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On the theory of discrete KMO-Langevin equations with reflection positivity (II)

Dedicated to Professor Seizo ITO on his sixtieth birthday

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§ 1. Introduction

In a series of papers ([3]~[7]), the author has discussed real continuous-time stationary Gaussian processes $X(t)$ with reflection positivity and developed a theory of generalized Langevin equations describing the time evolution of such processes. His original aim was two-fold:

1) Deep understanding of a mathematical structure behind significant Kubo's fluctuation-dissipation theorem in statistical physics ([2]);
2) Applications of the theory as a useful model in various fields of science.

The first aim was achieved in [6] and [7]. Indeed, we established two different kinds of Langevin equations—the first (resp. second) KMO-Langevin equation with a white (resp. Kubo) noise as a random force, and proved the generalized fluctuation-dissipation theorems based on these equations. For the simplest process with Markovian property, i. e., for Ornstein-Uhlenbeck Brownian motion, these two kinds of equations take the same form, and the classical Einstein relation is valid. For a general $X(t)$, however, the situation turns out to be not so simple; the Einstein relation still holds if we use the equation of the second type, but does not hold in the case of the first type. So we raised in [7] a question how this interesting deviation from the Einstein relation can be measured experimentally in the remarkable case of Stokes-Boussinesque-Langevin equation of the first type.

With the second aim in mind, the author proceeded to investigate the discrete-time case in the previous paper [8], and the present paper is a continuation of [8]. Let us recall that we have considered in [8] stationary Gaussian time-series $X(n)$ with reflection positivity and established a discrete analogue of the results for the first KMO-Langevin equation obtained in [6] and [7]. As in the continuous-time case mentioned above, we have obtained the generalized Einstein relation based on the first KMO-Langevin

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equation with discrete time, to observe the expected deviation from the classical Einstein relation valid only in the Markovian case.

The purpose of the present paper is to derive the second KMO-Langevin equation for the same time series $X(n)$ as in [8] and then to prove the generalized fluctuation-dissipation theorems based on this equation of the second type. To this end we have to introduce the Kubo noise $I(n)$ associated with $X(n)$, in addition to the familiar white noise $\xi(n)$. In the simplest Markovian case, the two noise processes coincide, which means that the two kinds of discrete KMO-Langevin equations take the same form. Now, the point is that, even if we use the equation of the second type, the generalized Einstein relation is also different from the classical one; this is a remarkable fact we have found in th discrete-time theory, in contrast to the situation in the continuous-time case explained above. We can go further by calculating the ratio of such two kinds of deviations concerning the generalized Einstein relations based on two kinds of discrete KMO-Langevin equations.

We now state the content of this paper. As in the previous paper [8], we will treat the covariance function R of $\boldsymbol{X}$ of the following form:

$$(1.1) \qquad R(n)=\int_{-1}^{1} t^{|n|}\sigma(dt) \qquad (n\in\boldsymbol{Z}),$$

where σ is a bounded Borel measure on $[-1,1]$ such that

$$(1.2) \qquad \sigma(\{-1,1\})=0$$

$$(1.3) \qquad \int_{-1}^{1}\left(\frac{1}{1+t}+\frac{1}{1-t}\right)\sigma(dt)<\infty.$$

Note that (1.2) and (1.3) imply that

$$(1.4) \qquad R\in l^1(\boldsymbol{Z}).$$

By (1.4),we can define a function $[R]$ on $\overline{U_1(0)}=\{z\in\boldsymbol{C}\,;\,|z|\le 1\}$ by

$$(1.5) \qquad [R](z)=\frac{1}{2\pi}\sum_{n=0}^{\infty}R(n)z^n.$$

The function $(2[R](-1))^{-1}[R](z)$ is called a **complex mobility function of** $\boldsymbol{X}$. By using Theorem 4.1 in [8] which gives a fundamental structure for the outer function h of $\boldsymbol{X}$, we will in § 2 obtain the following structure theorem for the function $[R]$.

THEOREM 2.1. *There exists a unique triple* $(\alpha_2, \beta_2, \rho_2)$ *such that*

(i) $\alpha_2>0$ *and* $\beta_2>0$

(ii) ρ_2 *is a bounded Borel measure on* $[-1,1]$ *with* $\rho_2(\{-1,1\})=0$

(iii) *for any* $z \in \overline{U_1(0)}$

$$[R](z)=\frac{\alpha_2}{\sqrt{2\pi}}\frac{1}{\beta_2(1+z)+1-z+(1-z^2)\int_{-1}^{1}\frac{1}{1-tz}\rho_2(dt)}.$$

We call the triple $(\alpha_2, \beta_2, \rho_2)$ or $(\alpha_2, \beta_2, \gamma_2)$ in Theorem 2.1 the **second KMO-Langevin data associated with** σ(or R), where γ_2 is a function on $\boldsymbol{Z}$ difined by

$$(1.6)\qquad \gamma_2=\frac{1}{2\pi}((1-e^{2i\cdot})\int_{-1}^{1}\frac{1}{1-te^{i\cdot}}\rho_2(dt))^{\wedge}.$$

It is noted that

$$(1.7)\qquad \gamma_2(n)=\begin{cases} 0 & \text{for } n\in\{-1,-2,\ldots\} \\ \int_{-1}^{1}t^n\rho_2(dt) & \text{for } n\in\{0,1\} \\ \int_{-1}^{1}t^{n-2}(t^2-1)\rho_2(dt) & \text{for } n\in\{2,3,\ldots\}, \end{cases}$$

which implies that

$$(1.8)\qquad \gamma_2\in l^1(\boldsymbol{Z}).$$

It will be found in § 3 that the correspondence between σ and $(\alpha_2, \beta_2, \rho_2)$ is bijective (Theorem 3.1). Furthermore, we will obtain a formula by which the triple $(\alpha_2, \beta_2, \gamma_2)$ can be calculated from σ (Theorem 3.2).

By using the Gaussian white noise $\boldsymbol{\xi}$ in (2.15) in [8] and the l^1 $(\boldsymbol{Z})$-function E_I defined by

$$(1.9)\qquad E_I=\left(\frac{1}{\sqrt{2\pi}}\frac{h}{[R]}\right)^{\wedge},$$

we will in § 4 introduce a real stationary Gaussian proces $\boldsymbol{I}=(I(n)\,;\,n\in\boldsymbol{Z})$ by

$$(1.10)\qquad I(n)=\frac{1}{\sqrt{2\pi}}\sum_{m=-\infty}^{n}E_I(n-m)\xi(m) \qquad \text{(a.s.)}.$$

Then we will obtain the following causal representation theorem for the process $\boldsymbol{X}$ in terms of the noise process $\boldsymbol{I}$:

THEOREM 4.1.

(i) $X(n)=\frac{1}{\sqrt{2\pi}}\sum_{m=-\infty}^{n}R(n-m)I(m)$ a.s. $(n\in\boldsymbol{Z})$

(ii) $\sigma(X(m)\,;\,m\leq n)=\sigma(I(m)\,;\,m\leq n)$ $(n\in\boldsymbol{Z})$.

The process $\boldsymbol{I}$ is called the **Kubo noise associated with $\boldsymbol{X}$**. The spectral density Δ_I and the covariance function R_I of $\boldsymbol{I}$ will be calculated. Then we will find that the process $\boldsymbol{I}$ has no reflection positivity except the case where $\boldsymbol{X}$ has a simple Markovian property.

By using Theorem 2.1, we will in § 5 derive a stochastic difference equation with the Kubo noise $\boldsymbol{I}$ as its random force; this is the desired description of the time evolution of $\boldsymbol{X}$ (cf. Theorem 6.1 in [8]).

THEOREM 1.1.

$$X(n)-X(n-1)=-\beta_2(X(n)+X(n-1))-(\gamma_2 * X)(n)+\alpha_2 I(n) \quad \text{a.s.} \ (n\in\boldsymbol{Z}). \tag{1.11}$$

We call equation (1.11) the **second KMO-Langevin equation for $\boldsymbol{X}$**.

As a basic example of $\boldsymbol{X}$, for each $p\in(-1,1)$, we will in § 3, § 4 and § 6 consider the Markov process $\boldsymbol{X}_p$ with covariance function R_p of the form

$$R_p(n)=p^{|n|}. \tag{1.12}$$

Note that

$$[R_p](z)=\frac{1}{2\pi}\frac{1}{1-pz} \qquad (z\in\overline{U_1(0)}).$$

By rewriting it into the form (iii) in Theorem 2.1, we have

$$[R_p](z)=\frac{\alpha_p^{(2)}}{\sqrt{2\pi}}\frac{1}{\beta_p^{(2)}(1+z)+1-z}, \tag{1.13}$$

where

$$\alpha_p^{(2)}=\sqrt{\frac{2}{\pi}}\frac{1}{1+p} \quad \text{and} \quad \beta_p^{(2)}=\beta_p^{(1)}=\frac{1-p}{1+p}. \tag{1.14}$$

Therefore, we find from Theorem 5.1 that the second KMO-Langevin equation for $\boldsymbol{X}_p$ becomes:

$$X_p(n)-X_p(n-1)=-\beta_p^{(2)}(X_p(n)+X_p(n-1))+\alpha_p^{(2)}I_p(n) \quad \text{a.s.} \ (n\in\boldsymbol{Z}), \tag{1.15}$$

where $\boldsymbol{I}_p=(I_p(n)\,;\, n\in\boldsymbol{Z})$ is the Kubo noise associated with $\boldsymbol{X}_p$. It is noted that $(\alpha_p^{(1)})^{-1}\alpha_p^{(2)}\boldsymbol{I}_p$ is a Gaussian white noise.

