On the theory of discrete KMO-Langevin equations with reflection positivity （Ⅰ）

To the memory of Professor Gishiro Maruyama

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§ 1. Introduction

In a series of papers （[6]~[10]）, the author has developed a theory of generalized Langevin equations for real continuous-time stationary Gaussian proccsses with reflection positivity. The time evolution of such a process X (t) can be described in terms of two kinds of Langevin equations with a notable difference in character of random forces （[9]）: One is the **first KMO-Langevin equation** having a **white noise** as a random force, and it has a root in his study （[7]） and （[8]） of the $[\alpha, \beta, \gamma]$-Langevin equations. The other is the **second KMO-Langevin equation** where a colored noise named the **Kubo noise** is taken to be a random force.

With the linear response theory of R. Kubo （[4]） in mind, we established in [10] the **fluctuation-dissipation theorems** based on these Langevin equations of the two types; our discovery was that the classical Einstein relation for Ornstein-Uhlenbeck processes holds for the second type, but does not hold for the first one. In addition, we calculated the deviation from the classical Einstein relation. As a concrete example in physics, we discussed the **Stokes-Boussinesque-Langevin equation** with the **Alder-Wainwright effect** within our framework of the theory of KMO-Langevin equations possessing reflection positivity.

The purpose of the present and subsequent papers is to establish the discrete analogues of the results mentioned above for the first and second KMO-Langevin equations. Further development of these results will be discussed in the author's forthcomming third paper, with the same title. In contrast to the continuous-time case, we will find that the Einstein relation for discrete-time series $X(n)$ always deviates from the classical one in the Markovian case, not only for the first type （see § 7 of the present （Ⅰ）), but also for the second one （see § 6 of （Ⅱ）). In the third paper （[12]）, we will discuss an entropy criterion and present an answer to the basic question in

the problem of modelling such as : " Which noise, white or Kubo, should be taken to be a random force in the equation with given coefficients (α, β, γ) called the KMO-Langevin data ? ". It should be noted that a variety of covariance functions arising from the theory of ARMA processes ([1]) and of one-dimensional transformations ([5]) are all realized as those of real discrete-time stationary Gaussian processes with reflection positivity. And so they enter into the present framework of the theory of KMO-Langevin equations.

Now we will state the content of this paper. Let $X = (X(n) ; n \in Z)$ be a real stationary Gaussian process with mean zero and covariance function R :

$$(1.1) \qquad R(n) = E(X(n)X(0)) \quad (n \in Z).$$

In § 2 we will briefly recall the spectral theory of X. In particular, for the Hardy spectral density Δ of X such that $\log \Delta \in L^1((-\pi, \pi))$, we define the outer function h of X on $U_1(0) \equiv \{z \in C ; |z| < 1\}$ by

$$(1.2) \qquad h(z) = \exp(\frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log \Delta(\theta) d\theta).$$

In a similar manner to the continuous-time case discussed in [6], we will in § 3 define the reflection positivity of X, which can be characterized by the condition : there exists a bounded Borel measure σ on $[-1, 1]$ such that

$$(1.3) \qquad R(n) = \int_{[-1,1]} t^{|n|} \sigma(dt) \quad (n \in Z).$$

The following conditions are assumed in what follows :

$$(1.4) \qquad \sigma(\{-1, 1\}) = 0$$

$$(1.5) \qquad \int_{-1}^{1} \left(\frac{1}{1+t} + \frac{1}{1-t} \right) \sigma(dt) < \infty.$$

From these conditions it follows that

$$(1.6) \qquad R \in l^1(Z).$$

Section 4 is devoted to the study of the structure of the outer function h as well as of the canonical representation kernel $E = \hat{h}$, which will play an important role in this paper. By using the result for the continuous-time case obtained in [9], we have

THEOREM 4.1.   *There exists a unique triple* $(\alpha_1, \beta_1, \rho_1)$ *such that*
(i) $\alpha_1 > 0$ *and* $\beta_1 > 0$
(ii) ρ_1 *is a bounded Borel measure on* $[-1, 1]$ *with* $\rho_1(\{-1, 1\}) = 0$

(iii) *for any* $z \in U_1(0)$

$$h(z) = \frac{\alpha_1}{\sqrt{2\pi}} \frac{1}{\beta_1(1+z) + 1 - z + (1-z^2)\int_{-1}^{1} \frac{1}{1-tz}\rho_1(dt)}.$$

We call the triple $(\alpha_1, \beta_1, \rho_1)$ in Theorem 4.1 the **first KMO-Langevin data** associated with σ(or R). Another triple $(\alpha_1, \beta_1, \gamma_1)$ is also called by the same name, where γ_1 is a function on Z defined by

$$(1.7) \qquad \gamma_1 = \frac{1}{2\pi}((1-e^{2i\cdot})\int_{-1}^{1}\frac{1}{1-te^{i\cdot}}\rho_1(dt))^{\wedge}.$$

More explicitly,

$$(1.8) \qquad \gamma_1(n) = \begin{cases} 0 & \text{for } n \in \{-1, -2, \ldots\} \\ \int_{-1}^{1} t^n \rho_1(dt) & \text{for } n \in \{0, 1\} \\ \int_{-1}^{1} t^{n-2}(t^2-1)\rho_1(dt) & \text{for } n \in \{2, 3, \ldots\}, \end{cases}$$

which implies that

$$(1.9) \qquad \gamma_1 \in l^1(Z).$$

It will be found in §5 that the correspondence between σ and $(\alpha_1, \beta_1, \rho_1)$ is bijective (Theorem 5.1). Furthermore we will obtain an explicit formula of the triple $(\alpha_1, \beta_1, \gamma_1)$ in terms of σ (Theorem 5.2).

By using Theorem 4.1, we will in §6 derive a stochastic difference equation with a white noise as its random force describing the time evolution of X.

THEOREM 6.1.

$$(1.10) \qquad X(n) - X(n-1) = -\beta_1(X(n) + X(n-1)) - (\gamma_1 * X)(n) + \alpha_1 \xi(n)$$
$$a.\ s.\ (n \in Z),$$

where $\boldsymbol{\xi} = (\xi(n) ; n \in Z)$ *is a real Gaussian white noise.*

This equation (1.10) is nothing but the **first KMO-Langevin equation** we are looking for in the discrete-time case. Conversely, we will show that the first KMO-Langevin equation can be uniquely solved for any given triple $(\alpha_1, \beta_1, \rho_1)$ with conditions (i) and (ii) in Theorem 4.1 and a real Gaussian white noise $\boldsymbol{\xi}$ (Theorem 6.2).

Fundamental examples of X are given by the Markov processes X_p, $-1 < p < 1$, with covariance functions R_p:

$$(1.11) \qquad R_p(n) = p^{|n|} \quad (n \in Z).$$

It then follows that the outer function h_p of X_p becomes

$$h_p(z) = \frac{1}{\sqrt{2\pi}} \frac{\sqrt{1-p^2}}{1-pz} \qquad (z \in U_1(0)).$$

By rewriting it in the form (iii) in Theorem 4. 1, we have

$$(1.12) \qquad h_p(z) = \frac{\alpha_p^{(1)}}{\sqrt{2\pi}} \frac{1}{\beta_p^{(1)}(1+z)+1-z},$$

where

$$(1.13) \qquad \alpha_p^{(1)} = 2\sqrt{\frac{1-p}{1+p}} \quad \text{and} \quad \beta_p^{(1)} = \frac{1-p}{1+p}.$$

Therefore, we find from Theorem 6. 1 that the time evolution of X_p is governed by

$$(1.14) \qquad X_p(n) - X_p(n-1) = -\beta_p^{(1)}(X_p(n) + X_p(n-1)) + \alpha_p^{(1)} X_0(n)$$
$$a. s. \ (n \in \mathbf{Z}).$$

We note that X_0 is a white noise.

Concerning a discrete analogue of the generalized fluctuation-dissipation theorem for the continuous-time case discussed in [10], we will in § 7 show

THEOREM 7. 1.

