscr{R}$ and $\mathscr{L}$——a second KMO-Langevin data

In the previous sections § 2, § 3 and § 4, we have considered a stationary curve A in a Hilbert space $\mathscr{H}^d$ and obtained a representation of the $M(d;\boldsymbol{C})$-valued mobility function $[R_A]\cdot(2\pi R(0))^{-1}$ of A by introducing a triple $(\alpha,\beta,\varkappa)$. In this section, apart from stationary curves in Hiblert spaces, we shall obtain a characterization of such triples $(\alpha,\beta,\varkappa)$, which gives a converse statement to Theorem 4.1. For that purpose, we define two sets $\mathscr{R}$ and $\mathscr{L}$ by

(5.1) $\mathscr{R}\equiv\{R:\boldsymbol{R}\longrightarrow M(d;\boldsymbol{C})$;

(i) R is a continuous and non-negative definite function

(ii) $R(0)\in GL(d;\boldsymbol{C})$

(iii) $D\equiv\lim_{\varepsilon\downarrow 0}\int_0^\infty e^{-\varepsilon t}R(t)dt$ exists and $D\in GL(d;\boldsymbol{C})\}$

and

(5.2) $\mathscr{L}\equiv\{(\alpha,\beta,\varkappa)$; (i), (ii), (iii), (iv) and (v) in Theorem 4.1 hold$\}$.

For any $R\in\mathscr{R}$ we define an $M(d;\boldsymbol{C})$-valued holomorphic function $[R]$ on $\boldsymbol{C}^+$ by

(5.3) $$[R](\zeta)\equiv\frac{1}{2\pi}\int_0^\infty e^{i\zeta t}R(t)\,dt.$$

We denote by Δ a bounded Borel spectral measure matrix of R on $\boldsymbol{R}$:

(5.4) $$R(t)=\int_{\boldsymbol{R}} e^{-it\lambda}\,\Delta(d\lambda).$$

Then we see that for any $\zeta\in\boldsymbol{C}^+$

(5.5) $$[R](\zeta)=\frac{1}{2\pi i}\int_{\boldsymbol{R}}\frac{1}{\lambda-\zeta}\,\Delta(d\lambda).$$

Now we shall show the following main theorem in this paper.

THEOREM 5.1 *There exists a bijective mapping Φ from $\mathscr{R}$ onto $\mathscr{L}$ such that*

(5.6) $$[R](\zeta)=(\beta-i\zeta+L(\zeta))^{-1}\cdot\frac{\alpha}{\sqrt{2\pi}}\qquad \textit{for any } \zeta\in\boldsymbol{C}^+,$$

where $R\in\mathscr{R}$, $(\alpha,\beta,\varkappa)=\Phi(R)\in\mathscr{L}$ and

(5.7) $$L(\zeta)\equiv(-i\zeta)\lim_{\varepsilon\downarrow 0}\int_{\boldsymbol{R}}\frac{1}{(\lambda-\zeta-i\varepsilon)(\lambda-i\varepsilon)}\varkappa(d\lambda)\cdot(\sqrt{2\pi}\,\alpha)^{-1}.$$

PROOF Let R be any element of $\mathscr{R}$. We take a $\boldsymbol{C}^d$-valued stationary process $X=(X(t);t\in\boldsymbol{R})$ on a probability space $(\Omega,\mathscr{B},P)$ such that $E(X_p(t))=0$ and $E(X_p(t)\overline{X_q(s)})=R_{pq}(t-s)$ $(1\le p,\ q\le d,\ s,\ t\in\boldsymbol{R})$, where $X(t)={}^t(X_1(t),X_2(t),\cdots,X_d(t))$. Denoting by $\mathscr{H}$ the Hilbert space defined by

(5.8) $\mathscr{H}\equiv$ the closed linear hull of $\{X_p(t);1\le p\le d,\ t\in\boldsymbol{R}\}$ in $L^2(\Omega,\mathscr{B},P)$,

we can obtain a strongly continuous one-parameter group of unitary operators $(U(t);t\in\boldsymbol{R})$ on $\mathscr{H}$ such that

(5.9) $$U(t)(X_p(s))=X_p(s+t)\quad(1\le p\le d,\ s,\ t\in\boldsymbol{R}).$$

We define a stationary curve $A=(A(t);t\in\boldsymbol{R})$ in $\mathscr{H}^d$ by $A(t)\equiv{}^t(U(t)X_1(0), U(t)X_2(0),\cdots,U(t)X_d(0))$. Then, since the covariance matrix of A is equal to R, we can apply Theorem 4.1 to obtain a mapping Φ from $\mathscr{R}$ into $\mathscr{L}$ satisfying relation (5.6). By using the uniqueness of Laplace transform, we see that Φ is injective.

In order to show that Φ is surjective, we take any element $(\alpha,\beta,\varkappa)$ of $\mathscr{L}$. As we have shown in the proof of Theorem 4.1, we note that L is well-defined as an $M(d;\boldsymbol{C})$-valued holomorphic function on $\boldsymbol{C}^+$. We define an $M(d;\boldsymbol{C})$-valued holomorphic function Z on $\boldsymbol{C}^+$ by

(5.10) $$Z(\zeta)\equiv\beta-i\zeta+L(\zeta).$$

Then we can define for each $u\in\boldsymbol{C}^d$ a holomorphic function h_u on $\boldsymbol{C}^+$ by

(5.11) $$h_u(\zeta)\equiv\left(Z(\zeta)^{-1}\cdot\frac{\alpha}{\sqrt{2\pi}}u,\ u\right)_{\boldsymbol{C}^d}.$$