Concerning a discrete analogue of R. Kubo's fluctuation-dissipation theorem for the continuous-time case in [7], we will in § 6 obtain the following

THEOREM 6.1.

(i) *For any* $\theta \in (-\pi, \pi)$

$$\frac{1}{\beta_2(1+e^{i\theta})+1-e^{i\theta}+2\pi\tilde{\gamma}_2(\theta)} = (2[R](-1))^{-1}[R](e^{i\theta}).$$

(ii) *The spectral measure of the stationary process* $\alpha_2 \boldsymbol{I}$

$$= \frac{R(0)}{2\pi}\{2(1+\beta_2+\gamma_2(0))\mathrm{Re}(\beta_2(1+e^{i\theta})+1-e^{i\theta}+2\pi\tilde{\gamma}_2(\theta)) - |\beta_2(1+e^{i\theta})+1-e^{i\theta}+2\pi\tilde{\gamma}_2(\theta)|^2\}d\theta$$

(iii) $D = \dfrac{R(0)}{2\beta_2}(1+\gamma_2(0))$,

where D is the diffusion constant of $\boldsymbol{X}$.

By taking into account the physical meaning of R. Kubo's fluctuation-dissipation theorem for the continuous-time case given in [7], we will call the relations (i), (ii) and (iii) in Theorem 6.1 the **generalized first fluctuation-dissipation theorem,** the **generalized second fluctuation-dissipation theorem** and the **generalized Einstein relation,** respectively.

Since the diffusion constants D_p of the Markov processes $\boldsymbol{X}_p$ ($p \in (-1, 1)$) become

$$D_p = \frac{R_p(0)}{2\beta_p^{(2)}}\left(= \frac{1+p}{2(1-p)}\right), \tag{1.16}$$

we conclude from the generalized Einstein relation (iii) in Theorem 6.1 that for non-Markovian process there occures a deviation from the classical Einstein relation (1.16) for the Markov processes, even if we use the second KMO-Langevin equation (1.11). In § 7 we will estimate the ratio of the two deviations from the classical Einstein relation arising from the use of the first and second KMO-Langevin equations, (6.1) in [8] and (1.11).

We will derive in § 8 the first KMO-Langevin equation describing the time evolution of the Kubo noise $\boldsymbol{I}$ associated with $\boldsymbol{X}$ (Theorem 8.2). Conversely, it will be proved that the Gaussian white noise $\boldsymbol{\xi}$ itself satisfies some stochastic difference equation with $\boldsymbol{I}$ as its random force (Theorem 8.3). These theorems tell us that the two noise processes $\boldsymbol{\xi}$ and $\boldsymbol{I}$ can play an exchangeable role as a random force in our dscription of the time evolution of $\boldsymbol{X}$.

Finally, we will in § 9 obtain the generalized fluctuation-dissipation theorem based on the first KMO-Langevin equation for $\boldsymbol{I}$ and then estimate how large the deviation from the classical Einstein relation is (Theorems 9.1 and 9.2).

In Appendix, we will prove a useful expression of the outer function in the continuous-time case, which will be used in § 2 and has been implicitly used in [3], [4] and [5].

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§ 2. Complex mobility function

As was stated in § 1, we will consider a non-negative definite function R of the form

$$(2.1)\qquad R(n)=\int_{-1}^{1} t^{|n|}\sigma(dt) \qquad (n\in \boldsymbol{Z}),$$

where σ is a bounded Borel measure on $[-1,1]$ such that

$$(2.2)\qquad \sigma(\{-1,1\})=0$$

$$(2.3)\qquad \int_{-1}^{1}\left(\frac{1}{1+t}+\frac{1}{1-t}\right)\sigma(dt)<\infty.$$

The conditions (2. 2) and (2. 3) imply that

$$(2.4)\qquad R\in l^1(\boldsymbol{Z}).$$

We can then define a function $[R]$ on $\overline{U_1(0)}$ by

$$(2.5)\qquad [R](z)=\frac{1}{2\pi}\sum_{n=0}^{\infty}R(n)z^n,$$

which is also expressed in terms of σ:

$$(2.6)\qquad [R](z)=\frac{1}{2\pi}\int_{-1}^{1}\frac{1}{1-tz}\sigma(dt).$$

DEFINITION 2. 1. The function $(2[R](-1))^{-1}[R]$ is called the **complex mobility function associated with** R or $\boldsymbol{X}$.

In order to give a useful expression of $[R](z)$ as the outer function of some Hardy density, we now consider a bounded Borel measure σ_1 on $[-1, 1]$ defined by

$$(2.7)\qquad \sigma_1(dt)=\frac{1}{2\pi}\left(\int_{-1}^{1}\frac{1}{1-ts}\sigma(ds)\right)\sigma(dt).$$

Note that

$$(2.8)\qquad \sigma_1(\{-1,1\})=0$$

$$(2.9)\qquad \int_{-1}^{1}\left(\frac{1}{1+t}+\frac{1}{1-t}\right)\sigma_1(dt)<\infty.$$

By using this measure σ_1, we define another non-negative definite function R_1 on $\boldsymbol{Z}$ by

$$(2.10)\qquad R_1(n)=\int_{-1}^{1} t^{|n|}\sigma_1(dt).$$

It was shown in [8] that R_1 has the Hardy spectral density $\Delta_1=\Delta_1(\theta)(\theta\in(-\pi,\pi))$ such that

$$(2.11)\qquad \Delta_1(\theta)=\frac{1}{2\pi}\int_{-1}^{1}\frac{1-t^2}{|1-te^{i\theta}|^2}\sigma_1(dt)$$

$$(2.12)\qquad \log \Delta_1\in L^1((-\pi,\pi)).$$

LEMMA 2.1. *$[R]$ coincides with the outer function of Δ_1, i. e.,*

$$[R](z)=\exp\Big(\frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\log \Delta_1(\theta)d\theta\Big)\qquad (z\in U_1(0)).$$

PROOF. We denote by F the outer function of Δ_1. It then follows from Lemma 3.2 in [8] that

$$(2.13)\qquad F(z)=\frac{\sqrt{2}}{1+z}h_{1,c}\Big(i\frac{1-z}{1+z}\Big)\qquad (z\in U_1(0)),$$

where $h_{1,c}$ is the outer function of the Hardy density $\Delta_{1,c}$ of the form

$$(2.14)\qquad \Delta_{1,c}(\xi)=\frac{1}{\pi}\int_0^{\infty}\frac{\lambda}{\lambda^2+\xi^2}\sigma_{1,c}(d\lambda)\qquad (\xi\in\boldsymbol{R})$$

with a bounded Borel measure $\sigma_{1,c}$ on $[0,\infty)$ given by

$$(2.15)\qquad \sigma_{1,c}=\phi(\sigma_1);$$

Here ϕ is a familiar homeomorphism from $(-1,1]$ onto $[0,\infty)$ such that

$$(2.16)\qquad \phi(t)=\frac{1-t}{1+t}.$$

Define a bounded Borel measure ν_c on $[0,\infty)$ by

$$(2.17)\qquad \nu_c=\phi\Big(\frac{\sqrt{2}}{1+\cdot}\sigma\Big).$$

Then, we claim

$$(2.18)\qquad \sigma_{1,c}(d\lambda)=\frac{1}{2\pi}\Big(\int_0^{\infty}\frac{1}{\lambda+\lambda'}\nu_c(d\lambda')\Big)\nu_c(d\lambda).$$

Let g be any real valued bounded Borel function on $[0,\infty)$. We see from (2.7), (2.15) and (2.17) that

$$\int_0^\infty g(\lambda)\sigma_{1,c}(d\lambda)=\frac{1}{2\pi}\int_{-1}^1 g(\phi(t))(\int_{-1}^1\frac{1}{1-ts}\sigma(ds))\sigma(dt)$$
$$=\frac{1}{2\pi}\int_0^\infty\frac{1+\phi^{-1}(\lambda)}{\sqrt{2}}g(\lambda)(\int_0^\infty\frac{1+\phi^{-1}(\lambda')}{\sqrt{2}(1-\phi^{-1}(\lambda)\phi^{-1}(\lambda'))}\nu_c(d\lambda'))\nu_c(d\lambda).$$

On the other hand, by (2.16),

$$\frac{1+\phi^{-1}(\lambda)}{\sqrt{2}}\frac{1+\phi^{-1}(\lambda')}{\sqrt{2}(1-\phi^{-1}(\lambda)\phi^{-1}(\lambda'))}=\frac{1}{\lambda+\lambda'}.$$

Hence, we have (2.18).