(i) *For any* $\theta \in (-\pi, \pi)$

$$\frac{1}{\beta_1(1+e^{i\theta})+1-e^{i\theta}+2\pi\tilde{\gamma}_1(\theta)} = \frac{h(e^{i\theta})}{2 \lim_{\tau \downarrow -\pi} h(e^{i\tau})}$$

(ii) $\dfrac{\alpha_1^2}{2} = R(0) C_{\beta_1, \gamma_1},$

where

$$(1.15) \qquad C_{\beta_1, \gamma_1} = \pi \left(\int_{-\pi}^{\pi} |\beta_1(1+e^{i\theta})+1-e^{i\theta}+2\pi\tilde{\gamma}_1(\theta)|^{-2} d\theta \right)^{-1}$$

(iii) $D = \dfrac{\alpha_1^2}{2(2\beta_1)^2},$

where

$$(1.16) \qquad D = \lim_{N \to \infty} \frac{1}{2N} E\left(\left(\sum_{n=0}^{N} X(n) \right)^2 \right).$$

(iv) $D = \dfrac{R(0)}{2\beta_1} \dfrac{C_{\beta_1, \gamma_1}}{2\beta_1}$

(v) $\dfrac{C_{\beta_1, \gamma_1}}{2\beta_1} - 1 = \dfrac{1}{R(0)} \displaystyle\int_{-1}^{1} \int_{-1}^{1} \frac{1+t}{1-tu} \sigma(dt) \rho_1(du).$

By taking account of the physical meaning of the generalized fluctuation-dissipation theorem for the continuous-time case given in [10], we call the relations (ⅰ), (ⅱ) and (ⅳ) in Theorem 7.1 the **generalized first fluctuation-dissipation theorem**, the **generalized second fluctuation-dissipation theorem** and the **generalized Einstein relation**, respectively. And we call the constants C_{β_1, γ_1} in (1.15) and D in (1.16) the **generalized friction coefficient** and the **diffusion constant** of X, respectively. We note that this D is also expressed in terms of R :

$$(1.17) \quad D = \sum_{n=0}^{\infty} R(n) - \frac{R(0)}{2}.$$

In particular, the diffusion constant D_p of X_p is given by

$$(1.18) \quad D_p = \frac{R_p(0)}{2\beta_p^{(1)}} \left(= \frac{1+p}{2(1-p)} \right) \qquad (p \in (-1, 1)).$$

This expression is the classical Einstein relation. And so we conclude from the generalized Einstein relation (ⅳ) in Theorem 7.1 that there occures a deviation from the classical Einstein relation (1.18) in general non-Markovian cases. Such a deviation can be calculated explicitly by the formula (ⅴ) in Theorem 7.1.

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§ 2. Preliminaries

Let $\boldsymbol{X} = (X(n) \; ; \; n \in \boldsymbol{Z})$ be a real stationary Gaussian process with discrete time on a probability space $(\Omega, \mathscr{F}, P)$ and let R be its covariance function :

$$(2.1) \quad R(n) = E(X(n)X(0)) \quad (n \in \boldsymbol{Z}).$$

We assume that the spectral measure of $\boldsymbol{X}$ has a spectral density $\varDelta = \varDelta(\theta)$ such that

$$(2.2) \quad \log \varDelta \in L^1((-\pi, \pi)),$$

$$(2.3) \quad R(n) = \int_{-\pi}^{\pi} e^{-in\theta} \varDelta(\theta) d\theta.$$

Then we can define an outer function h of $\varDelta$ by

$$(2.4) \quad h(z) = \exp\{\frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log \varDelta(\theta) d\theta\}$$

for $z \in U_1(0) = \{z \in \boldsymbol{C} \; ; \; |z| < 1\}$. Such a function h has the following prop-

erties ([2]) :

(2.5) $\displaystyle\sup_{0<r<1} \int_{-\pi}^{\pi} |h(re^{i\theta})|^2 d\theta < \infty$;

(2.6) $h(e^{i\theta}) = \displaystyle\lim_{r\uparrow 1} h(re^{i\theta})$ exists $\qquad a.\,e.\ \theta \in (-\pi, \pi)$;

(2.7) $|h(e^{i\theta})|^2 = \Delta(\theta)$ $\qquad\qquad\qquad a.\,e.\ \theta \in (-\pi, \pi)$;

(2.8) $\displaystyle\lim_{r\uparrow 1} h(re^{i\cdot}) = h(e^{i\cdot})$ $\qquad\qquad in\ L^2((-\pi, \pi))$.

Next we define a function E on $\mathbf{Z}$ by

(2.9) $E(n) = \hat{h}(n) = \displaystyle\int_{-\pi}^{\pi} e^{-in\theta} h(e^{i\theta}) d\theta.$

It then follows from the above properties of h that

(2.10) $E \in l^2(\mathbf{Z})$,

(2.11) $E(n) = 0$ $(n = -1, -2, -3, \ldots)$,

(2.12) $\displaystyle\lim_{n\to\infty} E(n) = 0,$ and

(2.13) $R(n) = \dfrac{1}{2\pi} \displaystyle\sum_{m=0}^{\infty} E(|n|+m)E(m)$ $(n \in \mathbf{Z})$.

As is well known ([3]), there exists a normalized Gaussian white noise $\boldsymbol{\xi} = (\xi(n)\ ;\ n \in \mathbf{Z})$ such that

(2.14) $X(n) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\sum_{m=-\infty}^{n} E(n-m)\xi(m)$ $\qquad in\ L^2(\Omega, \mathscr{F}, P).$

(2.15) $\sigma(X(m)\ ;\ m \le n) = \sigma(\xi(m)\ ;\ m \le n)$ $\qquad (n \in \mathbf{Z}).$

By taking account of what have been obtained so far, the function E is said to be a canonical representation kernel of X.

§ 3. Reflection positivity

Let $X = (X(n)\ ;\ n \in \mathbf{Z})$ be a real stationary Gaussian process on $(\Omega, \mathscr{F}, P)$ with covariance function R. We denote by M (resp. M^+) the closed linear hull of $\{X(n)\ ;\ n \in \mathbf{Z}\}$ (resp. $\{X(n)\ ;\ n \ge 0\}$) in $L^2(\Omega, \mathscr{F}, P)$ and by P_{M^+} the orthogonal projection on M^+. The time reflection operator T is a unitary and self-adjoint operator on M defined by

$T(X(n)) = X(-n)$ $\quad (n \in \mathbf{Z}).$

As in the continuous-time case ([6]), we say that X has **reflection positivity** (T-**positivity**) if and only if $P_{M^+}TP_{M^+}$ is non-negative.

In the sequel, we assume that X has reflection positivity. By taking the same consideration as § 2 in [6], we see that there exists a unique Borel

measure σ on $[-1, 1]$ such that

(3.1) $$R(n) = \int_{[-1,1]} t^{|n|} \sigma(dt) \quad (n \in \mathbf{Z}).$$

EXAMPLE 3.1   For each $p \in (-1, 1)$ consider a non-negative definite function R_p corresponding to the case $\sigma = \delta_{\{p\}}$, i. e.,

(3.2) $$R_p(n) = p^{|n|}.$$

Then the spectral density Δ_p of R_p is given by

(3.3) $$\Delta_p(\theta) = \frac{1}{2\pi} \frac{1-p^2}{|1-pe^{i\theta}|^2} \quad (\theta \in (-\pi, \pi)).$$

We now impose the additional assumption on the measure σ in (3.1):

(3.4) $$\sigma(\{-1, 1\}) = 0.$$

Then it immediately follows that the spectral measure of R has the density Δ of the form

(3.5) $$\Delta(\theta) = \frac{1}{2\pi} \int_{-1}^{1} \frac{1-t^2}{|1-te^{i\theta}|^2} \sigma(dt) \quad (\theta \in (-\pi, \pi)).$$

By using a homeomorphism φ from $(-1, 1]$ onto $[0, \infty)$:

(3.6) $$\varphi(t) = \frac{1-t}{1+t},$$

we define a bounded Borel measure σ_c on $[0, \infty)$ by

(3.7) $$\sigma_c = \varphi(\sigma).$$

A direct calculation yields that

(3.8) $$\Delta(2\tan^{-1}\xi) = \frac{1+\xi^2}{2\pi} \int_0^{\infty} \frac{\lambda}{\xi^2+\lambda^2} \sigma_c(d\lambda) \quad (\xi \in \mathbf{R}),$$

which leads us to consider

(3.9) $$\Delta_c(\xi) = \frac{1}{\pi} \int_0^{\infty} \frac{\lambda}{\xi^2+\lambda^2} \sigma_c(d\lambda).$$

Since $\sigma_c(\{0\}) = 0$, it follows from Lemma 2.12 in [6] that Δ_c is a Hardy weight, that is,

(3.10) $$\frac{\log \Delta_c(\xi)}{1+\xi^2} \in L^1(\mathbf{R}).$$

And so we can define an outer function h_c of Δ_c on $\mathbf{C}^+$ by

(3.11) $h_c(\zeta) = \exp\{\dfrac{1}{2\pi i} \displaystyle\int_R \dfrac{1+\xi\zeta}{\xi-\zeta} \dfrac{\log\Delta_c(\xi)}{1+\xi^2} d\xi\}.$

Noting that

$$\int_{-\pi}^{\pi} |\log\Delta(\theta)| d\theta = 2 \int_R \left| \frac{\log\Delta(2\tan^{-1}\xi)}{1+\xi^2} \right| d\xi,$$

we see from (3.8), (3.9) and (3.10) that the spectral density Δ of R satisfies condition (2.2). By a direct calculation, the outer function h of Δ can be rewritten into the form

(3.12) $h(z) = \exp\{\dfrac{1}{2\pi i} \displaystyle\int_R \dfrac{1+\lambda\zeta}{\lambda-\zeta} \dfrac{\log\Delta(2\tan^{-1}\lambda)}{1+\lambda^2} d\lambda\},$

where $z \in U_1(0)$ and $\zeta = i\dfrac{1-z}{1+z} \in C^+$.

Since for any $\zeta \in C^+$

$$\exp\{\frac{1}{2\pi i} \int_R \frac{1+\xi\zeta}{\xi-\zeta} \frac{\log(2/1+\xi^2)}{1+\xi^2} d\xi\} = \frac{\sqrt{2}}{1-i\zeta},$$

it follows from (3.8), (3.9), (3.11) and (3.12) that

Iemma 3.1.