Since for any $\zeta\in\boldsymbol{C}^+$

(5.12) $$\begin{aligned}&Z(\zeta)^{-1}\cdot\alpha+(Z(\zeta)^{-1}\cdot\alpha)^*\\&=Z(\zeta)^{-1}\cdot\alpha\cdot\{\alpha^{-1}\cdot Z(\zeta)+(\alpha^{-1}\cdot Z(\zeta))^*\}\cdot(Z(\zeta)^{-1}\cdot\alpha)^*,\end{aligned}$$

we see from (4.60) in the proof of Theorem 4.1 and condition (v) in $\mathscr{L}$ that

(5.13) Re h_u is a non-negative harmonic function on $\boldsymbol{C}^+$

and

(5.14) $$\sup_{\eta>0}\int_{\boldsymbol{R}}|\mathrm{Re}\ h_u(\xi+i\eta)|\,d\xi<\infty.$$

Therefore, we find ([1]) that there exists a unique bounded Borel measure $\varDelta_u$ on $\boldsymbol{R}$ such that for any $\zeta=\xi+i\eta\in\boldsymbol{C}^+$

(5.15) $$\mathrm{Re}\ h_u(\xi+i\eta)=\frac{1}{2\pi}\int_{\boldsymbol{R}}\frac{\eta}{(\lambda-\xi)^2+\eta^2}\varDelta_u(d\lambda).$$

By using vectors e_p, e_{pq} and $\tilde{e}_{pq}(1\leq p<q\leq d)$ in (4.24), (4.25) and (4.26), we define $\boldsymbol{C}$-valued bounded signed measures $\varDelta_{pp}$, $\varDelta_{pq}$ and $\varDelta_{qp}(1\leq p<q\leq d)$ on $\boldsymbol{R}$ by

(5.16) $$\varDelta_{pp}\equiv\varDelta_{e_p}$$

(5.17) $$\varDelta_{pq}\equiv\frac{1}{2}\{\varDelta_{e_{pq}}-\varDelta_{e_p}-\varDelta_{e_q}-i(\varDelta_{\tilde{e}_{pq}}-\varDelta_{e_p}-\varDelta_{e_q})\}$$

and

(5.18) $$\varDelta_{qp}\equiv\overline{\varDelta}_{pq}$$

And we define $\varDelta$ by

(5.19) $$\varDelta\equiv(\varDelta_{pq})_{1\leq p\leq q\leq d}$$

and then an $M(d;\boldsymbol{C})$-valued continuous function R on $\boldsymbol{R}$ by

(5.20) $$R(t)\equiv\int_{\boldsymbol{R}}e^{-it\lambda}\varDelta(d\lambda).$$

Then we shall show that $R\in\mathscr{R}$ and $\Phi(R)=(\alpha,\beta,\kappa)$.

At first we prove that for any $u={}^t(u_1, u_2, \cdots, u_d)\in\boldsymbol{C}^d$

$$(5.21)\qquad \sum_{p,q=1}^{d} \bar{u}_p u_q \Delta_{pq}=\Delta_u.$$

We denote by μ the left hand side of (5.21). We note that μ is an $\boldsymbol{R}$-valued bounded signed measure on $\boldsymbol{R}$. Then it follows from (5.15) that for any $\zeta=\xi+i\eta\in\boldsymbol{C}^+$

$$(5.22)\qquad \frac{1}{2\pi}\int_{\boldsymbol{R}}\frac{\eta}{(\lambda-\xi)^2+\eta^2}\mu(d\lambda)$$
$$=\sum_{p=1}^{d}|u_p|^2\cdot\frac{1}{2\pi}\int_{\boldsymbol{R}}\frac{\eta}{(\lambda-\xi)^2+\eta^2}\Delta_{e_p}(d\lambda)$$
$$+2\mathrm{Re}\Big(\sum_{1\le p<q\le d}\bar{u}_p u_q\frac{1}{2\pi}\int_{\boldsymbol{R}}\frac{\eta}{(\lambda-\xi)^2+\eta^2}((\Delta_{e_{pq}}-\Delta_{e_p}-\Delta_{e_q})$$
$$-i(\Delta_{\tilde{e}_{pq}}-\Delta_{e_p}-\Delta_{e_q}))(d\lambda)\Big)$$
$$=\mathrm{Re}\Big\{\sum_{p=1}^{d}|u_p|^2h_{e_p}(\zeta)+2\sum_{1\le p<q\le d}\bar{u}_p u_q(\mathrm{Re}\ h_{e_{pq}}(\zeta)-\mathrm{Re}\ h_{e_p}(\zeta)$$
$$-\mathrm{Re}\ h_{e_q}(\zeta)-i(\mathrm{Re}\ h_{\tilde{e}_{pq}}(\zeta)-\mathrm{Re}\ h_{e_p}(\zeta)-\mathrm{Re}\ h_{e_q}(\zeta)))\Big\}.$$

On the other hand, we see from (4.24), (4.25), (4.26) and (5.11) that for any $\zeta\in\boldsymbol{C}^+$ and $1\le p<q\le d$

$$(5.23)\qquad \mathrm{Re}\ h_{e_{pq}}(\zeta)-\mathrm{Re}\ h_{e_p}(\zeta)-\mathrm{Re}\ h_{e_q}(\zeta)-i(\mathrm{Re}\ h_{\tilde{e}_{pq}}(\zeta)-\mathrm{Re}\ h_{e_p}(\zeta)-\mathrm{Re}\ h_{e_q}(\zeta))$$
$$=\frac{1}{2}((Z(\zeta)^{-1}\cdot\frac{\alpha}{\sqrt{2\pi}}+(Z(\zeta)^{-1}\cdot\frac{\alpha}{\sqrt{2\pi}})^*)e_q, e_p)_{\boldsymbol{C}^d}$$