By (2.14) and (2.18), we appeal to Theorem A in Appendix to conclude that

$$h_{1,c}(\xi)=\frac{1}{2\pi}\int_0^\infty\frac{1}{\lambda-i\xi}\nu_c(d\lambda)\qquad(\xi\in \boldsymbol{C}^+).$$

By combining this with (2.13), and using (2.16) and (2.17),

$$F(z)=\frac{\sqrt{2}}{2\pi}\int_0^\infty\frac{1}{\lambda(1+z)+1-z}\nu_c(d\lambda)$$
$$=\frac{\sqrt{2}}{2\pi}\int_{-1}^1\frac{1}{1-tz}\sigma(dt)=[R](z).$$

Thus, we have completed the proof of Lemma 2.1. (Q. E. D.)

By (2.8), (2.9) and Lemma 2.1, we can apply Theorem 4.1 in [8] to the function $[R]$ to obtain the basic structure

THEOREM 2.1. *There exists a unique triple* $(\alpha_2, \beta_2, \rho_2)$ *such that*

(i) $\alpha_2>0$ *and* $\beta_2>0$

(ii) ρ_2 *is a bounded Borel measure on* $[-1,1]$ *with* $\rho_2(\{-1,1\})=0$

(iii) *for any* $z\in\overline{U_1(0)}$

$$[R](z)=\frac{\alpha_2}{\sqrt{2\pi}}\frac{1}{\beta_2(1+z)+1-z+(1-z^2)\int_{-1}^1\frac{1}{1-tz}\rho_2(dt)}.$$

§ 3. The second KMO-Langevin data

In § 5 of [8] we have defined two sets Σ_1 and $\mathscr{L}_1$:

(3.1) $\Sigma_1=\{\sigma;\ \sigma$ is a bounded Borel measure on $[-1,1]$ such that $\sigma(\{-1,1\})=0$ and $\int_{-1}^1\left(\frac{1}{1+t}+\frac{1}{1-t}\right)\sigma(dt)<\infty\}$

(3.2) $\mathscr{L}_1=\{(\alpha,\beta,\rho);\ \alpha>0,\beta>0$ and ρ is a bounded Borel measure on $[-1,1]$ such that $\rho(\{-1,1\})=0\}$, and studied the bijective mapping L_1 from Σ_1 onto $\mathscr{L}_1$.

Similarly to (5.3) in [8], for each $\sigma\in\Sigma_1$, we put

$$(3.3)\qquad \begin{cases} R_\sigma(n)=\int_{-1}^{1} t^{|n|}\sigma(dt) & (n\in\boldsymbol{Z})\\ \Delta_\sigma(\theta)=\frac{1}{2\pi}\int_{-1}^{1}\frac{1-t^2}{|1-te^{i\theta}|^2}\sigma(dt) & (\theta\in(-\pi,\pi))\\ h_\sigma(z)=\exp(\frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\log\Delta_\sigma(\theta)\,d\theta) & (z\in U_1(0)).\end{cases}$$

We begin with the study of another correspondence between $\sigma\in\Sigma_1$ and $(\alpha_2,\beta_2,\rho_2)\in\mathscr{L}_1$ established in Theorem 2.1.

THEOREM 3.1. *There exists a bijective mapping L_2 from Σ_1 onto $\mathscr{L}_1$ such that for any $\sigma\in\Sigma_1$ and $(\alpha_2,\beta_2,\rho_2)=L_2(\sigma)\in\mathscr{L}_1$,*

$$(3.4)\qquad [R_\sigma](z)=\frac{\alpha_2}{\sqrt{2\pi}}\frac{1}{\beta_2(1+z)+1-z+(1-z^2)\int_{-1}^{1}\frac{1}{1-tz}\rho_2(dt)}.$$

PROOF. By Theorem 2.1, there exists an injective mapping L_2 from Σ_1 into $\mathscr{L}_1$ satisfying relation (3.4), and so we have only to show that L_2 is surjective. Let $(\alpha_2,\beta_2,\rho_2)$ be any element of $\mathscr{L}_1$. By Theorem 4.1 in [8], we obtain an element σ_1 of Σ_1 such that $L_1(\sigma_1)=(\alpha_2,\beta_2,\rho_2)$. Further, it follows from Theorems 4.2 and 4.4 in [8] that this measure σ_1 can be expressed in the form

$$(3.5)\qquad \sigma_1(dt)=\frac{1}{2\pi}(\int_{-1}^{1}\frac{1}{1-ts}\sigma(ds))\sigma(dt)$$

with some $\sigma\in\Sigma_1$. Therefore, we can apply Lemma 2.1 to see that $[R_\sigma]$ is equal to h_{σ_1}. Recalling the definition of L_1 in [8], we conclude that $L_2(\sigma)=(\alpha_2,\beta_2,\rho_2)$. (Q.E.D.)

DEFINITION 3.1. For each $\sigma\in\Sigma_1$, we call a triple $(\alpha_2,\beta_2,\rho_2)(=\mathrm{L}_2(\sigma))$ or $(\alpha_2,\beta_2,\gamma_2)$ the **second KMO-Langevin data associated with** σ or R_σ, where γ_2 is a function on $\boldsymbol{Z}$ defined by

$$(3.6)\qquad \gamma_2=\frac{1}{2\pi}((1-e^{2i\cdot})\int_{-1}^{1}\frac{1}{1-te^{i\cdot}}\rho_2(dt))^{\wedge}.$$

From Proposition 5.1 in [8], we note that

$$(3.7)\qquad \gamma_2(n)=\begin{cases}0 & \text{for } n\in\{-1,-2,\ldots\}\\ \int_{-1}^{1}t^n\rho_2(dt) & \text{for } n\in\{0,1\}\\ \int_{-1}^{1}(t^n-t^{n-2})\rho_2(dt) & \text{for } n\in\{2,3,\ldots\}\end{cases}$$

(3.8) $\gamma_2 \in l^1(\boldsymbol{Z})$

(3.9) $\sum_{n=0}^{\infty} \gamma_2(n) = 0$

(3.10) $\sum_{n=0}^{\infty} (-1)^n \gamma_2(n) = 0.$

The following theorem gives a formula concerning the second KMO-Langevin data $(\alpha_2, \beta_2, \gamma_2)$ associated with a fixed $\sigma \in \Sigma_1$.

THEOREM 3.2.

(i) $\alpha_2 = \sqrt{\frac{2}{\pi}} \int_{-1}^{1} \frac{1}{1+t} \sigma(dt).$

(ii) $\beta_2 = \left(\int_{-1}^{1} \frac{1}{1+t} \sigma(dt)\right)\left(\int_{-1}^{1} \frac{1}{1-t} \sigma(dt)\right)^{-1}$

(iii) $\gamma_2(n) = \frac{\alpha_2}{(2\pi)^{3/2}} ([R_\sigma](e^{i\cdot})^{-1})^\wedge(n) - (\beta_2+1)\delta_{n,0} - (\beta_2-1)\delta_{n,1} \quad (n \in \boldsymbol{Z}).$

PROOF. By substituting $z=-1$ (resp. $z=1$) into (3.4), we have (i) (resp. (ii)). Furthermore, by substituting $z=e^{i\theta}$ into (3.4),

$$(1-e^{2i\theta}) \int_{-1}^{1} \frac{1}{1-te^{i\theta}} \rho_2(dt) = \frac{\alpha_2}{\sqrt{2\pi}} ([R_\sigma](e^{i\theta})^{-1}) - \beta_2(1+e^{i\theta}) - (1-e^{i\theta}).$$

The Fourier transform of this equality is nothing but (iii). (Q. E. D.)

REMARK 3.1. By Remark 5.1 in [8] and Theorem 3.1, we see that $[R_\sigma]^{-1}$ is equal to the outer function of the Hardy density $\Delta_{\sigma_1}^{-1}$ with σ_1 of the form (3.5).

EXAMPLE 3.1. For each $p \in (-1, 1)$, let us consider R_p corresponding to the Dirac measure $\sigma = \delta_{\{p\}}$:

(3.11) $R_p(n) = p^{|n|} \quad (n \in \boldsymbol{Z}).$

Observing that

(3.12) $$[R_p](z) = \frac{1}{\sqrt{2\pi}} \left(\sqrt{\frac{2}{\pi}} \frac{1}{1+p}\right) \frac{1}{\frac{1-p}{1+p}(1+z) + 1 - z} \quad (z \in U_1(0)),$$

we can see that the second KMO-Langevin data $(\alpha_p^{(2)}, \beta_p^{(2)}, \rho_p^{(2)})$ associated with R_p becomes

(3.13) $\alpha_p^{(2)} = \sqrt{\frac{2}{\pi}} \frac{1}{1+p}, \ \beta_p^{(2)} = \frac{1-p}{1+p}$ and $\rho_p^{(2)} = 0.$

EXAMPLE 3.2. Let R be the non-negative definite function given by

(3.14) $$R(n)=\sigma_1 p_1^{|n|}+\sigma_2 p_2^{|n|} \qquad (n\in\boldsymbol{Z}),$$

where σ_1, $\sigma_2>0$ and $-1<p_1<p_2<1$. Since

(3.15) $$[R](z)=\frac{\sigma_1+\sigma_2}{2\pi}\frac{1-q_2 z}{(1-p_1 z)(1-p_2 z)} \qquad (z\in\overline{U_1(0)}),$$

where

(3.16) $$q_2=\frac{\sigma_1 p_2+\sigma_2 p_1}{\sigma_1+\sigma_2} \qquad (\in(p_1, p_2)),$$

we find that the second KMO-Langevin data $(\alpha_2, \beta_2, \rho_2)$ associated with this R becomes