(ⅰ) $h(z) = \dfrac{\sqrt{2}}{1+z} h_c\left(i\dfrac{1-z}{1+z}\right)$ $(z \in U_1(0))$

(ⅱ) $h_c(\zeta) = \dfrac{\sqrt{2}}{1-i\zeta} h\left(\dfrac{i-\zeta}{i+\zeta}\right)$ $(\zeta \in C^+).$

Furthermore, immediately from (3.7), we obtain

Lemma 3.2.

(ⅰ) $\displaystyle\int_{-1}^{1} \dfrac{1+t}{1-t} \sigma(dt) = \int_0^\infty \lambda^{-1} \sigma_c(d\lambda)$

(ⅱ) $\displaystyle\int_{-1}^{1} \dfrac{1-t}{1+t} \sigma(dt) = \int_0^\infty \lambda \sigma_c(d\lambda).$

§4. Outher function h and canonical representation kernel E

Let $X = (X(n)\,;\, n \in Z)$ be a real stationary Gaussian process on $(\Omega, \mathscr{F}, P)$ satisfying reflection positivity. It then follows that the covariance function R of X has such a representation as (3.1) with a bounded Borel measure σ on $[-1, 1]$. In the sequel, we assume the following conditions:

(4.1) $\sigma(\{-1, 1\}) = 0$

(4.2) $\displaystyle\int_{-1}^{1} \left(\dfrac{1}{1+t} + \dfrac{1}{1-t}\right) \sigma(dt) < \infty.$

At first we will show the following structure theorem for the outer function h of X, which will be the key to derive a description of the time evolution of X in § 6.

THEOREM 4.1. *There exists a unique triple* $(\alpha_1, \beta_1, \rho_1)$ *such that*

（ⅰ） $\alpha_1 > 0$ *and* $\beta_1 > 0$

（ⅱ） ρ_1 *is a bounded Borel measure on* $[-1, 1]$ *with* $\rho_1(\{-1, 1\}) = 0$

（ⅲ） *for any* $z \in U_1(0)$

$$h(z) = \frac{\alpha_1}{\sqrt{2\pi}} \; \frac{1}{\beta_1(1+z) + 1 - z + (1-z^2) \int_{-1}^{1} \frac{1}{1-tz}\rho_1(dt)}.$$

PROOF. Since it follows from Lemma 3.2 and condition (4.2) that $\int_0^\infty (\lambda^{-1} + \lambda)\sigma_c(d\lambda) < \infty$, we can apply Theorem 2.2 in [9] to see that there exists a unique triple $(\alpha_c, \beta_c, \rho_c)$ such that

(4.3) $\alpha_c > 0$ and $\beta_c > 0$

(4.4) ρ_c is a Borel measure on $[0, \infty)$ satisfying

$$\rho_c(\{0\}) = 0 \quad \text{and} \quad \int_0^\infty \frac{1}{1+\lambda}\rho_c(d\lambda) < \infty$$

(4.5) $h_c(\zeta) = \dfrac{\alpha_c}{\sqrt{2\pi}} \; \dfrac{1}{\beta_c - i\zeta - i\zeta \int_0^\infty \frac{1}{\lambda - i\zeta}\rho_c(d\lambda)}$ for any $\zeta \in C^+$.

Then we define a triple $(\alpha_1, \beta_1, \rho_1)$ by

(4.6) $\alpha_1 = \sqrt{2}\,\alpha_c$

(4.7) $\beta_1 = \beta_c$

(4.8) $\rho_1(dt) = \dfrac{1+t}{2}(\varphi^{-1}\rho_c)(dt).$

In particular, we have

(4.9) $\rho_1([-1, 1]) = \displaystyle\int_0^\infty \frac{1}{\lambda + 1}\rho_c(d\lambda).$

Therefore, we find that the triple $(\alpha_1, \beta_1, \rho_1)$ satisfies （ⅰ） and （ⅱ） in Theorem 4.1. Furthermore, it follows from Lemma 3.1 （ⅰ） and (4.5) that for any $z \in U_1(0)$

$$h(z) = \frac{\sqrt{2}}{1+z} \; \frac{\alpha_c}{\sqrt{2\pi}} \; \frac{1}{\beta_c + \dfrac{1-z}{1+z} + \dfrac{1-z}{1+z} \int_0^\infty \left(\lambda + \dfrac{1-z}{1+z}\right)^{-1}\rho_c(d\lambda)}.$$

On the other hand, by the definition of φ,

$$\int_0^\infty \left(\lambda + \frac{1-z}{1+z}\right)^{-1} \rho_c(d\lambda) = \int_{-1}^1 \left(\frac{1-t}{1+t} + \frac{1-z}{1+z}\right)^{-1} (\varphi^{-1}\rho_c)(dt)$$

$$= (1+z) \int_{-1}^1 \frac{1}{1-tz} \frac{1+t}{2} (\varphi^{-1}\rho_c)(dt).$$

Therefore, we find from (4.6), (4.7) and (4.8) that the triple $(\alpha_1, \beta_1, \rho_1)$ satisfies relation (iii) in Theorem 4.1.

The uniqueness of such a triple $(\alpha_1, \beta_1, \rho_1)$ can be proved as follows: since $0 \le \dfrac{(1+x)(1-x)}{1-tx} \le 2$ for any t, $x \in (-1, 1)$, we see that

(4.10) $$\lim_{x \downarrow -1} h(x) = \frac{\alpha_1}{2\sqrt{2\pi}}$$

and

(4.11) $$\lim_{x \uparrow 1} h(x) = \frac{\alpha_1}{2\sqrt{2\pi}\beta_1},$$

which determine uniquely the pair (α_1, β_1). Furthermore, since

$$\int_{-1}^1 \frac{1}{1+xt} \rho_1(dt) = \frac{1}{x} \int_0^1 \frac{1}{\dfrac{1-x}{x}+s} \rho_1(d(s-1)) \quad \text{for any } x \in (0, 1),$$

it follows from the uniqueness of Stieltjes transform that the measure ρ_1 is uniquely determined. (Q. E. D.)

Next we will show the following expression of h, closely related to the one established in Theorem 4.1.

THEOREM 4.2. *There exists a unique Borel measure v on* $[-1, 1]$ *such that*

(i) $v(\{-1, 1\}) = 0$ *and* $v([-1, 1]) < \infty$

(ii) $\displaystyle\int_{-1}^1 \left(\frac{1}{1-t} + \frac{1}{1+t}\right) v(dt) < \infty$

(iii) $h(z) = \dfrac{1}{2\pi} \displaystyle\int_{-1}^1 \frac{1}{1-tz} v(dt)$ *for any* $z \in U_1(0)$.

PROOF. By Theorem 2.1 and (2.19) in [9], we see that there exists a unique Borel measure v_c on $[0, \infty)$ such that

(4.12) $v_c(\{0\}) = 0$ and $v_c([0, \infty]) < \infty$

(4.13) $\displaystyle\int_0^\infty \lambda^{-1} v_c(d\lambda) < \infty$

$$(4.14) \qquad h_c(\zeta) = \frac{1}{2\pi} \int_0^\infty \frac{1}{\lambda - i\zeta} \nu_c(d\lambda) \qquad \text{for any } \zeta \in C^+.$$

Then we define a Borel measure ν on $[-1, 1]$ by

$$(4.15) \qquad \nu(dt) = \frac{1+t}{\sqrt{2}} (\varphi^{-1}\nu_c)(dt).$$

Since

$$\int_0^\infty \lambda^{-1}\nu_c(d\lambda) = \int_{-1}^1 (\varphi(t))^{-1}(\varphi^{-1}\nu_c)(dt)$$

$$= \sqrt{2} \int_{-1}^1 \frac{1}{1-t} \nu(dt),$$

it follows from (4.12), (4.13) and (4.15) that the measure ν satisfies （ⅰ） and（ⅱ） in Theorem 4.2. Furthermore, we see from Lemma 3.1 （ⅰ） and (4.14) that for any $z \in U_1(0)$

$$h(z) = \frac{\sqrt{2}}{1+z} \frac{1}{2\pi} \int_0^\infty \left(\lambda + \frac{1-z}{1+z}\right)^{-1} \nu_c(d\lambda)$$

$$= \frac{\sqrt{2}}{1+z} \frac{1}{2\pi} \int_{-1}^1 \left(\frac{1-t}{1+t} + \frac{1-z}{1+z}\right)^{-1} (\varphi^{-1}\nu_c)(dt)$$

$$= \frac{1}{2\pi} \int_{-1}^1 \frac{1}{1-tz} \frac{1+t}{\sqrt{2}} (\varphi^{-1}\nu_c)(dt),$$

which, together with (4.15), yields （ⅲ） in Theorem 4.2. The uniqueness of a Borel measure ν satisfying （ⅰ）, （ⅱ） and （ⅲ） can be proved by using the uniqueness of Stieltjes transform.   (Q. E. D.)

By taking a boundary value of the outer function h, we can rephrase Theorems 4.1 and 4.2 as follows.