Therefore, by (5.11), (5.22) and (5.23), for any $\zeta=\xi+i\eta\in\boldsymbol{C}^+$,

$$(5.24)\qquad \frac{1}{2\pi}\int_{\boldsymbol{R}}\frac{\eta}{(\lambda-\xi)^2+\eta^2}\mu(d\lambda)$$
$$=\sum_{p=1}^{d}|u_p|^2\mathrm{Re}(Z(\zeta)^{-1}\cdot\frac{\alpha}{\sqrt{2\pi}}e_p, e_p)_{\boldsymbol{C}^d}$$
$$+\mathrm{Re}\Big(\sum_{1\le p<q\le d}\bar{u}_p u_q((Z(\zeta)^{-1}\cdot\frac{\alpha}{\sqrt{2\pi}}+(Z(\zeta)^{-1}\cdot\frac{\alpha}{\sqrt{2\pi}})^*)e_q, e_p)_{\boldsymbol{C}^d}$$
$$=\sum_{p,q=1}^{d}\bar{u}_p u_q\ \mathrm{Re}(Z(\zeta)^{-1}\cdot\frac{\alpha}{\sqrt{2\pi}}e_p, e_q)_{\boldsymbol{C}^d}$$
$$=\mathrm{Re}(Z(\zeta)^{-1}\cdot\frac{\alpha}{\sqrt{2\pi}}u, u)_{\boldsymbol{C}^d}.$$

Thus, it follows from (5.11), (5.15) and (5.24) that for any $\zeta=\xi+i\eta\in\boldsymbol{C}^+$

$$(5.25)\qquad \int_{\boldsymbol{R}}\frac{\eta}{(\lambda-\xi)^2+\eta^2}\mu(d\lambda)=\int_{\boldsymbol{R}}\frac{\eta}{(\lambda-\xi)^2+\eta^2}\Delta_u(d\lambda).$$

By using the same consideration as (4.62), we find that (5.25) gives $\mu=\Delta_u$ and so (5.21) holds. In particular, we find (5.20) and (5.21) that

(5.26) R is an $M(d;\boldsymbol{C})$-valued continuous and non-negative definite function.

Next we shall show

(5.27) $R(0)\in GL(d;\boldsymbol{C})$.

By using (5.20) and (5.21) again, we note that for any $u\in\boldsymbol{C}^d$

(5.28) $(R(0)u,u)_{\boldsymbol{C}^d}=\Delta_u(\boldsymbol{R})$.

Let u be any vector in $\boldsymbol{C}^d$ such that $(R(0)u,u)_{\boldsymbol{C}^d}=0$. It then follows from (5.11), (5.12), (5.15) and (5.28) that for any $\zeta\in\boldsymbol{C}^+$

$$\mathrm{Re}(Z(\zeta)^{-1}\cdot\frac{\alpha}{\sqrt{2\pi}}u,u)_{\boldsymbol{C}^d}=0,$$

which implies $u=0$. Hence we have (5.27).

Next we claim that for any $\zeta\in\boldsymbol{C}^+$

(5.29) $[R](\zeta)=Z(\zeta)^{-1}\cdot\dfrac{\alpha}{\sqrt{2\pi}}$

Similarly as (5.5), we see from (5.20) and (5.21) that for any $u\in\boldsymbol{C}^d$

$$([R](\zeta)u,u)_{\boldsymbol{C}^d}=\frac{1}{2\pi i}\int_{\boldsymbol{R}}\frac{1}{\lambda-\zeta}\Delta_u(d\lambda)$$

and so by (5.15)

(5.30) $\mathrm{Re}([R](\zeta)u,u)_{\boldsymbol{C}^d})=(\mathrm{Re}\,h_u)(\zeta)$.

Similarly as Lemma 4.22 (i), we see that for any $\eta>0$

(5.31) $L(i\eta)=L(i)+\displaystyle\int_{\boldsymbol{R}}\frac{\eta-1}{(\lambda-i\eta)(\lambda-i)}\varkappa(d\lambda)\cdot(\sqrt{2\pi}\,\alpha)^{-1}$

and so

(5.32) $\displaystyle\lim_{\eta\to\infty}\frac{L(i\eta)}{\eta}=0.$

By (5.10), (5.11) and (5.32), we have

(5.33) $\displaystyle\lim_{\eta\to\infty}\eta\,h_u(i\eta)=(\frac{\alpha}{\sqrt{2\pi}}u,u)_{\boldsymbol{C}^d}$ for any $u\in\boldsymbol{C}^d$.

On the other hand, similarly as Lemma 4.5, we have

(5.34) $\lim_{\eta\to\infty} \eta([R](i\eta)u, u)_{C^d}=(\frac{R(0)}{2\pi}u, u)_{C^d}$ for any $u\in \boldsymbol{C}^d$.

Since $R(0)^*=R(0)$ and $\alpha^*=\alpha$, it follows from (5.30), (5.33) and (5.34) that

(5.35) $\frac{\alpha}{\sqrt{2\pi}}=\frac{R(0)}{2\pi}$.

Therefore, since $([R](\cdot)u, u)_{C^d}$ and h_u are holomorphic on $\boldsymbol{C}^+$ for any $u\in \boldsymbol{C}^d$, we can apply Cauchy-Riemann's relation to (5.30), (5.33), (5.34) and (5.35) to find that for any $u\in \boldsymbol{C}^d$

$$([R](\cdot)u, u)_{C^d}=h_u \quad \textit{on } \boldsymbol{C}^+,$$

which with (5.11) gives (5.29).

Finally we shall show

(5.36) $\lim_{\varepsilon\downarrow 0}[R](i\varepsilon)=\beta^{-1}\cdot\frac{\alpha}{\sqrt{2\pi}}\in GL(d\,;\boldsymbol{C})$.

As we have seen in (4.57) in the proof of Theorem 4.1, condition (iii)(f) in $\mathscr{L}$ is equivalent to

(5.37) $\lim_{\eta\downarrow 0} L(i\eta)=0$,

which with (5.10) and (5.20) gives (5.36).