(3.17) $$\begin{cases} \alpha_2=\sqrt{\dfrac{2}{\pi}}\dfrac{\sigma_1(1+p_2)+\sigma_2(1+p_1)}{(1+p_1)(1+p_2)} \\ \beta_2=\dfrac{\sigma_1(1+p_2)+\sigma_2(1+p_1)}{\sigma_1(1-p_2)+\sigma_2(1-p_1)}\dfrac{(1-p_1)(1-p_2)}{(1+p_1)(1+p_2)} \\ \rho_2(dt)=2\dfrac{(p_2-q_2)(q_2-p_1)(\sigma_1+\sigma_2)}{(1+p_1)(1+p_2)\{\sigma_1(1-p_2)+\sigma_2(1-p_1)\}}\delta_{\{q_2\}}(dt). \end{cases}$$

§ 4. The Kubo noise (1)

Let $\boldsymbol{X}=(X(n)\,;\, n\in\boldsymbol{Z})$ be a real stationary Gaussian process on a probability space $(\Omega, \mathscr{B}, P)$; the covariance function R takes the form (2.1) with a bounded Borel measure σ on $[-1, 1]$ satisfying conditions (2.2) and (2.3). As usual, $h=h_\sigma$ and $E=\hat{h}$ denote the outer function and the canonical representation kernel, respectively. Then, in terms of a normalized Gaussian white noise $\boldsymbol{\xi}=(\xi(n)\,;\, n\in\boldsymbol{Z})$, $\boldsymbol{X}$ is expressed in the form

(4.1) $$X(n)=\frac{1}{\sqrt{2\pi}}\sum_{m=-\infty}^{n}E(n-m)\xi(m) \qquad \text{a.s.}\ (n\in\boldsymbol{Z}),$$

which is canonical in the sense that

(4.2) $$\sigma(X(m)\,;\, m\leq n)=\sigma(\xi(m)\,;\, m\leq n) \qquad (n\in\boldsymbol{Z}).$$

In this section we will establish another important expression of $\boldsymbol{X}$:

(4.3) $$X(n)=\frac{1}{\sqrt{2\pi}}\sum_{m=-\infty}^{n}R(n-m)I(m) \qquad \text{a.s.}\ (n\in\boldsymbol{Z}).$$

Such a representation is naturally introduced (Kubo [2]) by taking, as a representation kernel, (4.4) $E_0(n)=\chi_{[0,\infty)}(n)R(n) \qquad (n\in \boldsymbol{Z}).$

In order to define explicitly a real stationary Gaussian process $\boldsymbol{I}=(I(n);n\in\boldsymbol{Z})$ in (4.3), we first note that

(4.5) $\tilde{E}_0(\theta)=[R](e^{i\theta}) \qquad (\theta\in(-\pi,\pi)).$

By virtue of Theorem 5.1 in [8] and Theorem 3.1, we can define a function $h_I\in\mathcal{O}(U_1(0))\cap C(\overline{U_1(0)})$ by

(4.6) $h_I(z)=\frac{1}{\sqrt{2\pi}}\frac{h(z)}{[R](z)} \qquad (z\in\overline{U_1(0)}),$

where $\mathcal{O}(U_1(0))$ stands for the set of all holomorphic functions on $U_1(0)$. We denote by E_I the Fourier transform of h_I:

(4.7) $E_I(n)=\hat{h}_I(n)=\int_{-\pi}^{\pi}e^{-in\theta}h_I(e^{i\theta})d\theta.$

We note that

(4.8) E_I is real valued and $E_I\in l^1(\boldsymbol{Z})$

(4.9) $E_I=0$ on $\{-1,-2,-3,\ldots\}$.

With this E_I as a representation kernel, we set

(4.10) $I(n)=\frac{1}{\sqrt{2\pi}}\sum_{m=-\infty}^{n}E_I(n-m)\xi(m) \qquad \text{a.s. } (n\in\boldsymbol{Z}).$

Note that the right hand side of (4.10) is convergent almost surely and in the mean ($L^2(\Omega,\mathcal{B},P)$).

We are now in a position to prove the desired representation (4.3) of $\boldsymbol{X}$.

THEOREM 4.1.

(i) $X(n)=\frac{1}{\sqrt{2\pi}}\sum_{m=-\infty}^{n}R(n-m)I(m) \qquad \text{a.s. } (n\in\boldsymbol{Z})$

(ii) $\sigma(X(m);m\le n)=\sigma(I(m);m\le n) \qquad (n\in\boldsymbol{Z}).$

PROOF. By (2.4), (4.8) and (4.10), we have

$$\frac{1}{\sqrt{2\pi}}\sum_{m=-\infty}^{n}R(n-m)I(m)$$
$$=\frac{1}{\sqrt{2\pi}}\sum_{l=-\infty}^{n}\Big(\frac{1}{\sqrt{2\pi}}\sum_{m=l}^{n}R(n-m)E_I(m-l)\Big)\xi(l) \qquad \text{a.s.}$$

The kernel $K(n,l)=\frac{1}{\sqrt{2\pi}}\sum_{m=l}^{n}R(n-m)E_I(m-l)$ $(l<n)$ is computed by using (4.4)~(4.9):

$$K(n, l)=\frac{1}{\sqrt{2\pi}}(E_0(n-\cdot), E_I(\cdot-l))_{l^2(Z)}$$
$$=\hat{h}(n-l)=E(n-l).$$

This yields (i). (ii) follows easily from (4.2), (4.10) and (i).

(Q. E. D.)

By taking Definition 8.1 in [6] into account, we will introduce the following

DEFINITION 4.1. The stationary Gaussian process $\boldsymbol{I}$ given by (4.10) is called the **Kubo noise associated with** $\boldsymbol{X}$.

Let $(\alpha_2, \beta_2, \rho_2)$ (and $(\alpha_2, \beta_2, \gamma_2)$) be the second KMO-Langevin data associated with R. We now calculate the covariance function R_I of the Kubo noise $\boldsymbol{I}$ by means of these data.

THEOREM 4.2.

(i) *The spectral density* Δ_I *is given by*

$$\Delta_I(\theta)=\frac{4}{\sqrt{2\pi}\,\alpha_2(1+\beta_2+\gamma_2(0))}\{\sin^2\frac{\theta}{2}\int_{-1}^{1}\frac{(1+t)^2}{|1-te^{i\theta}|^2}\rho_2(dt)$$
$$+\beta_2(1+\cos^2\frac{\theta}{2}\int_{-1}^{1}\frac{(1-t)^2}{|1-te^{i\theta}|^2}\rho_2(dt))$$
$$+\frac{\sin^2\theta}{2}\int_{-1}^{1}\int_{-1}^{1}\frac{(t-s)^2}{|1-te^{i\theta}|^2|1-se^{i\theta}|^2}\rho_2(dt)\rho_2(ds)\}.$$

(ii) Δ_I *is a Hardy density with* h_I *in* (4.6) *as its outer function.*

(iii) $$R_I(n)=\begin{cases}\frac{2\sqrt{2\pi}}{\alpha_2(1+\beta_2+\gamma_2(0))}\{\gamma_2(0)+\gamma_2(1)+\beta_2(2+\gamma_2(0)-\gamma_2(1)) \\ \qquad +\gamma_2(0)^2+\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\gamma_2(l+2j)\gamma_2(l)\} & \text{for } n=0 \\ \frac{\sqrt{2\pi}}{\alpha_2(1+\beta_2+\gamma_2(0))}\{(1-\beta_2)\gamma_2(|n|+1) \\ \qquad -\sum_{l=1}^{\infty}\gamma_2(l)\gamma_2(|n|+l)\} & \text{for } n\neq 0.\end{cases}$$

PROOF. By (4.7) and (4.10), we see that

$$R_I(n-m)=\int_{-\pi}^{\pi}e^{-i(n-m)\theta}|h_I(e^{i\theta})|^2d\theta \qquad (n, m\in\boldsymbol{Z}),$$

which implies that the spectral density Δ_I is given by

$$(4.11)\qquad \Delta_I(\theta)=|h_I(e^{i\theta})|^2 \qquad (\theta\in(-\pi, \pi)).$$

We have to calculate it in terms of the data $(\alpha_2, \beta_2, \rho_2)$. For that purpose, we first claim

(4.12) $$|h(e^{i\theta})|^2=2\mathrm{Re}([R](e^{i\theta}))-\frac{1}{2\pi}R(0) \qquad (\theta\in(-\pi,\pi)).$$

Since $\tilde{R}(\theta)=\Delta(\theta)=|h(e^{i\theta})|^2$, (4.12) follows from

$$[R](e^{i\theta})=\frac{1}{2\pi}\sum_{n=0}^{\infty}R(n)e^{i\theta n}$$

and

$$\overline{[R](e^{i\theta})}=\frac{1}{2\pi}\sum_{n=-\infty}^{-1}R(n)e^{i\theta n}+\frac{1}{2\pi}R(0).$$