COROLLARY 4.1.   *For almost all* $\theta \in (-\pi, \pi)$,

（ⅰ） $h(e^{i\theta}) = \dfrac{\alpha_1}{\sqrt{2\pi}} \dfrac{1}{\beta_1(1+e^{i\theta}) + 1 - e^{i\theta} + (1-e^{2i\theta}) \displaystyle\int_{-1}^1 \dfrac{1}{1-te^{i\theta}} \rho_1(dt)}$

（ⅱ） $h(e^{i\theta}) = \dfrac{1}{2\pi} \displaystyle\int_{-1}^1 \dfrac{1}{1-te^{i\theta}} \nu(dt).$

As a consequence of the above expression （ⅱ） of h, we can obtain an expression of the canonical representation kernel E in (2.9), which says that $E(n)$, $n \geq 0$, is nothing but the moment sequence of the measure ν.

THEOREM 4.3.

$$E(n) = \chi_{[0,\infty)}(n) \int_{-1}^1 t^n \nu(dt) \qquad \text{for any } n \in Z.$$

PROOF.   By Theorem 4.2 (ii) and Corollary 4.1 (ii), the following series

$$h(e^{i\theta}) = \frac{1}{2\pi} \sum_{n=0}^{\infty} e^{i\theta n} \left(\int_{-1}^{1} t^n \nu(dt) \right)$$

is absolutely convergent (a.e.).   Therefore, by (2.9), we have the assertion.   (Q. E. D.)

Concerning a relation between the measures σ in (3.1) and ν in Theorem 4.2, we will show

THEOREM 4.4.

$$(4.16) \qquad \sigma(dt) = \frac{1}{2\pi} \left(\int_{-1}^{1} \frac{1}{1-ts} \nu(ds) \right) \nu(dt).$$

PROOF.   By Theorem 4.3, for any $n \in \{0, 1, 2, \dots\}$,

$$\sum_{m=0}^{\infty} E(n+m) E(m) = \int_{-1}^{1} t^n \left(\int_{-1}^{1} \frac{1}{1-ts} \nu(ds) \right) \nu(dt).$$

Therefore, by (2.14) and (3.1), we have Theorem 4.4.   (Q. E. D.)

REMARK 4.1.   By Lemma 2.6 (i) in [9], the measure σ_c in (3.7) is related to the measure ν_c in (4.12) as follows:

$$(4.17) \qquad \sigma_c(d\lambda) = \frac{1}{2\pi} \left(\int_{0}^{\infty} \frac{1}{\lambda + \lambda'} \nu_c(d\lambda') \right) \nu_c(d\lambda).$$

In view of (3.7) and (4.15), it turns out that this equality is equivalent to (4.16).

EXAMPLE 4.1.   For each fixed $p \in (-1, 1)$, consider the non-neagtive definite function R_p in (3.2).   Then we see that the outer function h_p and the canonical representation kernel E_p become

$$(4.18) \qquad h_p(z) = \frac{1}{\sqrt{2\pi}} \frac{\sqrt{1-p^2}}{1-pz} \qquad\qquad (z \in U_1(0))$$

$$(4.19) \qquad E_p(n) = \chi_{[0,\infty)}(n) \sqrt{2\pi(1-p^2)}\, p^n \qquad\qquad (n \in \mathbf{Z}).$$

Furthermore we observe that

$$(4.20) \qquad h_p(z) = \frac{1}{\sqrt{2\pi}} 2\sqrt{\frac{1-p}{1+p}} \frac{1}{\dfrac{1-p}{1+p}(1+z) + 1 - z} \qquad (z \in U_1(0),$$

which implies that the triple $(\alpha_p^{(1)}, \beta_p^{(1)}, \rho_p^{(1)})$ in Theorem 4.1 associated with R_p becomes

$$(4.21) \quad \alpha_p^{(1)} = 2\sqrt{\frac{1-p}{1+p}}, \quad \beta_p^{(1)} = \frac{1-p}{1+p} \text{ and } \rho_p^{(1)} = 0.$$

EXAMPLE 4.2.   Let R be the non-negative definite function given by

$$(4.22) \quad R(n) = \sigma_1 p_1^{|n|} + \sigma_2 p_2^{|n|} \quad (n \in \mathbf{Z}),$$

where σ_1, $\sigma_2 > 0$ and $-1 < p_1 < p_2 < 1$.   Put

$$(4.23) \quad a_1 = \sigma_1(1-p_1^2)p_2 + \sigma_2(1-p_2^2)p_1$$
$$(4.24) \quad a_2 = \sigma_1(1-p_1^2)(1+p_2^2) + \sigma_2(1-p_2^2)(1+p_1^2).$$

By a direct calculation, we can conclude that the outer function h, the triple $(\alpha_1, \beta_1, \rho_1)$ and the measure ν associated with this R are given as follows:

（ i ）   The case $a_1 = 0 (-1 < p_1 < 0 < p_2 < 1)$:

$$(4.25) \quad h(z) = \sqrt{\frac{a_2}{2\pi}} \frac{1}{(1-p_1 z)(1-p_2 z)} \quad (z \in (U_1(0)))$$

$$(4.26) \quad \begin{cases} \alpha_1 = \dfrac{2\sqrt{a_2}}{(1+p_1)(1+p_2)} \\[2mm] \beta_1 = \dfrac{(1-p_1)(1-p_2)}{(1+p_1)(1+p_2)} \\[2mm] \rho_1(dt) = \dfrac{-2p_1 p_2}{(1+p_1)(1+p_2)} \delta_{\{0\}}(dt) \end{cases}$$

$$(4.27) \quad \nu = \frac{\sqrt{2\pi a_2}}{p_2 - p_1} \{ (-p_1)\delta_{\{p_1\}} + p_2 \delta_{\{p_2\}} \}$$

（ ii ）   The case $a_1 \neq 0$:

$$(4.28) \quad h(z) = \sqrt{\frac{r_1}{2\pi}} \frac{1-q_1 z}{(1-p_1 z)(1-p_2 z)} \quad (z \in U_1(0))$$

$$(4.29) \quad \begin{cases} \alpha_1 = \dfrac{2\sqrt{r_1}(1+q_1)}{(1+p_1)(1+p_2)} \\[2mm] \beta_1 = \dfrac{(1+q_1)(1-p_1)(1-p_2)}{(1-q_1)(1+p_1)(1+p_2)} \\[2mm] \rho_1(dt) = \dfrac{2(q_1-p_1)(p_2-q_1)}{(1-q_1)(1+p_1)(1+p_2)} \delta_{\{q_1\}}(dt) \end{cases}$$

$$(4.30) \quad \nu = \frac{\sqrt{2\pi r_1}}{p_2 - p_1} \{ (q_1-p_1)\delta_{\{p_1\}} + (p_2-q_1)\delta_{\{p_2\}} \},$$

where

$$(4.31) \quad q_1 = \frac{1}{2}\left(\frac{a_2}{a_{1(+)}} \sqrt{\left(\frac{a_2}{a_1}\right)^2 - 4} \right) \quad \text{if } a_1 \underset{(<)}{>} 0$$

$$(4.32) \quad r_1 = \frac{a_1}{q_1}.$$

We note that $p_1 < q_1 < p_2$ and $q_1 \underset{(<)}{>} 0$ if $a_1 \underset{(<)}{>} 0$.

§5. The first KMO-Langevin data

This section is devoted to the further study of the key expression of h in Theorem 4.1. In order to investigate the correspondence between σ in (3.1) and $(\alpha_1, \beta_1, \rho_1)$ in Theorem 4.1, we introduce two classes Σ_1 and $\mathscr{L}_1$ by

(5.1) $\Sigma_1 = \{\sigma \ ; \ \sigma$ is a bounded Borel measure on $[-1, 1]$ such that

$$\sigma(\{-1, 1\}) = 0 \text{ and } \int_{-1}^{1} \left(\frac{1}{1+t} + \frac{1}{1-t}\right) \sigma(dt) < \infty\}$$

and

(5.2) $\mathscr{L}_1 = \{(\alpha, \beta, \rho) \ ; \ \alpha > 0, \ \beta > 0$ and ρ is a bounded Borel measure on $[-1, 1]$ such that $\rho(\{-1, 1\}) = 0 \}$.