Thus, we find from (5.26), (5.27), (5.29) and (5.36) that $R\in\mathscr{R}$ and $\Phi(R)=(\alpha,\beta,\kappa)$, that is, Φ is surjective. (Q.E.D.)

REMARK 5.1 By (4.54) and (4.55) in the proof of Theorems 4.1 and 5.1, we note that the following conditions (5.38) and (5.39) are equivalent to condition (v) in $\mathscr{L}$:

(5.38) $\int_{\boldsymbol{R}}$ trace $(Z(\xi+i\eta)^{-1}\cdot\alpha+(Z(\xi+i\eta)^{-1}\cdot\alpha)^*)d\xi$ is independent of $\eta>0$.

(5.39) For any p, $q\in\{1, 2, \cdots, d\}$ and $\eta>0$

$$((Z(\cdot+i\eta)^{-1}\cdot\alpha+(Z(\cdot+i\eta)^{-1}\cdot\alpha)^*)_{pq}\in L^1(\boldsymbol{R})$$

and

$$\int_{\boldsymbol{R}}(Z(\xi+i\eta)^{-1}\cdot\alpha+(Z(\xi+i\eta)^{-1}\cdot\alpha)^*)_{pq}d\xi \text{ is independent of } \eta.$$

By combining Theorem 5.1 with Theorem 4.1 and then noting (4.57), (4.59) and (4.62), we have

THEOREM 5.2 *Let R and (α, β, κ) be any element of $\mathscr{R}$ and $\mathscr{L}$, respectively, such that $\Phi(R)=(\alpha, \beta, \kappa)$. Then,*

(i) $\alpha=\frac{R(0)}{\sqrt{2\pi}}$

(ii) $\beta=R(0)\cdot D^{-1}$

(iii) $\kappa(d\xi)=\frac{1}{(2\pi)^2}\lim_{\eta\downarrow 0} R(0)\cdot([R](\xi+i\eta)^{-1}+([R](\xi+i\eta)^{-1})^*)\cdot R(0)d\xi$ *in* $\mathscr{D}'(\boldsymbol{R})$.

By taking account of Definition 3.1 in [10], we shall give

DEFINITION 5.1 We call such a triple $(\alpha,\beta,\kappa)\in\mathscr{L}$ that $\Phi^{-1}(\alpha,\beta,\kappa))=R\in\mathscr{R}$ *a second KMO-Langevin data* associated with R.

§ 6. $\mathscr{R}_{sm}$ and $\mathscr{L}_{sm}$

By using Theorem 5.1, we shall characterize the set of second KMO-Langevin data associated with covariance matrices of smooth stationary curves treated in § 2 and § 3. For that purpose, we shall define two subsets $\mathscr{R}_{sm}$ and $\mathscr{L}_{sm}$ of $\mathscr{R}$ and $\mathscr{L}$, respectively, by

(6.1) $\mathscr{R}_{sm}\equiv\{R\in\mathscr{L}; \int_{\boldsymbol{R}} \lambda^2(\Delta(d\lambda)u, u)<\infty$ for any $u\in\boldsymbol{C}^d$,
where Δ is a spectral measure matrix of $R\}$

and

(6.2) $\mathscr{R}_{sm}\equiv\{(\alpha,\beta,\kappa)\in\mathscr{L};$
(iii-a)′ each component of κ is a $\boldsymbol{C}$-valued bounded Borel measure on $\boldsymbol{R}$

(iii-f)′ $\lim_{\varepsilon\downarrow 0}\int_{\boldsymbol{R}}\frac{\lambda}{\lambda^2+\varepsilon^2}\kappa(d\lambda)$ exists

(iv) $\lim_{n\to\infty}(\lim_{\eta\downarrow 0}\int_{\boldsymbol{R}}(n\wedge\xi^2)\text{trace}(Z(\xi+i\eta)^{-1}\cdot\alpha+(Z(\xi+i\eta)^{-1}\cdot\alpha)^*d\xi)$ exists$\}$

At first we shall prepare

LEMMA 6.1 *Let R and (α, β, κ) be any element of $\mathscr{R}$ and $\mathscr{L}$, respectively, such that $\Phi(R)=(\alpha, \beta, \kappa)$. Then, we have, for any $p\in\{1, 2, \cdots, d\}$ and $n\in\boldsymbol{N}$*

(6.3) $$\lim_{\eta\downarrow 0}\int_{R}(n\wedge\xi^2)(Z(\xi+i\eta)^{-1}\cdot\alpha+(Z(\xi+i\eta)^{-1}\cdot\alpha)^*)_{pp}d\xi = \int_{R}(n\wedge\xi^2)\,\Delta_{pp}(d\lambda).$$

PROOF By (4.52), we have, for any $p\in\{1,2,\cdots,d\}$, $n\in \boldsymbol{N}$ and $\eta>0$,

(6.4) $$\int_{R}(n\wedge\xi^2)(Z(\xi+i\eta)^{-1}\cdot\alpha+(Z(\xi+i\eta)^{-1}\cdot\alpha)^*)_{pp}d\xi = \sqrt{2\pi}\int_{R}\left(\int_{R}\frac{1}{\pi}\frac{\eta}{(\lambda-\xi)^2+\eta^2}(n\wedge\xi^2)d\xi\right)\Delta_{pp}(d\lambda).$$

Since for any $n\in\boldsymbol{N}$

$$\lim_{\eta\downarrow 0}\int_{R}\frac{1}{\pi}\frac{\eta}{(\lambda-\xi)^2+\eta^2}(n\wedge\xi^2)d\xi=n\wedge\lambda^2 \text{ boundedly in } \lambda\in\boldsymbol{R},$$

we see that (6.3) follows from (6.4). (Q.E.D.)

Now we shall show

THEOREM 6.1 $\Phi(\mathscr{R}_{sm})=\mathscr{L}_{sm}$.