By (4.6) and (4.12), we have

(4.13) $$\Delta_I(\theta)=\frac{1}{2\pi}\left(2\mathrm{Re}\left(\frac{1}{[R](e^{i\theta})}\right)-\frac{R(0)}{2\pi}\left|\frac{1}{[R]e^{i\theta}}\right|^2\right) \qquad (\theta\in(-\pi,\pi)).$$

The key expession (3.4) enables us to write

(4.14) $$\mathrm{Re}\left(\frac{1}{[R](e^{i\theta})}\right)=\frac{\sqrt{2\pi}}{\alpha_2}\left(\beta_2(1+\cos\theta)+1-\cos\theta+\int_{-1}^{1}\frac{1-\cos 2\theta}{|1-te^{i\theta}|^2}\rho_2(dt)\right)$$

(4.15) $$\mathrm{Im}\left(\frac{1}{[R](e^{i\theta})}\right)=\frac{\sqrt{2\pi}}{\alpha_2}\left(\beta_2\sin\theta-\sin\theta+\int_{-1}^{1}\frac{2t\sin\theta-\sin 2\theta}{|1-te^{i\theta}|^2}\rho_2(dt)\right).$$

By a simple calculation, we see that

(4.16) $$\begin{aligned}&\left|\frac{1}{[R](e^{i\theta})}\right|^2\\ &=\frac{2\pi}{\alpha_2^2}\Big(4\beta_2^2\cos^2\frac{\theta}{2}+4\sin^2\frac{\theta}{2}+4\sin^2\theta\int_{-1}^{1}\frac{\beta_2(1+t)+1-t}{|1-te^{i\theta}|^2}\rho_2(dt)\\ &\qquad+4\sin^2\theta\int_{-1}^{1}\int_{-1}^{1}\frac{1-(t+s)\cos\theta+ts}{|1-te^{i\theta}|^2|1-se^{i\theta}|^2}\rho_2(dt)\rho_2(ds)\Big).\end{aligned}$$

Substituting (4.14) and (4.16) into (4.13), we get

(4.17) $$\begin{aligned}&\Delta_I(\theta)\\ &=2\pi^{-1}(\sqrt{2\pi}-R(0)\alpha_2^{-1}\beta_2)\alpha_2^{-1}\beta_2\cos^2\frac{\theta}{2}\\ &\quad+2\pi^{-1}(\sqrt{2\pi}-R(0)\alpha_2^{-1})\alpha_2^{-1}\sin^2\frac{\theta}{2}\\ &\quad+(2\pi\alpha_2)^{-1}\sin^2\theta\Big(\int_{-1}^{1}\frac{\sqrt{2\pi}}{|1-te^{i\theta}|^2}\rho_2(dt)\\ &\quad-\int_{-1}^{1}\frac{R(0)\alpha_2^{-1}(\beta_2(1+t)+1-t)}{|1-te^{i\theta}|^2}\rho_2(dt)\end{aligned}$$

$$
\begin{aligned}
&-\int_{-1}^{1}\int_{-1}^{1}\frac{R(0)\alpha_2^{-1}(1-(t+s)\cos\theta+ts)}{|1-te^{i\theta}|^2|1-se^{i\theta}|^2}\rho_2(dt)\rho_2(ds))\\
&=2\pi^{-1}(\sqrt{2\pi}-R(0)\alpha_2^{-1}\beta_2)\alpha_2^{-1}\beta_2\cos^2\frac{\theta}{2}\\
&+2\pi^{-1}(\sqrt{2\pi}-R(0)\alpha_2^{-1})\alpha_2^{-1}\sin^2\frac{\theta}{2}\\
&+(2\pi\alpha_2)^{-1}\sin^2\theta\int_{-1}^{1}\int_{-1}^{1}\frac{1}{|1-te^{i\theta}|^2|1-se^{i\theta}|^2}[\frac{\sqrt{2\pi}}{2\rho_2((-1,1))}(2+t^2+s^2\\
&-2(t+s)\cos\theta)-\frac{R(0)\alpha_2^{-1}}{2\gamma_2(0)}\{(\beta_2+1)(2+t^2+s^2-2ts\cos\theta)\\
&+(\beta_2-1)(t|1-se^{i\theta}|^2+s|1-te^{i\theta}|^2)\}\\
&-R(0)\alpha_2^{-1}(1-(t+s)\cos\theta+ts)]\rho_2(dt)\rho_2(ds).
\end{aligned}
$$

On the other hand, by substituting $z=0$ into (3.4), we get

$$
(4.18)\qquad R(0)=\sqrt{2\pi}\,\alpha_2\frac{1}{1+\beta_2+\gamma_2(0)}.
$$

Hence

$$
\begin{aligned}
(4.19)\qquad &\Delta_I(\theta)\\
&=\frac{4}{\sqrt{2\pi}\,\alpha_2(1+\beta_2+\gamma_2(0))}\{\gamma_2(0)\sin^2\frac{\theta}{2}+\sin^2\theta\int_{-1}^{1}\frac{t}{|1-te^{i\theta}|^2}\rho_2(dt)\\
&\quad+\frac{\sin^2\theta}{2}\int_{-1}^{1}\int_{-1}^{1}\frac{(t-s)^2}{|1-te^{i\theta}|^2|1-se^{i\theta}|^2}\rho_2(dt)\rho_2(ds)\\
&\quad+\beta_2((1+\gamma_2(0))\cos^2\frac{\theta}{2}-\sin^2\theta\int_{-1}^{1}\frac{t}{|1-te^{i\theta}|^2}\rho_2(dt))\}.
\end{aligned}
$$

Furthermore, we note that

$$
\gamma_2(0)\sin^2\frac{\theta}{2}+\sin^2\theta\int_{-1}^{1}\frac{t}{|1-te^{i\theta}|^2}\rho_2(dt)=\sin^2\frac{\theta}{2}\int_{-1}^{1}\frac{(1+t)^2}{|1-te^{i\theta}|^2}\rho_2(dt)
$$

and

$$
\gamma_2(0)\cos^2\frac{\theta}{2}-\sin^2\theta\int_{-1}^{1}\frac{t}{|1-te^{i\theta}|^2}\rho_2(dt)=\cos^2\frac{\theta}{2}\int_{-1}^{1}\frac{(1-t)^2}{|1-te^{i\theta}|^2}\rho_2(dt).
$$

Therefore, by substituting these into (4.19), we arrive at the equaliy (i).

Next we will show (ii). By Remark 5.1 in [8], Remark 3.1, (4.6) and (4.11), we see that there exist positive constants c_3 and c_4 such that

$$
(4.20)\qquad c_3\le\Delta_I(\theta)\le c_4\qquad(\theta\in(-\pi,\pi)),
$$

which implies that Δ_I is a Hardy density. And so we can define its outer function h_I by

$$h_I(z)=\exp\Big(\frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\log\Delta_I(\theta)\,d\theta\Big)\qquad(z\in U_1(0)).$$

By using Lemma 2.1, (4.6) and (4.11) again, we see that

$$\begin{aligned}h_I(z)&=\exp\Big(\frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\log\Big(\frac{1}{2\pi}\Big)d\theta\Big)\cdot\\&\quad\cdot\exp\Big(\frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\log|h(e^{i\theta})|^2d\theta\Big)\cdot\\&\quad\cdot\exp\Big(\frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\log|[R](e^{i\theta})|^2d\theta\Big)^{-1}\\&=\frac{1}{\sqrt{2\pi}}\frac{h(e^{i\theta})}{[R](e^{i\theta})},\end{aligned}$$

which, with (4.6), gives (ii).

Now we will proceed to the proof of (iii). By using (i), we see that for any $n\in\boldsymbol{Z}$,

(4.21) $R_I(n)$

$$\begin{aligned}&=\frac{4}{\sqrt{2\pi}\,\alpha_2(1+\beta_2+\gamma_2(0))}\Big\{\int_{-1}^{1}\Big(\int_{-\pi}^{\pi}e^{-in\theta}\frac{\sin^2\frac{\theta}{2}}{|1-te^{i\theta}|^2}d\theta\Big)(1+t)^2\rho_2(dt)\\&\quad+\beta_2\Big(\int_{-\pi}^{\pi}e^{-in\theta}d\theta+\int_{-1}^{1}\Big(\int_{-\pi}^{\pi}e^{-in\theta}\frac{\cos^2\frac{\theta}{2}}{|1-te^{i\theta}|^2}d\theta\Big)(1-t)^2\rho_2(dt)\Big)\\&\quad+\frac{1}{2}\int_{-1}^{1}\int_{-1}^{1}\Big(\int_{-\pi}^{\pi}e^{-in\theta}\frac{\sin^2\theta}{|1-te^{i\theta}|^2|1-se^{i\theta}|^2}d\theta\Big)(t-s)^2\rho_2(dt)\rho_2(ds)\Big\}.\end{aligned}$$