For each $\sigma \in \Sigma_1$, we denote by R_σ, $\varDelta_\sigma$ and h_σ the non-negative definite function, the spectral density and the outer function associated with σ, respectively:

(5.3)
$$\begin{cases} R_\sigma(n) = \displaystyle\int_{-1}^{1} t^{|n|} \sigma(dt) & (n \in \mathbf{Z}) \\[2mm] \varDelta_\sigma(\theta) = \dfrac{1}{2\pi} \displaystyle\int_{-1}^{1} \dfrac{1-t^2}{|1-te^{i\theta}|^2} \sigma(dt) & (\theta \in (-\pi, \pi)) \\[2mm] h_\sigma(z) = \exp\{\dfrac{1}{4\pi} \displaystyle\int_{-\pi}^{\pi} \dfrac{e^{i\theta}+z}{e^{i\theta}-z} \log \varDelta_\sigma(\theta) d\theta\} & (z \in U_1(0)). \end{cases}$$

THEOREM 5.1.   *There exists a bijective mapping L_1 from Σ_1 onto $\mathscr{L}_1$ such that for any $\sigma \in \Sigma_1$ and $(\alpha_1, \beta_1, \rho_1) = L_1(\sigma) \in \Sigma_1$*

(5.4) $$h_\sigma(z) = \frac{\alpha_1}{\sqrt{2\pi}} \ \frac{1}{\beta_1(1+z) + 1 - z + (1-z^2) \displaystyle\int_{-1}^{1} \dfrac{1}{1-tz}\rho_1(dt)} (z \in U_1(0)).$$

PROOF.   By Theorem 4.1, we have an injective mapping L_1 from Σ_1 into $\mathscr{L}_1$ satisfying relation (5.4) and so we have only to show that L_1 is surjective. Let $(\alpha_1, \beta_1, \rho_1)$ be any element of $\mathscr{L}_1$. We define a new triple $(\alpha_c, \beta_c, \rho_c)$ by

(5.5) $$\alpha_c = \frac{\alpha_1}{\sqrt{2}}, \ \beta_c = \beta_1 \text{ and } \rho_c = \varphi\left(\frac{2}{1+\cdot}\rho_1\right).$$

By noting (4.9), we see from Theorem 3.1 in [9] that there exists a unique bounded Borel measure σ_c on $[0, \infty)$ such taht

(5.6) $\sigma_c(\{0\})=0$ and $\displaystyle\int_0^\infty (\lambda+\lambda^{-1})\sigma_c(d\lambda)<\infty$

(5.7) $h_c(\zeta)=\dfrac{\alpha_c}{\sqrt{2\pi}}\ \dfrac{1}{\beta_c-i\zeta-i\zeta\displaystyle\int_0^\infty \frac{1}{\lambda-i\zeta}\rho_c(d\lambda)}$ $(\zeta\in C^+)$,

where h_c is the outer function of the Hardy weight Δ_c of the form

(5.8) $\Delta_c(\xi)=\dfrac{1}{\pi}\displaystyle\int_0^\infty \dfrac{\lambda}{\xi^2+\lambda^2}\sigma_c(d\lambda)$ $(\xi\in R)$.

We then define a bounded Borel measure σ on $[-1,1]$ by

(5.9) $\sigma=\varphi^{-1}(\sigma_c)$.

By Lemma 3.2, (5.6) and (5.9), we see that $\sigma\in\Sigma_1$. Furthermore, it follows from Lemma 3.1 (ⅰ) and (3.12) that

$$h_\sigma(z)=\dfrac{\sqrt{2}}{1+z}h_c\Big(i\dfrac{1-z}{1+z}\Big)\qquad (z\in U_1(0))\,.$$

By combining this with (5.7) and then noting (5.5), we conclude that $L_1(\sigma)=(\alpha_1,\beta_1,\rho_1)$.   (Q. E. D.)

For each bounded Borel measure ρ on $[-1,1]$ with $\rho(\{-1,1\})=0$, we define a function γ on Z by

(5.10) $\gamma=\dfrac{1}{2\pi}((1-e^{2i\cdot})\displaystyle\int_{-1}^1 \dfrac{1}{1-te^{i\cdot}}\rho(dt))^\wedge$.

We note that γ is well-defined, because

(5.11) $\left|\dfrac{(1+e^{i\theta})(1-e^{i\theta})}{1-te^{i\theta}}\right|\le 2$ for any $t\in(-1,1)$ and $\theta\in(-\pi,\pi)$.

Some properties of γ are listed as

PROPOSITION 5.1.

(ⅰ) $\gamma(n)=\begin{cases}0 & \text{for } n\in\{-1,-2,\ldots\}\\[4pt]\displaystyle\int_{-1}^1 t^n\rho(dt) & \text{for } n\in\{0,1\}\\[10pt]\displaystyle\int_{-1}^1 (t^n-t^{n-2})\rho(dt) & \text{for } n\in\{2,3,\ldots\}\end{cases}$

(ⅱ) $\gamma\in l^1(Z)$

(ⅲ) $\displaystyle\sum_{n=0}^\infty \gamma(n)=0$

(ⅳ) $\displaystyle\sum_{n=0}^\infty (-1)^n\gamma(n)=0$.

PROOF.   By the definition of γ, we have

$$\gamma(n) = \frac{1}{2\pi} \int_{-1}^{1} \left(\int_{-\pi}^{\pi} e^{-in\theta} \frac{1-e^{i2\theta}}{1-te^{i\theta}} d\theta \right) \rho(dt) \qquad (n \in \mathbf{Z}).$$

Hence (i) follows from the simple fact that for any $t \in (-1, 1)$

$$\int_{-\pi}^{\pi} e^{-in\theta} \frac{1-e^{i2\theta}}{1-te^{i\theta}} d\theta = \begin{cases} 0 & \text{for } n \in \{-1, -2, ...\} \\ 2\pi t^n & \text{for } n \in \{0, 1\} \\ 2\pi(t^n - t^{n-2}) & \text{for } n \in \{2, 3, ...\}. \end{cases}$$

Since

$$\sum_{n=2}^{\infty} |\gamma(n)| \le \int_{-1}^{1} \left(\sum_{n=2}^{\infty} |t|^{n-2} \right) (1-t^2) \rho(dt) = \int_{-1}^{1} (1+|t|) \rho(dt) < \infty,$$

we have (ii). By (ii), we can take the inverse Fourier transform of (5.10) to get

$$2\pi \tilde{\gamma}(\theta) = (1 - e^{2i\theta}) \int_{-1}^{1} \frac{1}{1-te^{i\theta}} \rho(dt) \qquad \text{for any } \theta \in (-\pi, \pi).$$

By taking $\theta = 0$ and $\theta = -\pi$ in the above expression, we have (iii) and (iv), respectively.   (Q. E. D.)

DEFINITION 5. 2.   For each $\sigma \in \Sigma_1$, we call a triple $(\alpha_1, \beta_1, \rho_1)(=L_1(\sigma))$ or $(\alpha_1, \beta_1, \gamma_1)$ the **first KMO-Langevin data** associated with σ or R_σ.

We will give a formula concerning the first KMO-Langevin data $(\alpha_1, \beta_1, \gamma_1)$ associated with a fixed $\sigma(\in \Sigma_1)$.

THEOREM 5. 2.

(i) $\alpha_1 = 2\sqrt{\int_{-1}^{1} \frac{1-t}{1+t} \sigma(dt)}$

(ii) $\beta_1 = \sqrt{\left(\int_{-1}^{1} \frac{1-t}{1+t} \sigma(dt) \right) \left(\int_{-1}^{1} \frac{1+t}{1-t} \sigma(dt) \right)^{-1}}$

(iii) $\gamma_1(n) = \frac{\alpha_1}{(2\pi)^{3/2}} \left(\frac{1}{h_\sigma(e^{i\cdot})} \right)^{\wedge}(n) - (\beta_1 + 1)\delta_{n,0} - (\beta_1 - 1)\delta_{n,1} \qquad (n \in \mathbf{Z})$.

PROOF.   By noting (4.10), we see from (2.6), (2.7) and (3.5) that

$$\alpha_1^2 = (2\sqrt{2\pi})^2 \lim_{r \uparrow 1} |h_\sigma(r)|^2$$
$$= (2\sqrt{2\pi})^2 \Delta(-\pi)$$
$$= 4 \int_{-1}^{1} \frac{1-t}{1+t} \sigma(dt),$$

which implies （ⅰ）. Similarly, by noting (4.11), we have

$$\beta_1^2 = \Delta(-\pi)(\Delta(0))^{-1}$$
$$= \int_{-1}^{1} \frac{1-t}{1+t}\sigma(dt)(\int_{-1}^{1} \frac{1+t}{1-t}\sigma(dt))^{-1},$$

which implies （ⅱ）. By Corollary 4.1 （ⅰ）, for a.e. $\theta \in (-\pi, \pi)$,

$$(1-e^{2i\theta}) \int_{-1}^{1} \frac{1}{1-te^{i\theta}}\rho_1(dt)$$

$$= \frac{\alpha_1}{\sqrt{2\pi}} \frac{1}{h_\sigma(e^{i\theta})} - \beta_1(1+e^{i\theta}) - (1-e^{i\theta}).$$

By taking Fourier transform of both hand sides, we have （ⅲ）.　(Q. E. D.)

REMARK 5.1.　We see from (2.4), (3.5) and (4.2) that

（ⅰ）　$\dfrac{1}{h_\sigma(z)} = \exp\{\dfrac{1}{4\pi} \displaystyle\int_{-\pi}^{\pi} \dfrac{e^{i\theta}+z}{e^{i\theta}-z} \log (\Delta_\sigma(\theta)^{-1})d\theta\}$　$(z \in U_1(0))$

（ⅱ）　there exist positive constants c_1 and c_2 such that
$$c_1 \le \Delta_\sigma(\theta) \le c_2 \quad \text{for any } \theta \in (-\pi, \pi)$$

（ⅲ）　$\dfrac{1}{h_\sigma}$ is the outer function of the Hardy density Δ_σ^{-1}.

§6.　The first KMO-Langevin equation

Returning to §4, we will derive a stochastic difference equation describing the time evolution of a real stationary Gaussian process $X = (X(n) ; n \in Z)$ on a probability space $(\Omega, \mathscr{F}, P,)$; the covariance function R takes the form (3.1) with some bounded Borel measure $\sigma \in \Sigma_1$.