PROOF Let R be any element of $\mathscr{R}_{sm}$ and $(\alpha,\beta,\kappa)\equiv\Phi(R)$. We take a stationary curve $A=(A(t)\,;t\in\boldsymbol{R})$ with covariance matrix R constructed in the proof of Theorem 5.1. Since $R\in\mathscr{R}_{sm}$ implies that $A(t)$ is differentiable at t, we can apply (3.2), Theorems 3.1 and 5.1 to get (iii-a)′ in $\mathscr{L}_{sm}$. Therefore, we can rewrite Lemma 4.21(i) into

(6.5) $$L_\varepsilon(\zeta)=\frac{1}{i}\int_{R}\frac{1}{\lambda-\zeta-i\varepsilon}\kappa(d\lambda)\cdot(\sqrt{2\pi}\,\alpha)^{-1} -\frac{1}{i}\int_{R}\frac{1}{\lambda-i\varepsilon}\kappa(d\lambda)\cdot(\sqrt{2\pi}\,\alpha)^{-1} \text{ for any } \zeta\in\boldsymbol{C}^+.$$

Since

(6.6) $$\lim_{\varepsilon\downarrow 0}\int_{R}\frac{1}{\lambda-\zeta-i\varepsilon}\kappa(d\lambda)=\int_{R}\frac{1}{\lambda-\zeta}\kappa(d\lambda) \quad \text{for any } \zeta\in\boldsymbol{C}^+,$$

we see from (4.44) and (6.5) that

(6.7) $$\lim_{\varepsilon\downarrow 0}\int_{R}\frac{1}{\lambda-i\varepsilon}\kappa(d\lambda) \text{ exists.}$$

By combining (6.5) with (ii-e) in $\mathscr{L}$, we find that (iii-f)′ in $\mathscr{L}_{sm}$ holds. By applying Lebesgue's monotone convergence theorem to (6.3), we see that $R\in\mathscr{R}_{sm}$ implies (vi) in $\mathscr{L}_{sm}$.

Conversely, let (α,β,κ) be any element of $\mathscr{L}_{sm}$ and $R\equiv\Phi^{-1}((\alpha,\beta,\kappa))$. By applying Lebesgue's monotone convergence theorem to (6.3) again, we see that condition (vi) in $\mathscr{L}_{sm}$ implies that

(6.8) $\sum_{p=1}^{d}\int_{R}\lambda^2 \Delta_{pp}(d\lambda)<\infty.$

Since for any $u={}^t(u_1, u_2, \cdots, u_d)\in C^d$ $\sum_{p,q=1}^{d} \bar{u}_p u_q \Delta_{pq}$ is a bounded Borel measure on $\boldsymbol{R}$, we note that

(6.9) $|\Delta_{pq}(B)|\leq\sum_{r=1}^{d}\Delta_{rr}(B)$ for any $p, q\in\{1, 2, \cdots, d\}$

and Borel set B in $\boldsymbol{R}$.
Therefore, (6.8) with (6.9) gives that R belongs to $\mathscr{R}_{sm}$. (Q. E. D.)

REMARK 6.1 We note that conditions (iii-f)$'$ in $\mathscr{L}_{sm}$ with condition (iii-e) in $\mathscr{L}$ implies condition (iii-f) in $\mathscr{L}$.

Next we shall consider any element $R\in\mathscr{R}_{sm}$ and obtain a continued fraction expansion of $[R]$. We set $(\alpha, \beta, \kappa)\equiv\Phi(R)$. By condition (iii-e) in $\mathscr{L}$ and (6.7), we have

LEMMA 6.2

$$\lim_{\varepsilon\downarrow 0}\frac{1}{i}\int_{R}\frac{1}{\lambda-i\varepsilon}\kappa(d\lambda)=\sqrt{\frac{\pi}{2}}(\beta\cdot\alpha+(\beta\cdot\alpha)^*) -i\lim_{\varepsilon\downarrow 0}\int_{R}\frac{\lambda}{\lambda^2+\varepsilon^2}\kappa(d\lambda).$$

In the sequel in this section, we shall suppose the following condition:

(6.10) $\lim_{\varepsilon\downarrow 0}\frac{1}{i}\int_{R}\frac{1}{\lambda-i\varepsilon}\kappa(d\lambda)\in GL(d\,;C)$

REMARK 6.2 If κ is a symmetric Borel measure on $\boldsymbol{R}$, then, we see from Lemma 6.2 that

(6.11) $\lim_{\varepsilon\downarrow 0}\frac{1}{i}\int_{R}\frac{1}{\lambda-i\varepsilon}\kappa(d\lambda)=\sqrt{\frac{\pi}{2}}(\beta\cdot\alpha+(\beta\cdot\alpha)^*).$

We define an $M(d\,;C)$-valued continuous function ϕ on $\boldsymbol{R}$ by

(6.12) $\phi(t)\equiv\int_{R}e^{-it\lambda}\kappa(d\lambda)\cdot R(0)^{-1}.$

Then we shall show

LEMMA 6.3

(i) $\phi(0)=\kappa(\boldsymbol{R})\cdot R(0)^{-1}$

(ii) $\lim_{\varepsilon\downarrow 0}[\phi](i\varepsilon)=\frac{1}{2\pi i}\lim_{\varepsilon\downarrow 0}\int_{R}\frac{1}{\lambda-i\varepsilon}\kappa(d\lambda)\cdot R(0)^{-1}$

(iii) $\phi(\ \cdot\ )\cdot R(0)\in\mathscr{R}$.