For each $t\in(-1,1)$ we put

$$f_t(\theta)=\frac{1}{|1-te^{i\theta}|^2}.$$

As we have seen in Example 3.1 of [8],

$$\hat{f}_t(n)=\frac{2}{1-t^2}t^{|n|}\qquad(n\in\boldsymbol{Z}).$$

Since $\sin^2\frac{\theta}{2}=\frac{1}{2}-\frac{e^{i\theta}+e^{-i\theta}}{4}$ and $\cos^2\frac{\theta}{2}=\frac{1}{2}+\frac{e^{i\theta}+e^{-i\theta}}{4}$, we see that for any $n\in\{0,1,2,\dots\}$ and $s,\ t\in(-1,1)$,

$$(4.22)\qquad \int_{-\pi}^{\pi}e^{-in\theta}\sin^2\frac{\theta}{2}\,f_t(\theta)\,d\theta=\frac{\pi}{2(1-t^2)}(2t^{|n|}-t^{|n-1|}-t^{|n+1|})$$

$$(4.23)\qquad \int_{-\pi}^{\pi}e^{-in\theta}\cos^2\frac{\theta}{2}\,f_t(\theta)\,d\theta=\frac{\pi}{2(1-t^2)}(2t^{|n|}+t^{|n-1|}+t^{|n+1|})$$

$$(4.24)\quad \int_{-\pi}^{\pi} e^{-in\theta}\sin^2\theta\, f_t(\theta)f_s(\theta)\,d\theta$$
$$=\frac{1}{2}\{\widehat{f_tf_s}(n)-\frac{1}{2}(\widehat{f_tf_s}(n-2)+\widehat{f_tf_s}(n+2))\}$$
$$=\frac{1}{4\pi}\{\hat{f}_t*\hat{f}_s(n)-\frac{1}{2}(\hat{f}_t*\hat{f}_s(n-2)+\hat{f}_t*\hat{f}_s(n+2))\}$$
$$=\frac{\pi}{2(1-t^2)(1-s^2)}\sum_{l=-\infty}^{\infty}(2t^{|n-l|}-t^{|n-2-l|}-t^{|n+2-l|})s^{|l|}$$
$$=\frac{\pi}{2(1-t^2)(1-s^2)}(\mathrm{I}_n+\mathrm{II}_n+\mathrm{III}_n+\mathrm{IV}_n),$$

where

$$\mathrm{I}_n=\sum_{l=-\infty}^{n-2}(2t^{|n-l|}-t^{|n-2-l|}-t^{|n+2-l|})s^{|l|}$$
$$\mathrm{II}_n=\sum_{l=n-1}^{n}(2t^{|n-l|}-t^{|n-2-l|}-t^{|n+2-l|})s^{|l|}$$
$$\mathrm{III}_n=\sum_{l=n+1}^{n+2}(2t^{|n-l|}-t^{|n-2-l|}-t^{|n+2-l|})s^{|l|}$$
$$\mathrm{IV}_n=\sum_{l=n+3}^{\infty}(2t^{|n-l|}-t^{|n-2-l|}-t^{|n+2-l|})s^{|l|}.$$

For $n=0$, we can see that

$$\mathrm{I}_0=-\frac{(1-t^2)^2s^2}{1-ts},\quad \mathrm{II}_0=(1-t^2)(ts+2),\quad \mathrm{III}_0=(1-t^2)s(t-s+t^2s)$$

and $\mathrm{IV}_0=-\dfrac{(1-t^2)^2ts^2}{1-ts}$, which imply that

$$(4.25)\quad \mathrm{I}_0+\mathrm{II}_0+\mathrm{III}_0+\mathrm{IV}_0=2\frac{(1-t^2)(-s^2)}{1-ts}.$$

For $n=1$, we can see that

$$\mathrm{I}_1=-\frac{(1-t^2)^2s}{1-ts},\quad \mathrm{II}_1=(1-t^2)(t+2s),\quad \mathrm{III}_1=(1-t^2)s^2(t-(1-t^2)s)$$

and $\mathrm{IV}_1=-\dfrac{(1-t^2)^2ts^4}{1-ts}$, which imply that

$$(4.26)\quad \mathrm{I}_1+\mathrm{II}_1+\mathrm{III}_1+\mathrm{IV}_1=2\frac{(1-t^2)(1-s^2)(t+s)}{1-ts}.$$

For $n\geq 2$, we can see that

$$\mathrm{I}_n=\frac{ts}{1-ts}(2t^n-t^{n-2}-t^{n+2})+\sum_{l=0}^{n-2}(2t^{n-l}-t^{n-2-l}-t^{n+2-l})s^l$$

$$\mathrm{II}_n=\sum_{l=n-1}^{n}(2t^{n-l}-t^{l-n+2}-t^{n+2-l})s^l$$

$$\mathrm{III}_n=\sum_{l=n+1}^{n+2}(2t^{l-n}-t^{l-n+2}-t^{n+2-l})s^l$$

$$\mathrm{IV}_n=\sum_{l=n+3}^{\infty}(2t^{l-n}-t^{l-n+2}-t^{l-n-2})s^l.$$

And so

$$(4.27)\qquad \begin{aligned} &\mathrm{I}_n+\mathrm{II}_n+\mathrm{III}_n+\mathrm{IV}_n\\ &=(2t^n-t^{n-2}-t^{n+2})\frac{ts}{1-ts}+2\sum_{l=0}^{n}t^{n-l}s^l-\sum_{l=0}^{n-2}t^{n-2-l}s^l\\ &\quad-\sum_{l=0}^{n+2}t^{n+2-l}s^l-\sum_{l=n-1}^{\infty}t^{l-n+2}s^l+2\sum_{l=n+1}^{\infty}t^{l-n}s^l\\ &\quad-\sum_{l=n+3}^{\infty}t^{l-n-2}s^l\\ &=(2t^n-t^{n-2}-t^{n+2})\frac{ts}{1-ts}+2\frac{t^{n+1}-s^{n+1}}{t-s}-\frac{t^{n-1}-s^{n-1}}{t-s}-\frac{t^{n+3}-s^{n+3}}{t-s}\\ &\quad-\frac{ts^{n-1}}{1-ts}+2\frac{ts^{n+1}}{1-ts}-\frac{ts^{n+3}}{1-ts}\\ &=\frac{(1-t^2)(1-s^2)}{(1-ts)(t-s)}\{(1-s^2)s^{n-1}-(1-t^2)t^{n-1}\}. \end{aligned}$$

We note that (4.26) implies that (4.27) holds for $n\geq 1$. By (4.21)~(4.25), we see that

$$\begin{aligned} R_I(0)&=\frac{4}{\sqrt{2\pi}\,\alpha_2(1+\beta_2+\gamma_2(0))}\{\frac{\pi}{2}\int_{-1}^{1}(2-2t)\frac{(1+t)^2}{1-t^2}\rho_2(dt)\\ &\quad+\beta_2(2\pi+\frac{\pi}{2}\int_{-1}^{1}(2+2t)\frac{(1-t)^2}{1-t^2}\rho_2(dt))\\ &\quad+\frac{\pi}{2}\int_{-1}^{1}\int_{-1}^{1}\frac{(t-s)^2}{1-ts}\rho_2(dt)\rho_2(ds)\}\\ &=\frac{2\sqrt{2\pi}}{\alpha_2(1+\beta_2+\gamma_2(0))}\{\int_{-1}^{1}(1+t)\rho_2(dt)\\ &\quad+\beta_2(2+\int_{-1}^{1}(1-t)\rho_2(dt))\\ &\quad+\frac{1}{2}\int_{-1}^{1}\int_{-1}^{1}\frac{(t-s)^2}{1-ts}\rho_2(dt)\rho_2(ds)\}. \end{aligned}$$

This fact together with (3.7) implies that (iii) holds for $n=0$.

In case $n\geq 1$, we use (4.21)~(4.24) and (4.27) to get

$$(4.28)\qquad R_I(n)=\frac{4}{\sqrt{2\pi}\,\alpha_2(1+\beta_2+\gamma_2(0))}\{\frac{\pi}{2}\int_{-1}^{1}(2t^n-t^{n-1}-t^{n+1})\frac{(1+t)^2}{1-t^2}\rho_2(dt)$$

$$+\beta_2\frac{\pi}{2}\int_{-1}^{1}(2t^n+t^{n-1}+t^{n+1})\frac{(1-t)^2}{1-t^2}\rho_2(dt)$$
$$+\frac{\pi}{4}\int_{-1}^{1}\int_{-1}^{1}\frac{((1-s)^2s^{n-1}-(1-t^2)t^{n-1})}{1-ts}(t-s)\rho_2(dt)\rho_2(ds)\}.$$

By (3.7), we have

$$(4.29)\qquad \int_{-1}^{1}(2t^n-t^{n-1}-t^{n+1})\frac{(1+t)^2}{1-t^2}\rho_2(dt)=\gamma_2(n+1)$$

$$(4.30)\qquad \int_{-1}^{1}(2t^n+t^{n-1}+t^{n+1})\frac{(1-t)^2}{1-t^2}\rho_2(dt)=-\gamma_2(n+1).$$