Let the triple $(\alpha_1, \beta_1, \gamma_1)$ be the first KMO-Langevin data associated with σ. By using the normalized Gaussian white noise $\boldsymbol{\xi} = (\xi(n) ; n \in Z)$ in (2.14), we will show

THEOREM 6.1.

$$(6.1) \quad X(n) - X(n-1) = -\beta_1(X(n)+X(n-1)) - (\gamma_1 * X)(n) + \alpha_1\xi(n)$$
$$a.s. \ (n \in Z).$$

PROOF　By Theorems 4.2 （ⅱ） and 4.3, we have

$$(6.2) \quad E \in l^1(Z).$$

Furthermore, by noting Proposition 5.1 （ⅱ）, we see from (2.14) that the following two random series are absolutely convergent (a.s.) : for any $n \in Z$

$$(6.3) \quad X(n) - X(n-1) + \beta_1(X(n)+X(n-1))$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} \{E(n-m) - E(n-1-m) + \beta_1(E(n-m)$$

$$+ E(n-m-1))\} \xi(m)$$

(6.4) $(\gamma_1 * X)(n) = \sum_{l=-\infty}^{\infty} \gamma_1(n-l) X(l)$

$$= \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} (\sum_{l=-\infty}^{\infty} \gamma_1(n-l) E(l-m)) \xi(m).$$

On the other hand, it follows from (2.9), (5.10), Proposition 5.1 (ii) and (6.2) that for any n, $m \in Z$

(6.5) $E(n-m) - E(n-1-m) + \beta_1(E(n-m) + E(n-m-1))$

$$= \int_{-\pi}^{\pi} e^{-i(n-m)\theta} \{1 - e^{i\theta} + \beta_1(1 + e^{i\theta})\} h(e^{i\theta}) d\theta$$

(6.6) $\sum_{l=-\infty}^{\infty} \gamma_1(n-l) E(l-m)$

$$= \int_{-\pi}^{\pi} e^{-i(n-m)\theta} (1 - e^{2i\theta}) \int_{-1}^{1} \frac{1}{1 - te^{i\theta}} \rho_1(dt) h(e^{i\theta}) d\theta.$$

Therefore, by substituting (6.5) and (6.6) into (6.3) and (6.4), respectively, we conclude from Corollary 4.1 (i) that for any $n \in Z$

$$X(n) - X(n-1) + \beta_1(X(n) + X(n-1)) + (\gamma_1 * X)(n)$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} (\int_{-\pi}^{\pi} e^{-i(n-m)\theta} \frac{\alpha_1}{\sqrt{2\pi}} d\theta) \xi(m)$$

$$= \alpha_1 \xi(n) \quad \text{a.s.,}$$

which completes the proof of Theorem 6.1.   (Q. E. D.)

DEFINITION 6.1.   We call the stochastic difference equation (6.1) the **first KMO-Langevin equation** associated with X.

As the converse of Theorem 6.1, we will show

THEOREM 6.2.   *For each triple* $(\alpha_1, \beta_1, \rho_1) \in \mathscr{L}_1$ *and each normalized Gaussian white noise* $\xi = (\xi(n) ; n \in Z)$ *on a probability space* $(\Omega, \mathscr{F}, P)$, *there exists a unique real stationary Gaussian process* $X = (X(n) ; n \in Z)$ *on* $(\Omega, \mathscr{F}, P)$ *with reflection positivity such that* X *satisfies the first KMO-Langevin equation* (6.1) ; *the covariance function of this* X *coincides with* R_σ, $\sigma = L_1^{-1}((\alpha_1, \beta_1, \rho_1))$.

PROOF   By Theorem 5.1, we get a bounded Borel measure $\sigma = L_1^{-1}((\alpha_1, \beta_1, \rho_1))$ in Σ_1 such that the outer function h_σ associated with σ takes the form (5.4).   Choose an $l^2(Z)$-function $E = \hat{h}_\sigma$ as a canonical representation kernel to define a real stationary Gaussian process $X = (X(n) ; n \in Z)$:

$$X(n) = (\sqrt{2\pi})^{-1} \sum_{m=-\infty}^{\infty} E(n-m)\xi(m).$$

Then it follows from Theorem 6.1 that this X is our desired process.

To prove the uniqueness of such a process X, let $Y = (Y(n) \ ; \ n \in Z)$ be another real stationary Gaussian process on $(\Omega, \mathscr{F}, P)$ satisfying the same equation (6.1). Fix any $m \in Z$. By multiplying both hand sides of equation (6.1) by $(h_\sigma(\cdot)e^{-im\cdot})^\sim(n)$ and then summing up with respect to n, we can observe from (2.10), Corollary 4.1, (5.10), Proposition 5.1 (ii) and (6.2) that

$$Y(m) = (\sqrt{2\pi})^{-1} \sum_{n=-\infty}^{m} E(m-n)\xi(n),$$

which implies $Y = X$. (Q. E. D.)

EXAMPLE 6.1. Let $X_p = (X_p(n) \ ; \ n \in Z)$ be a real stationary Gaussian process with the covariance function R_p given in (3.2) $(p \in (-1, 1))$. We note that each X_p has the simple Markov property, and in particular X_0 represents a normalized Gaussian white noise ξ. The first KMO-Langevin equation of X_p takes the simplest from

(6.13) $\quad X_p(n) - X_p(n-1) = -\beta_p^{(1)}(X_p(n) + X_p(n-1)) + \alpha_p^{(1)}\xi(n)$

$$\text{a.s.} \ (n \in Z),$$

where the pair $(\alpha_p^{(1)}, \beta_p^{(1)})$ was given by (4.21). In case $p=0$, the above form (6.13) for the white noise ξ becomes trivial:

(6.14) $\quad \xi(n) - \xi(n-1) = -(\xi(n) + \xi(n-1)) + 2\xi(n) \qquad \text{a.s.} \ (n \in Z).$

EXAMPLE 6.2. Let $X = (X(n) \ ; \ n \in Z)$ be a real stationary Gaussian process with the covariance function R of the form (4.22). It follows from Proposition 5.1 (i) that

(6.15) $\quad \gamma_1(n) = \begin{cases} 0 & \text{for } n \in \{-1, -2, \ldots\} \\ \rho_0 q_1^n & \text{for } n \in \{0, 1\} \\ \rho_0 q_1^{n-2}(q_1^2 - 1) & \text{for } n \in \{2, 3, \ldots\}, \end{cases}$

where

(6.16) $\quad \rho_0 = \dfrac{2(q_1 - p_1)(p_2 - q_1)}{(1 - q_1)(1 + p_1)(1 + p_2)}.$

We note that (i) in Example 4.2 is a special case $q_1 = 0$ of (ii) in Example 4.2. Therefore, we see from Theorem 6.1 that X satisfies the following KMO-Langevin equation:

(6.17) $X(n) - X(n-1) = -\beta_1(X(n) + X(n-1)) - (\gamma_1 * X)(n) + \alpha_1 \xi(n)$

$$\text{a.s. } (n \in \mathbf{Z}),$$

where the pair (α_1, β_1) was given by (4.26) or (4.29). It deserves mention that the second term on the right hand side of equation (6.17) depends upon the whole past of X in case $q_1 \neq 0$, i.e. $a_1 \neq 0$.

§ 7. Generalized fluctuation-dissipation theorems

In this final section we will prove a couple of relations between our objects-the first KMO-Langevin data $(\alpha_1, \beta_1, \gamma_1) \in \mathscr{L}_1$, the outer function h of X and other important quantities. The physical meaning of these relations will be explained in detail later (see Remarks 7.2~7.4).

Let $(\alpha_1, \beta_1, \rho_1)$ be any element of $\mathscr{L}_1$ and $\boldsymbol{\xi} = (\xi(n) \; ; \; n \in \mathbf{Z})$ be a normalized Gaussian white noise. By Theorem 6.2, we obtain a real stationary Gaussian process $\boldsymbol{X} = (X(n) \; ; \; n \in \mathbf{Z})$ as the unique solution of the first KMO-Langevin equation:

(7.1) $X(n) - X(n-1) = -\beta_1(X(n) + X(n-1)) - (\gamma_1 * X)(n) + \alpha_1 \xi(n)$

$$\text{a.s. } (n \in \mathbf{Z}),$$

where γ_1 is given by (5.10). This process has the covariance function $R = R_\sigma \in l^1(\mathbf{Z})$, with $\sigma = L_1^{-1}((\alpha_1, \beta_1, \rho_1))$.

We will begin with

LEMMA 7.1.   *The following limit exists*:

(7.2) $D \equiv \lim_{N \to \infty} \frac{1}{2N} E((\sum_{n=0}^{N} X(n))^2) = \sum_{n=0}^{\infty} R(n) - \frac{R(0)}{2}$.