PROOF (i) and (ii) follow immediately from (6.12). Since κ satisfies condition (iii-b) in $\mathscr{L}$, we see that $\phi(\ \cdot\ )\cdot R(0)$ is an $M(d;\boldsymbol{C})$-valued non-negative definite function on $\boldsymbol{R}$. Therefore, by condition (6.10), (i) and (ii), for the proof of (iii), it suffices to prove

$$(6.13)\qquad \kappa(\boldsymbol{R})\in GL(d;\boldsymbol{C}).$$

Let u be any element of $\boldsymbol{C}^d$ such that $(\kappa(\boldsymbol{R})u, u)_{\boldsymbol{C}^d}=0$. By using condition (iii-b) in $\mathscr{L}$ again, we can see that $\sum_{p,q=1}^{d} u_p\bar{u}_q\kappa_{pq}(d\lambda)=0$ and so for any $\zeta\in\boldsymbol{C}^+$

$$\sum_{p,q=1}^{d}\bar{u}_p u_q\int_{\boldsymbol{R}}\frac{1}{\lambda-\zeta}\kappa_{pq}(d\lambda)=0.$$

By combining this with condition (6.10), we find that $u=0$ and so (6.12) holds. (Q. E. D.)

As we have seen in § 2 and § 3, it follows from Theorem 2.1, Lemma 3.1(ii), Theorem 5.2(i) and Lemma 6.3(ii) that

LEMMA 6.4 *For any* $\zeta\in\boldsymbol{C}^+$

$$[R](\zeta)=(\beta+i\lim_{\varepsilon\downarrow 0}\int_{\boldsymbol{R}}\frac{1}{\lambda-i\varepsilon}\kappa(d\lambda)-i\zeta+2\pi[\phi](\zeta))^{-1}\cdot\frac{\alpha}{\sqrt{2\pi}}.$$

By virtue of Lemmas 6.3(iii) and 6.4, we can apply Theorems 5.1 and 5.2 (i) to the function $\phi(\ \cdot\ )\cdot R(0)$ to obtain the following continued fraction expansion of the function $[R]$ for $R\in\mathscr{R}_{sm}$.

THEOREM 6.2 *Let R be any element of $\mathscr{R}_{sm}$ and $(\alpha,\beta,\kappa)\equiv\Phi(R)$. Furthermore we suppose condition* (6.10). *Then there exists a unique triple $(\alpha_2,\beta_2,\kappa_2)$ of $\mathscr{L}$ such that for any $\zeta\in\boldsymbol{C}^+$*

$$(6.14)\qquad [R](\zeta)$$
$$=(\beta+i\lim_{\varepsilon\downarrow 0}\int_{\boldsymbol{R}}\frac{1}{\lambda-i\varepsilon}\kappa(d\lambda)-i\zeta-(\beta_2-i\zeta+L_2(\zeta))^{-1}\cdot\alpha_2\cdot\alpha^{-1})^{-1}\cdot\frac{\alpha}{\sqrt{2\pi}},$$

where

$$(6.15)\qquad L_2(\zeta)=(-i\zeta)\lim_{\varepsilon\downarrow 0}\int_{\boldsymbol{R}}\frac{1}{(\lambda-\zeta-i\varepsilon)(\lambda-i\varepsilon)}\kappa_2(d\lambda)\cdot(\sqrt{2\pi}\ \alpha_2)^{-1}.$$

Finally we shall give an expression of the triple $(\alpha_2,\beta_2,\kappa_2)$ in terms of the second KMO-Langevin data (α,β,κ) associated with R.

THEOREM 6.3

(i) $\alpha_2=\frac{\varkappa(\boldsymbol{R})}{\sqrt{2\pi}}$

(ii) $\beta_2=\varkappa(\boldsymbol{R})\cdot(\sqrt{2\pi}\ \alpha)^{-1}\cdot(\lim_{\varepsilon\downarrow 0}\frac{1}{i}\int_{\boldsymbol{R}}\frac{1}{\lambda-i\varepsilon}\varkappa(d\lambda))^{-1}$

(iii) $\varkappa_2(d\xi)$

$$=\lim_{\eta\downarrow 0}\varkappa(\boldsymbol{R})\cdot(\int_{\boldsymbol{R}}\frac{1}{\tau-\xi-i\eta}\varkappa(d\tau))^{-1}\cdot\int_{\boldsymbol{R}}\frac{1}{\pi}\frac{\eta}{(\tau-\xi)^2+\eta^2}\varkappa(d\tau)\cdot\cdot(\int_{\boldsymbol{R}}\frac{1}{\tau-\xi+i\eta}\varkappa(d\tau))^{-1}\cdot\varkappa(\boldsymbol{R})d\xi \qquad in\ \mathscr{D}'(\boldsymbol{R}).$$

PROOF Since it follows from Theorem 5.2(i) that $\alpha_2=\frac{\phi(0)\cdot R(0)}{\sqrt{2\pi}}$, we find that (i) follows from Lemma 6.2(i). By Theorem 5.2(ii), $\beta_2=\phi(0)\cdot R(0)\cdot(2\pi[\phi](0+i0)\cdot R(0))^{-1}$. Therefore, we see from Theorems 5.2 (i), (ii), Lemmas 6.3(i) and 6.3(ii) that (ii) holds. By Theorem 5.2 (iii), we have

(6.16) $\varkappa_2(d\xi)$

$$=\frac{1}{(2\pi)^2}\lim_{\eta\downarrow 0}\phi(0)\cdot R(0)\cdot([\phi(\ \cdot\ )R(0)](\xi+i\eta)^{-1}+([\phi(\ \cdot\ )R(0)](\xi+i\eta)^{-1})^*)\phi(0)\cdot R(0)d\xi \qquad in\ \mathscr{D}'(\boldsymbol{R}).$$

On the other hand, we can see from (6.12) that for any $\zeta=\xi+i\eta\in\boldsymbol{C}^+$

(6.17) $[\phi(\ \cdot\ )\cdot R(0)](\xi+i\eta)^{-1}+([\phi(\ \cdot\ )\cdot R(0)](\xi+i\eta)^{-1})^*$

$$=(2\pi)^2(\int_{\boldsymbol{R}}\frac{1}{\tau-\xi-i\eta}\varkappa(d\tau))^{-1}\cdot\int_{\boldsymbol{R}}\frac{1}{\pi}\frac{\eta}{(\tau-\xi)^2+\eta^2}\varkappa(d\tau)\cdot\cdot(\int_{\boldsymbol{R}}\frac{1}{\tau-\xi+i\eta}\varkappa(d\tau))^{-1}.$$

Therefore, by Lemma 6.3(i), (6.16) and (6.17), we have (iii). (Q.E.D.)