Moreover,

$$
\begin{aligned}
(4.31)\qquad & \int_{-1}^{1}\int_{-1}^{1}\frac{((1-s)^2s^{n-1}-(1-t^2)t^{n-1})}{1-ts}(t-s)\rho_2(dt)\rho_2(ds)\\
&=\sum_{l=0}^{\infty}\int_{-1}^{1}\int_{-1}^{1}((t^2-1)t^{n-1}-(s^2-1)s^{n-1})(ts)^l(t-s)\rho_2(dt)\rho_2(ds)\\
&=2\sum_{l=0}^{\infty}\Big(\int_{-1}^{1}(t^2-1)t^{n+l}\rho_2(dt)\int_{-1}^{1}s^l\rho_2(ds)\\
&\qquad\qquad -\int_{-1}^{1}(s^2-1)s^{l+n-1}\rho_2(ds)\int_{-1}^{1}t^{l+1}\rho_2(dt)\Big)\\
&=2\sum_{l=0}^{\infty}\Big(\gamma_2(n+l+2)\int_{-1}^{1}s^l\rho_2(ds)-\gamma_2(n+l+1)\int_{-1}^{1}t^{l+1}\rho_2(dt)\Big)\\
&=2\sum_{l=2}^{\infty}\gamma_2(n+l)\int_{-1}^{1}(t^{l-2}-t^l)\rho_2(dt)-\gamma_2(n+1)\int_{-1}^{1}t\rho_2(dt))\\
&=-2\Big(\sum_{l=2}^{\infty}\gamma_2(n+l)\gamma_2(l)+\gamma_2(n+1)\gamma_2(1)\Big)\\
&=-2\sum_{l=1}^{\infty}\gamma_2(n+l)\gamma_2(l).
\end{aligned}
$$

By substituting (4.29)~(4.31) into (4.28), we find the expression (iii) for $n\geq 1$. Thus we have proved Theorem 4.2. (Q. E. D.)

EXAMPLE 4.1. For each $p\in(-1,1)$, let $\boldsymbol{X}_p=(X_p(n)\,;\,n\in\boldsymbol{Z})$ be a real stationary Gaussian process with covariance function R_p of the form (3.11) in Example 3.1. We denote by $(\alpha_p^{(1)},\beta_p^{(1)},\rho_p^{(1)})$, $(\alpha_p^{(2)},\beta_p^{(2)},\rho_p^{(2)})$ and $\boldsymbol{I}_p=(I_p(n)\,;\,n\in\boldsymbol{Z})$ the first KMO-Langevin data, the second KMO-Langevin data associated with R_p and the Kubo noise associated with $\boldsymbol{X}_p$, respectively. We then know from (4.21) in [8] and (3.13) that

$$(4.32)\qquad \begin{cases}\alpha_p^{(1)}=2\sqrt{\dfrac{1-p}{1+p}},\ \alpha_p^{(2)}=\sqrt{\dfrac{2}{\pi}}\dfrac{1}{1+p}\\ \beta_p^{(1)}=\beta_p^{(2)}=\dfrac{1-p}{1+p}\\ \rho_p^{(1)}=\rho_p^{(2)}=0.\end{cases}$$

By a simple calculation, we have

$$(4.33)\quad h_{I_p}=\frac{1}{\sqrt{2\pi}}\frac{\sqrt{2\pi}\,\alpha_p^{(1)}}{1+\beta_p^{(1)}}=\frac{1}{\sqrt{2\pi}}\frac{\alpha_p^{(1)}}{\alpha_p^{(2)}}=\sqrt{1-p^2},\ \text{constant function}$$

$$(4.34)\quad I_p(n)=\frac{\alpha_p^{(1)}}{\alpha_p^{(2)}}\xi(n)=\sqrt{2\pi(1-p^2)}\,\xi(n),\ \text{white noise}\ (n\in\boldsymbol{Z}).$$

EXAMPLE 4.2. Let $\boldsymbol{X}=(X(n)\,;\,n\in\boldsymbol{Z})$ be a real stationary Gaussian process with covariance function R of the form (3.14) in Example 3.2. And let $\boldsymbol{I}=(I(n)\,;\,n\in\boldsymbol{Z})$ be the Kubo noise associated with $\boldsymbol{X}$. Put

$$(4.35)\quad a_1=\sigma_1(1-p_1^2)p_2+\sigma_2(1-p_2^2)p_1$$
$$(4.36)\quad a_2=\sigma_1(1-p_1^2)(1+p_2^2)+\sigma_2(1-p_2^2)(1+p_1^2).$$

(i) The case where $a_1=0$: It then follows from (4.25) in [8], (4.6) and Theorem 4.2(ii) that the outer function h_I of $\boldsymbol{I}$ becomes

$$(4.37)\quad h_I(z)=\frac{\sqrt{a_2}}{\sigma_1+\sigma_2}\frac{1}{1-q_2z}=c_5h_{q_2}(z),$$

where h_{q_2} is the outer function corresponding to $\boldsymbol{X}_{q_2}$ in the previous Example 4.1 and c_5 is a positive constant given by

$$(4.38)\quad c_5=\frac{\sqrt{2\pi a_2}}{\sqrt{(\sigma_1+\sigma_2)^2-(\sigma_1p_2+\sigma_2p_1)^2}}.$$

Therefore, we see that

$$(4.39)\quad I(n)=c_5X_{q_2}(n)\qquad(n\in\boldsymbol{Z}),$$

which implies that the Kubo noise $\boldsymbol{I}$ is a colored noise with a simple Markovian property, if $q_2\neq0$.

(ii) The case where $a_1\neq0$: it then follows from (4.27) in [8], (4.6), (4.37) and Theorem 4.1(ii) that the outer function h_I of $\boldsymbol{I}$ becomes

$$(4.40)\quad h_I(z)=\frac{\sqrt{r_1}}{\sigma_1+\sigma_2}\frac{1-q_1z}{1-q_2z}\qquad(z\in\overline{U_1(0)}),$$

where q_1 and r_1 are the same in (4.31) and (4.32) of [8], respectively;

$$(4.41)\quad q_1=\frac{1}{2}\left(\frac{a_2}{a_1}\underset{(+)}{-}\sqrt{\left(\frac{a_2}{a_1}\right)^2-4}\right)\qquad\text{if } a_1\underset{(<)}{>}0,$$

$$(4.42)\quad r_1=\frac{a_1}{q_1}.$$

We note that

(4.43) $\quad q_1 \neq q_2$ if and only if $\quad \sigma_1 p_1(1-p_2^2)+\sigma_2 p_2(1-p_1^2)\neq 0.$

Therefore, the Kubo noise $\boldsymbol{I}$ is a colored noise with multiple Markovian property, if $q_1 \neq q_2$.

§ 5. The second KMO-Langevin equation

By taking the Kubo noise $\boldsymbol{I}$ in (4.10) as a random force, we will derive a stochastic difference equation which describes the time evolution of $\boldsymbol{X}$.

THEOREM 5.1.

(5.1) $$X(n)-X(n-1)=-\beta_2(X(n)+X(n-1))-(\gamma_2 * X)(n)+\alpha_2 I(n) \quad \text{a. s. } (n\in \boldsymbol{Z}).$$

PROOF. The idea of proof is similar to the one of Theorem 6.1 in [8], where we established the first KMO-Langevin equation describing the different time evolution of the same process $\boldsymbol{X}$ by taking the white noise as a random force. By (2.4) and (3.8), we see from Theorem 4.1(i) that the following two random series are absolutely convergent (a.s.): for any $n \in \boldsymbol{Z}$,

(5.2) $$\begin{aligned} &X(n)-X(n-1)+\beta_2(X(n)+X(n-1)) \\ &=\frac{1}{\sqrt{2\pi}}\sum_{m=-\infty}^{\infty}\{(\chi_{[0,\infty)}R)(n-m)-(\chi_{[0,\infty)}R)(n-m-1) \\ &\qquad +\beta_2((\chi_{[0,\infty)}R)(n-m)+(\chi_{[0,\infty)}R)(n-m-1))\}I(m)\end{aligned}$$

(5.3) $$\begin{aligned} (\gamma_2 * X)(n)&=\sum_{l=-\infty}^{\infty}\gamma_2(n-l)X(l) \\ &=\frac{1}{\sqrt{2\pi}}\sum_{m=-\infty}^{\infty}\Big(\sum_{l=-\infty}^{\infty}\gamma_2(n-l)(\chi_{[0,\infty)}R)(l-m)\Big)I(m).\end{aligned}$$

On the other hand, it follows from (2.4), (2.5), (3.6) and (3.8) that for any $n, m\in \boldsymbol{Z}$,

(5.4) $$\begin{aligned} &(\chi_{[0,\infty)}R)(n-m)-(\chi_{[0,\infty)}R)(n-m-1) \\ &\qquad +\beta_2((\chi_{[0,\infty)}R)(n-m)+(\chi_{[0,\infty)}R)(n-m-1)) \\ &=\int_{-\pi}^{\pi}e^{-i(n-m)\theta}(1-e^{i\theta}+\beta_2(1+e^{i\theta}))[R](e^{i\theta})\,d\theta\end{aligned}$$

(5.5) $$\begin{aligned} &\sum_{l=-\infty}^{\infty}\gamma_2(n-l)(\chi_{[0,\infty)}R)(l-m) \\ &=\int_{-\pi}^{\pi}e^{-i(n-m)\theta}(1-e^{2i\theta})\int_{-1}^{1}\frac{1}{1-te^{i\theta}}\rho_2(dt)[R](e^{i\theta})\,d\theta.\end{aligned}$$

Therefore, by substituting (5.4) and (5.5) into (5.2) and (5.3), respectively, we conclude from Theorem 3.1 that for any $n\in \boldsymbol{Z}$,

$$
\begin{aligned}
&X(n)-X(n-1)+\beta_2(X(n)+X(n-1))+(\gamma_2 * X)(n)\\
&=\frac{1}{\sqrt{2\pi}}\sum_{m=-\infty}^{\infty}\left(\int_{-\pi}^{\pi} e^{-i(n-m)\theta}\frac{\alpha_2}{\sqrt{2\pi}}d\theta\right)I(m)\\
&=\alpha_2 I(n) \qquad \text{a. s.,}
\end{aligned}
$$

which completes the proof of Theorem 5. 1. (Q. E. D.)