PROOF.   For any $N \in \mathbf{Z}$,

$$E((\sum_{n=0}^{N} X(n))^2) = (N+1)R(0) + 2\sum_{n=0}^{N-1} (\sum_{m=n+1}^{N} R(n-m))$$

$$= (N+1)R(0) + 2\sum_{n=1}^{N} (\sum_{l=1}^{n} R(l)).$$

Therefore, we have the assertion, noting that $R \in l^1(\mathbf{Z})$.

$$\text{(Q. E. D.)}$$

DEFINITION 7.1.   In view of the definition of diffusion constant for the continuous-time case (cf. (2.30) in [10]), the above limit D is called the **diffusion constant** of the procese X.

As a discrete analogue of Theorem 2.1 in [10], we will show

THEOREM 7.1.

（ⅰ）　*For any* $\theta \in (-\pi, \pi)$

$$\frac{1}{\beta_1(1+e^{i\theta})+1-e^{i\theta}+2\pi\tilde{\gamma}_1(\theta)}=\frac{h(e^{i\theta})}{2\lim_{\tau\downarrow\pi}h(e^{i\tau})}.$$

（ⅱ）　$\dfrac{\alpha_1^2}{2}=R(0)C_{\beta_1,\gamma_1},$

where

(7.3)　$C_{\beta_1,\gamma_1}=\pi(\int_{-\pi}^{\pi}|\beta_1(1+e^{i\theta})+1-e^{i\theta}+2\pi\tilde{\gamma}_1(\theta)|^{-2}d\theta)^{-1}.$

（ⅲ）　$D=\dfrac{\alpha_1^2}{2(2\beta_1)^2}$

（ⅳ）　$D=\dfrac{R(0)}{2\beta_1}\dfrac{C_{\beta_1,\gamma_1}}{2\beta_1}$

（ⅴ）　$\dfrac{C_{\beta_1,\gamma_1}}{2\beta_1}-1=\dfrac{1}{R(0)}\int_{-1}^{1}\int_{-1}^{1}\dfrac{1+t}{1-tu}\sigma(dt)\rho_1(du).$

PROOF.　By noting (5.11), we see from Corollary 4.1 that

(7.4)　$2\lim_{\theta\downarrow-\pi}h(e^{i\theta})=\dfrac{\alpha_1}{\sqrt{2\pi}}.$

Therefore, （ⅰ） follows from Corollary 4.1 （ⅰ）, (5.10) and Proposition 5. 1 （ⅱ）.

By using Corollary 4.1 （ⅰ）, (5.10) and Proposition 5.1 （ⅱ） again, we see from (2.7) that

$$\varDelta(\theta)=\frac{\alpha_1^2}{2\pi}|\beta_1(1+e^{i\theta})+1-e^{i\theta}+2\pi\tilde{\gamma}_1(\theta)|^{-2}\qquad \text{a.e.}\ \ \theta\in(-\pi,\pi).$$

By integrating both hand sides with respect to θ, we have （ⅱ）.

We now compute the diffusion constant D. By (3.1) and (3.7), we have

(7.5)　$\sum\limits_{n=0}^{\infty}R(n)=\int_{-1}^{1}\dfrac{1}{1-t}\sigma(dt)$

(7.6)　$\int_{-1}^{1}\dfrac{1}{1-t}\sigma(dt)=\dfrac{1}{2}\int_{0}^{\infty}(1+\dfrac{1}{\lambda})\sigma_c(d\lambda)$

(7.7)　$R(0)=\sigma_c([0,\infty)).$

Hence, appealing to the result in [10], Theorem 2.1 (ⅲ), we get

$$D=\frac{1}{2}\{\int_{0}^{\infty}(1+\frac{1}{\lambda})\sigma_c(d\lambda)-\sigma_c([0,\infty))\}=\frac{1}{2}\int_{0}^{\infty}\frac{1}{\lambda}\sigma_c(d\lambda)$$

$$=\frac{\alpha_c^2}{4\beta_c^2}=\frac{\alpha_1^2}{8\beta_1^2},$$

which completes the proof of (iii). (iv) is an immediate consequence of (ii) and (iii).

Now, we proceed to the proof of (v). We first claim

(7.8) $$E(0) = \frac{\sqrt{2\pi}\,\alpha_1}{1+\beta_1+\gamma_1(0)}$$

(7.9) $$E(n) = \frac{1-\beta_1}{1+\beta_1}E(0) - \frac{2\beta_1}{1+\beta_1}\sum_{m=1}^{n-1}E(m) - \frac{1}{1+\beta_1}\sum_{m=1}^{n}(\gamma_1*E)(m)$$
$$(n \geq 1).$$

By Theorems 4.2 and 4.3, we have

(7.10) $$\frac{1}{2\pi}\sum_{n=0}^{\infty}E(n)z^n = h(z) \quad (z \in U_1(0)).$$

In particular,

$$E(0) = 2\pi h(z)|_{z=0}.$$

Therefore, by taking $z=0$ (resp. $n=0$) in Theorem 4.1 (resp. Proposition 5.1 (i)), we get (7.8). Furthermore, by using Theorem 4.1 again, we see from (7.10) that for any $n \in N$

$$E(n) - E(n-1) = -\beta_1(E(n)+E(n-1)) - (\gamma_1*E)(n)$$

and so

$$E(n) - E(0) = \sum_{m=1}^{n}(E(m)-E(m-1))$$
$$= -\beta_1(E(0)+E(n)+2\sum_{m=1}^{n-1}E(m)) - \sum_{m=1}^{n}(\gamma_1*E)(m),$$

which implies (7.9).

Next we prove the key formula

(7.11) $$2\pi R(0) = \left(1 + \frac{\gamma_1(0)}{1+\beta_1}\right)E(0)\sum_{n=0}^{\infty}E(n) - \frac{4\pi\beta_1}{1+\beta_1}\sum_{n=1}^{\infty}R(n)$$
$$- \frac{2\pi}{1+\beta_1}\sum_{m=0}^{\infty}(\sum_{k=0}^{\infty}R(m+k)\gamma_1(k)).$$

By substituting (7.8) and (7.9) into (2.13), we have

$$2\pi R(0) = E(0)^2 + \frac{1-\beta_1}{1+\beta_1}E(0)\sum_{n=1}^{\infty}E(n) - \frac{2\beta_1}{1+\beta_1}\sum_{n=1}^{\infty}(\sum_{l=1}^{n-1}E(l))E(n)$$
$$- \frac{1}{1+\beta_1}\sum_{n=1}^{\infty}(\sum_{l=1}^{n}(\gamma_1*E)(l))E(n).$$

On the other hand, we see from (2.11), (2.13) and (6.2) that

$$\sum_{n=1}^{\infty} (\sum_{l=1}^{n-1} E(l))E(n) = \sum_{l=1}^{\infty} (\sum_{m=1}^{\infty} E(l+m))E(l)$$

$$= \sum_{m=1}^{\infty} (\sum_{l=0}^{\infty} E(m+l)E(l) - E(0)E(m+1))$$

$$= 2\pi \sum_{m=1}^{\infty} R(m) - E(0) \sum_{m=1}^{\infty} E(m),$$

and

$$\sum_{n=1}^{\infty} (\sum_{l=1}^{n} (\gamma_1 * E)(l))E(n)$$

$$= \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} (\sum_{l=1}^{\infty} E(l+m)E(l-k))\gamma_1(k)$$

$$= \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \gamma_1(k)\{\sum_{l=0}^{\infty} E(l+m)E(l-k) - E(m)E(-k)\}$$

$$= 2\pi \sum_{m=0}^{\infty} (\sum_{k=0}^{\infty} R(m+k)\gamma_1(k)) - \gamma_1(0)E(0) \sum_{m=0}^{\infty} E(m).$$

And so

$$2\pi R(0) = E(0)^2 + \frac{1-\beta_1}{1+\beta_1}E(0) \sum_{n=1}^{\infty} E(n)$$

$$- \frac{2\beta_1}{1+\beta_1} (2\pi \sum_{n=1}^{\infty} R(n) - E(0) \sum_{m=1}^{\infty} E(m))$$

$$- \frac{1}{1+\beta_1}\{2\pi \sum_{m=0}^{\infty} (\sum_{k=0}^{\infty} R(m+k)\gamma_1(k)) - \gamma_1(0)E(0) \sum_{m=0}^{\infty} E(m)\}$$

$$= E(0)^2 + E(0) \sum_{n=1}^{\infty} E(n) + \frac{\gamma_1(0)}{1+\beta_1}E(0) \sum_{m=0}^{\infty} E(m)$$

$$- \frac{4\pi\beta_1}{1+\beta_1} \sum_{n=1}^{\infty} R(n) - \frac{2\pi}{1+\beta_1} \sum_{m=0}^{\infty} (\sum_{k=0}^{\infty} R(m+k)\gamma_1(k)),$$

which implies (7.11).