REMARK 6.3 Let R and $(\alpha,\beta,\varkappa)$ be any element of $\mathscr{R}_{sm}$ and $\mathscr{L}_{sm}$, respectively, such that $\Phi(R)=(\alpha,\beta,\varkappa)$. We denote by $\phi_\varkappa$ a covariance matrix whose spectral measure matrix is $\varkappa$.

(i) We note that $\phi_\varkappa\cdot R(0)$ is equal to ϕ_M in (2.12)

(ii) If $\phi_\varkappa$ belongs to $\mathscr{R}_{sm}$, we can apply Theorem 2.2 in Mori's theory to obtain a continued fraction expansion for $[R]$, which is a fundamental idea in [7].

§ 7. $\mathscr{R}_{ds}$ and $\mathscr{L}_{ds}$

At first, we shall in this section characterize the set of second KMO-Langevin data associated with elements of $\mathscr{R}$ having spectral density matrices. For that purpose, we shall define two subsets $\mathscr{R}_{ds}$ and $\mathscr{L}_{ds}$ of $\mathscr{R}$ and $\mathscr{L}$, respectively, by

(7.1) $\mathscr{R}_{ds}\equiv\{R\in\mathscr{R};\ R$ has a spectral density matrix$\}$

and

(7.2) $\mathscr{L}_{ds}\equiv\{(\alpha,\beta,\varkappa)\in\mathscr{L};$

(v)′

(a) $\lim_{\eta\downarrow 0}(Z(\xi+i\eta)^{-1}\cdot\alpha+(Z(\xi+i\eta)^{-1}\cdot\alpha)^*)$ exists for a. e. $\xi\in\boldsymbol{R}$

(b) $\int_{\boldsymbol{R}}\text{trace }(Z(\xi+i\eta)^{-1}\cdot\alpha+(Z(\xi+i\eta)^{-1}\cdot\alpha)^*)d\xi$

$=\int_{\boldsymbol{R}}\lim_{\varepsilon\downarrow 0}\text{trace }(Z(\xi+i\varepsilon)^{-1}\cdot\alpha+(Z(\xi+i\varepsilon)^{-1}\cdot\alpha)^*)d\xi<\infty$

for any $\eta>0\}$.

THEOREM 7.1 $\Phi(\mathscr{R}_{ds})=\mathscr{L}_{ds}$.

PROOF Let R be any element of $\mathscr{R}_{ds}$ and $(\alpha,\beta,\varkappa)\equiv\Phi(R)$. We denote by $\Delta'=(\Delta'_{pq})_{1\leq p,q\leq d}$ a spectral density matrix of R. We define for each $u\in\boldsymbol{C}^d$ a function Δ'_u on $\boldsymbol{R}$ by

(7.3) $\Delta'_u(\xi)\equiv(\Delta'(\xi)u,u)_{\boldsymbol{C}^d}$.

Then we note that

(7.4) $\Delta'_u(\xi)\geq 0$ for a. e. $\xi\in\boldsymbol{R}$

(7.5) $\Delta'_u\in L'(\boldsymbol{R})$

and

(7.6) $(\Delta(d\lambda)u,u)_{\boldsymbol{C}^d}=\Delta'_u(\xi)d\xi$.

By (4.51) and (7.6), for any $u\in\boldsymbol{C}^d$ and $\eta>0$

(7.7) $$\frac{1}{\pi}\int_{\boldsymbol{R}}\frac{\eta}{(\lambda-\xi)^2+\eta^2}\Delta'_u(\xi)d\xi=2\ \mathrm{Re}([R](\xi+i\eta)u,u)_{\boldsymbol{C}^d}.$$

Therefore, it follows from (7.4), (7.5) and (7.7) that

(7.8) $$\Delta'_u(\xi)=2\lim_{\eta\downarrow 0}\mathrm{Re}([R](\xi+i\eta)u,u)_{\boldsymbol{C}^d}\quad\text{for a. e. }\xi\in\boldsymbol{R}$$

and

(7.9) $$\Delta'_u(\ \cdot\ )=2\lim_{\eta\downarrow 0}\mathrm{Re}([R](\ \cdot+i\eta)u,u)_{\boldsymbol{C}^d}\quad\text{in }L^1(\boldsymbol{R}).$$

In particular, we see from (7.7) and (7.9) that for any $u \in \boldsymbol{C}^d$ and $\eta > 0$

$$(7.10) \quad \int_{\boldsymbol{R}} \mathrm{Re}(([R](\xi+i\eta)u, u)_{\boldsymbol{C}^d})d\xi = \int_{\boldsymbol{R}} \lim_{\eta \downarrow 0} \mathrm{Re}(([R](\xi+i\eta)u, u)_{\boldsymbol{C}^d})d\xi.$$

On the other hand, by (5.6), for any $\zeta \in \boldsymbol{C}^+$,

$$(7.11) \quad Z(\zeta)^{-1}\cdot\alpha + (Z(\zeta)^{-1}\cdot\alpha)^* = \sqrt{2\pi}([R](\zeta)+[R](\zeta)^*).$$

Therefore, we find from (7.8), (7.9), (7.10) and (7.11) that (v)′ in $\mathscr{L}_{ds}$ holds.

Conversely, let (α, β, κ) be any element of $\mathscr{L}_{ds}$ and $R = \Phi^{-1}(\alpha, \beta, \kappa))$. We denote by Δ a spectral measure matrix of R. By condition (v)′ in $\mathscr{L}_{ds}$, we can define $m = (m_{pq}(d\xi))_{1 \le p, q \le d}$ by

$$(7.11) \quad m(d\xi) \equiv \frac{1}{\sqrt{2\pi}} \lim_{\eta \downarrow 0}(Z(\xi+i\eta)^{-1}\cdot\alpha + (Z(\xi+i\eta)^{-1}\cdot\alpha)^*)d\xi.$$

In particular, we see from (4.54) that for any $u = {}^t(u_1, u_2, \cdots, u_d) \in \boldsymbol{C}^d$

$$(7.12) \quad m_u \equiv \sum_{p,q=1}^{d} \bar{u}_p u_q m_{pq} \text{ is a bounded Borel measure on } \boldsymbol{R}.$$

Let any $u \in \boldsymbol{C}^d$ be fixed. Since it follows from (4.54) that for any $\zeta \in \boldsymbol{C}^+$,

$$|((Z(\zeta)^{-1}\cdot\alpha + (Z(\zeta)^{-1}\cdot\alpha)^*)u, u)_{\boldsymbol{C}^d}| \le (u, u)_{\boldsymbol{C}^d}\cdot\mathrm{trace}\,((Z(\zeta)^{-1}\cdot\alpha + (Z(\zeta)^{-1}\cdot\alpha)^*),$$

by condition (v″-b), we can apply a generalized Lebesgue's convergence theorem ([13]) to find that, for any $\boldsymbol{C}$-valued bounded Borel measurable function f on $\boldsymbol{R}$,

$$(7.13) \quad \lim_{\varepsilon \downarrow 0} \int_{\boldsymbol{R}} f(\lambda)((Z(\lambda+i\varepsilon)^{-1}\cdot\alpha + (Z(\lambda+i\varepsilon)^{-1}\cdot\alpha)^*)u, u)_{\boldsymbol{C}^d} d\lambda = \int_{\boldsymbol{R}} f(\lambda) \lim_{\varepsilon \downarrow 0}((Z(\lambda+i\varepsilon)^{-1}\cdot\alpha + (Z(\lambda+i\varepsilon)^{-1}\cdot\alpha)^*)u, u)_{\boldsymbol{C}^d} d\lambda.$$

Therefore, by (7.11), (7.12) and (7.13), for any $\zeta = \xi + i\eta \in \boldsymbol{C}^+$,

$$(7.14) \quad \frac{1}{\pi}\int_{\boldsymbol{R}} \frac{\eta}{(\lambda-\xi)^2+\eta^2} m_u(d\lambda) = \lim_{\varepsilon \downarrow 0} \int_{\boldsymbol{R}} \frac{1}{\pi} \frac{\eta}{(\lambda-\xi)^2+\eta^2} \frac{1}{\sqrt{2\pi}}((Z(\lambda+i\varepsilon)^{-1}\cdot\alpha + {} + (Z(\lambda+i\varepsilon)^{-1}\cdot\alpha)^*)u, u)_{\boldsymbol{C}^d} d\lambda.$$

On the other hand, by (5. 11) and (5. 15), for any $\lambda\in\boldsymbol{R}$ and $\varepsilon>0$,

$$(7.15)\qquad ((Z(\lambda+i\varepsilon)^{-1}\bullet\alpha+(Z(\lambda+i\varepsilon)^{-1}\bullet\alpha)^*)u,\,u)_{C^d} = \sqrt{\frac{2}{\pi}}\int_{\boldsymbol{R}}\frac{\varepsilon}{(\tau-\lambda)^2+\varepsilon^2}\Delta_u(d\tau),$$

where $\Delta_u\equiv\sum_{p,q=1}^{d}\bar{u}_p u_q\Delta_{pq}$.

By substituting (7. 15) into (7. 14), we find that for any $\zeta=\xi+i\eta\in\boldsymbol{C}^+$

$$\frac{1}{\pi}\int_{\boldsymbol{R}}\frac{\eta}{(\lambda-\xi)^2+\eta^2}m_u(d\lambda)$$
$$=\lim_{\varepsilon\downarrow 0}\int_{\boldsymbol{R}}\Big(\int_{\boldsymbol{R}}\frac{1}{\pi}\frac{\varepsilon}{(\tau-\lambda)^2+\varepsilon^2}\bullet\frac{1}{\pi}\frac{\eta}{(\lambda-\xi)^2+\eta^2}d\lambda\Big)\Delta_u(d\tau)$$
$$=\int_{\boldsymbol{R}}\frac{1}{\pi}\frac{\eta}{(\tau-\xi)^2+\eta^2}\Delta_u(d\tau),$$

which gives

$$(7.16)\qquad m_u(d\lambda)=\Delta_u(d\lambda).$$

Hence, it follows from (7. 11) and (7. 16) that R belongs to $\mathscr{R}_{ds}$. (Q. E. D.)

Finally, by noting (7. 8), (7. 9) and (7. 11), we can characterize the set $\Phi(\mathscr{R}_{sm}\cap\mathscr{R}_{ds})$ as follows.

THEOREM 7. 2

$$\Phi(\mathscr{R}_{sm}\cap\mathscr{R}_{ds})$$
$$=\{(\alpha,\beta,\kappa)\in\mathscr{L}_{ds};$$
$$(\mathrm{vi})'\int_{\boldsymbol{R}}\xi^2\lim_{\eta\downarrow 0}\mathrm{trace}\ (Z(\xi+i\eta)^{-1}\bullet\alpha+(Z(\xi+i\eta)^{-1}\bullet\alpha)^*)d\xi<\infty\}.$$

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