We are now ready to show

(7.12) $$\frac{C_{\beta_1, \gamma_1}}{2\beta_1} - 1 = \frac{1}{R(0)} \sum_{m=0}^{\infty} (\sum_{k=0}^{\infty} R(m+k)\gamma_1(k)),$$

which is proved in the following manner.
By (7.10),

$$\sum_{n=0}^{\infty} E(n) = 2\pi \lim_{x \uparrow 1} h(x).$$

And so by (4.11)

(7.13) $$\sum_{n=0}^{\infty} E(n) = \frac{\sqrt{2\pi}\,\alpha_1}{2\beta_1}.$$

Combining this with (7.8), we have

$$\left(1+\frac{\gamma_1(0)}{1+\beta_1}\right)E(0)\sum_{n=0}^{\infty}E(n)=\frac{\pi\alpha_1^2}{\beta_1(1+\beta_1)}.$$

On the other hand, by Lemmà 7.1 and Theorem 7.1 (iii),

$$\sum_{n=1}^{\infty}R(n)=\frac{\alpha_1^2}{8\beta_1^2}-\frac{R(0)}{2}.$$

Therefore, by combining these with (7.11), we see that

$$R(0)=\frac{\alpha_1^2}{2\beta_1(1+\beta_1)}-\frac{2\beta_1}{1+\beta_1}\left(\frac{\alpha_1^2}{8\beta_1^2}-\frac{R(0)}{2}\right)$$
$$-\frac{1}{1+\beta_1}\sum_{m=0}^{\infty}(\sum_{k=0}^{\infty}R(m+k)\gamma_1(k))$$

and so

$$R(0)=\frac{\alpha_1^2}{4\beta_1}-\sum_{m=0}^{\infty}(\sum_{k=0}^{\infty}R(m+k)\gamma_1(k)),$$

which, together with Theorem 7.1 (ii), implies (7.12).

Next we claim

(7.14) $$\sum_{m=0}^{\infty}(\sum_{k=0}^{\infty}R(m+k)\gamma_1(k))=\int_{-1}^{1}\int_{-1}^{1}\frac{1+t}{1-tu}\sigma(dt)\rho_1(du).$$

By (3.1) and Proposition 5.1 (i),
the left hand side of (7.14) $$=\sum_{k=0}^{\infty}(\sum_{m=1}^{\infty}R(m+k))\gamma_1(k)$$

$$=\sum_{k=0}^{\infty}(\int_{-1}^{1}\frac{t^k}{1-t}\sigma(dt))\gamma_1(k)$$

$$=\int_{-1}^{1}\{\sum_{k=0}^{\infty}\gamma_1(k)t^k\}\frac{\sigma(dt)}{1-t},$$

$$=\int_{-1}^{1}\{(1-t^2)\int_{-1}^{1}\frac{\rho_1(du)}{1-tu}\}\frac{\sigma(dt)}{1-t},$$

which implies (7.14). Thus, we conclude from (7.12) and (7.14) that
(v) holds.   (Q. E. D.)

Before we go into the explanation of the physical meaning of Theorem 7.1, we will consider the simplest

EXAMPLE 7.1.   Let X_p be the same stochastic process as in Example 6.1, and D_p be the diffusion constant of X_p. By (3.2) and Lemma 7.1,

(7.15) $$R_p(0)=1$$

(7.16) $$D_p=\frac{1+p}{2(1-p)}.$$

By combining these with (4.21), we see that

$$(7.19) \qquad \frac{(\alpha_p^{(1)})^2}{2} = R_p(0)(2\beta_p^{(1)})$$

$$(7.20) \qquad D_p = \frac{R_p^{(0)}}{2\beta_p^{(1)}},$$

which, together with Theorem 7.1 （ⅱ） or Theorem 7.1 （ⅳ）, imply

$$(7.21) \qquad C_{\beta_p^{(1)}, \gamma_p^{(1)}} = C_{\beta_p^{(1)}, 0} = 2\beta_p^{(1)}.$$

In addition, we see from (3.2), (4.19) and (4.21) that a remarkable relation between R_p and E_p holds:

$$(7.22) \qquad R_p(n) = \frac{1 + \beta_p^{(1)}}{\sqrt{2\pi} \alpha_p^{(1)}} E_p(n) \quad (n \geq 0).$$

We will return to the general case and give some characterization of the simple Markovian property. As a discrete analogue of Theorem 2.2 in [10], we can see from Theorem 7.1 that

THEOREM 7.2.

（ⅰ）　$\dfrac{C_{\beta_1, \gamma_1}}{2\beta_1} \geq 1$

（ⅱ）　*The following four statements are equivalent*;

（a）　$\dfrac{C_{\beta_1, \gamma_1}}{2\beta_1} = 1$

（b）　$\gamma_1 = 0$

（c）　$\rho_1 = 0$

（d）　$X = X_p$ *with some* $p(\in (-1, 1))$.

REMARK 7.1.　As we have seen in Theorem 2.2 in [10], the relation (7.22) characterizes the simple Markovian property for the continuous-time processes. However, this is no longer true for the present discrete-time processes. We will give such an example. Let X be a real stationary Gaussian process discussed in the case （ⅰ） of Example 6.2 such that

$$(7.23) \qquad p_1 = -p_2 \text{ and } \sigma_1 = \sigma_2 = \frac{1}{2(1 - p_1^2)}.$$

It then follows from Theorem 4.3 and (4.27) the canonical representation kernel E of X becomes

$$(7.24) \qquad E(n) = \chi_{[0, \infty)}(n) \sqrt{2\pi(1 + p_1^2)} (p_1^n + (-p_1)^n),$$

which implies the desired relation

(7.25) $R(n) = \dfrac{1}{2(1-p_1^2)\sqrt{2(1-p_1^2)}} E(n)$ $(n \geq 0)$.

Finally we will give three remarks concerning the physical meaning of Theorem 7.1 (cf. [10] for the continuous-time case).

REMARK 7.2.   In relation (i) in Theorem 7.1, the left hand side denotes a complex mobility of the system X described by equation (7.1), which represents the response of the system X to the external force $\alpha_1 \xi$. On the other hand, the right hand side is determined by the outer funciton of X, which represents the thermal fluctuation of the system in equilibrium without the external force. The relation (i) in Theorem 7.1 might be said to be the **generalized first fluctuation-dissipation theorem.**

REMARK 7.3.   We are now concerned with relation (ii) in Theorem 7. 1. The fluctuation power of a randow force $\alpha_1 \xi$ in equation (7.1) is $\dfrac{\alpha_1^2}{2}$. While, the positive constant C_{β_1, γ_1} is expressed in terms of the drift coefficient representing the systematic part of equation (7.1). And from the physical point of view we can regard $R(0)$ as the absolute constant $k\,T$, where k and T denote the Boltzman constant and absolute temperature of the system in equilibrium, respectively. This leads us to think of C_{β_1, γ_1} as the **generalized friction constant**, and the relation (ii) itself might be said to be the **generalized second fluctuation-dissipation theorem.**

REMARK 7.4.   For the Markov process X_p in Example 7.1, we found that the diffusion constant D_p is inversely proportional to the friction constant $\beta_p^{(1)}$. This relation (7.20) is analogous to the classical Einstein relation valid for the Ornstein-Uhlenbeck Brownian motion with continuous time (see (2.29) in [10]). For this reason, we call relation (7.20) for X_p the **Einstein relation.** In a general system described by equation (7.1) with $\gamma_1 \neq 0$, however, we found a significant deviation (iv) in Theorem 7.1 from the Einstein relation (7.20) with $\gamma_1 = 0$, and obtained the formula (v) in Theorem 7.1 expressing the degree of such a deviation. In view of the analogous fact in the continuous-time case (Theorem 2.1 in [10]), we call the relation (iv) in Theorem 7.1 the **generalized Einstein relation.**

References

[1]   BOX, G. E. P. and G. M. JENKINS., Time series analysis, Forecasting and Control, Holden-Day, 1976.

[2]   HOFFMAN, K., Banach spaces of analytic functions, Prentice Hall, Englewood Cliffs, N. J., 1962.

[3]   KARHUNEN, K., Über die struktur stationarer zufälliger funktionnen, Ark. Mat., 1

(1950), 141-160.

[4] KUBO, R., Statistical mechanical theory of irreversible processes I, general theory and simple applications to magnetic and conduction problem, J. Phys. Soc. Japan 12 (1957), 570-586.

[5] MORI, H., B. C. SO and T. OSE., Time-correlation functions of one-dimensional transformations, Prog. Theor. Phys., 66(1981), 1266-1283.

[6] OKABE, Y., On a stationary Gaussian process with T-positivity and its associated Langevin equation and S-matrix, J. Fac. Sci. Univ. Tokyo, Sect. IA 26(1979), 115 -165.

[7] OKABE, Y., On a stochastic differential equation for a stationary Gaussian process with T-positivity and the fluctuation-dissipation theorem, J. Fac. Sci. Univ. Tokyo, Sect. IA 28(1981), 169-213.

[8] OKABE, Y., A generalized fluctuation-dissipation theorem for the one-dimensional diffusion process, Comm. Math. Phys., 98(1985), 449-468.

[9] OKABE, Y., On KMO-Langevin equations for stationary Gaussian processes with T-positivity, J. Fac. Sci. Univ. Tokyo, Sect. IA 33(1986), 1-56.

[10] OKABE, Y., On the theory of Brownian motion with the Alder-Wainwright effect, J. Stat. Phys., 45(1986), 953-981.

[11] OKABE, Y., On the theory of discrete KMO-Langevin equations with reflection positivity （II）, to be submitted to Hokkaido Math. J.

[12] OKABE, Y., On the theory of discrete KMO-Langevin equations with reflection positivity （III）, in preparation.

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