DEFINITION 5. 1. We call the stochastic difference equation (5. 1) the **second KMO-Langevin equation for X.**

§ 6. Generalized fluctuation-dissipation theorems (1)

Let $(\alpha_1, \beta_1, \rho_1)$ be any element of $\mathscr{L}_1$ and $\boldsymbol{\xi}=(\xi(n)\ ;\ n\in \boldsymbol{Z})$ be a normalized Gaussian white noise. In [8], Theorem 6. 2, we have obtained a real stationary Gaussian process $\boldsymbol{X}=(X(n)\ ;\ n\in \boldsymbol{Z})$ which solves uniquely the first KMO-Langevin equation

$$
X(n)-X(n-1)=-\beta_1(X(n)+X(n-1))-(\gamma_1 * X)(n)+\alpha_1\xi(n) \quad \text{a. s. } (n\in \boldsymbol{Z}), \tag{6.1}
$$

where γ_1 is a function on $\boldsymbol{Z}$ defined by

$$
\gamma_1(n)=\begin{cases} 0 & \text{for } n\in\{-1,-2,\dots\}\\ \int_{-1}^{1} t^n\rho_1(dt) & \text{for } n\in\{0,1\}\\ \int_{-1}^{1}(t^n-t^{n-2})\rho_1(dt) & \text{for } n\in\{2,3,\dots\}. \end{cases} \tag{6.2}
$$

We denote by R the covariance function of $\boldsymbol{X}$. Let $(\alpha_2, \beta_2, \rho_2)$ and $\boldsymbol{I}=(I(n)\ ;\ n\in \boldsymbol{Z})$ be the second KMO-Langevin data and the Kubo noise associated with $\sigma=L_1^{-1}((\alpha_1, \beta_1, \rho_1))$ and $\boldsymbol{X}$, respectively. In addition to (6. 1), $\boldsymbol{X}$ also satisfies the second KMO-Langevin equation:

$$
X(n)-X(n-1)=-\beta_2(X(n)+X(n-1))-(\gamma_2 * X)(n)+\alpha_2 I(n) \quad \text{a. s. } (n\in \boldsymbol{Z}), \tag{6.3}
$$

where γ_2 is a function on $\boldsymbol{Z}$ given by (3. 7).

We denote by D the diffusion constant of $\boldsymbol{X}$:

$$
D=\lim_{N\to\infty}\frac{1}{2N}E\left(\left(\sum_{n=0}^{N}X(n)\right)^2\right). \tag{6.4}
$$

In [8] we have shown the following generalized fluctuation-dissipation theorems for $\boldsymbol{X}$ based on equation (6. 1) ;

A generalized first fluctuation-dissipation theorem:

$$(6.5)\qquad \frac{1}{\beta_1(1+e^{i\theta})+1-e^{i\theta}+2\pi\tilde{\gamma}_1(\theta)}=\frac{h(e^{i\theta})}{\lim_{\tau\downarrow-\pi}h(e^{i\tau})}\qquad (\theta\in(-\pi,\pi)).$$

A generalized second fluctuation-dissipation theorem:

$$(6.6)\qquad \frac{\alpha_1^2}{2}=R(0)\ C_{\beta_1,\gamma_1},$$

where

$$(6.7)\qquad C_{\beta_1,\gamma_1}=\pi\left(\int_{-\pi}^{\pi}|\beta_1(1+e^{i\theta})+1-e^{i\theta}+2\pi\tilde{\gamma}_1(\theta)|^{-2}d\theta\right)^{-1}.$$

A generalized Einstein relation:

$$(6.8)\qquad D=\frac{R(0)}{2\beta_1}\cdot\frac{C_{\beta_1,\gamma_1}}{2\beta_1}$$

$$(6.9)\qquad \frac{C_{\beta_1,\gamma_1}}{2\beta_1}-1=\frac{1}{R(0)}\int_{-1}^{1}\int_{-1}^{1}\frac{1-s}{1-st}\rho_1(dt)\sigma(ds).$$

In this section, we will prove results of the same type based on equation (6. 3).

THEOREM 6. 1.

(i) *For any* $\theta\in(-\pi,\pi)$

$$\frac{1}{\beta_2(1+e^{i\theta})+1-e^{i\theta}+2\pi\tilde{\gamma}_2(\theta)}=(2[R](-1))^{-1}[R](e^{i\theta}).$$

(ii) *The spectral density of the stationary process* $\alpha_2 \boldsymbol{I}$

$$\begin{aligned}&=\frac{R(0)}{2\pi}\{2(1+\beta_2+\gamma_2(0))\mathrm{Re}(\beta_2(1+e^{i\theta})+1-e^{i\theta}+2\pi\tilde{\gamma}_2(\theta))\\&\qquad -|\beta_2(1+e^{i\theta})+1-e^{i\theta}+2\pi\tilde{\gamma}_2(\theta)|^2\}\\&=\frac{2R(0)}{\pi}\{\sin^2\frac{\theta}{2}\int_{-1}^{1}\frac{(1+t)^2}{|1-te^{i\theta}|^2}\rho_2(dt)\\&\qquad +\beta_2(1+\cos^2\frac{\theta}{2}\int_{-1}^{1}\frac{(1-t)^2}{|1-te^{i\theta}|^2}\rho_2(dt))\\&\qquad +\frac{\sin^2\theta}{2}\int_{-1}^{1}\int_{-1}^{1}\frac{(t-s)^2}{|1-te^{i\theta}|^2|1-se^{i\theta}|^2}\rho_2(dt)\rho_2(ds)\}.\end{aligned}$$

(iii) $D=\dfrac{R(0)}{2\beta_2}(1+\gamma_2(0)).$

PROOF. As we have seen in Theorem 3. 2(i), we note that

$$(6.10)\qquad \alpha_2=2\sqrt{2\pi}[R](-1),$$

which, together with Theorem 2. 1(iii), gives (i). By substituting (4. 18) into (4. 13), we see that the first expression of (ii) follows from Theorems 2. 1(iii) and 4. 2(ii). The second expression follows immediately from Theorem 4. 2(i) and (4. 18).

In [8], Lemma 7. 1, we established

$$(6.11) \quad D=\sum_{n=0}^{\infty} R(n)-\frac{R(0)}{2}.$$

As is easily seen in Theorem 3. 2(ii),

$$(6.12) \quad \sum_{n=0}^{\infty} R(n)=\frac{\sqrt{2\pi}\,\alpha_2}{2\beta_2},$$

which, together with (4. 18), yields (iii). (Q. E. D.)

EXAMPLE 6. 1. For each $p\in(-1,1)$, let $\boldsymbol{X}_p=(X_p(n);\ n\in\boldsymbol{Z})$ be a real stationary Gaussian Markov process discussed in Example 4. 1. By (6. 13) in [8] and Theorem 5. 1, the first and second KMO-Langevin equations take the following foms, respectively:

$$(6.13) \quad X_p(n)-X_p(n-1)=-\beta_p^{(1)}(X_p(n)+X_p(n-1))+\alpha_p^{(1)}\xi(n)$$
$$(6.14) \quad X_p(n)-X_p(n-1)=-\beta_p^{(2)}(X_p(n)+X_p(n-1))+\alpha_p^{(2)}I_p(n).$$

Since it follows from (4. 32) and (4. 34) that

$$(6.15) \quad \beta_p^{(1)}=\beta_p^{(2)}$$
$$(6.16) \quad \alpha_p^{(1)}\xi(n)=\alpha_p^{(2)}I_p(n),$$

we note that two equations (6. 13) and (6. 14) actually coincide.

Since by (7. 21) in [8]

$$(6.17) \quad [R_p](z)=\frac{1+\beta_p^{(1)}}{\sqrt{2\pi}\,\alpha_p^{(1)}}h_p(z) \qquad (z\in U_1(0)),$$

the relation (i) in Theorem 6. 1 is also the same as (6. 5) for such a simple process $\boldsymbol{X}_p$.

Since $\rho_p^{(2)}=0$, we see from (6. 16) that Theorem 6. 1(ii) is rewritten in the form

$$\frac{(\alpha_p^{(1)})^2}{2\pi}=\frac{1}{\pi}R(0)(2\beta_p^{(2)}).$$

Since by (7. 20) in [8] and (6. 15)

$$(6.18) \quad C_{\beta_p^{(1)},\gamma_p^{(1)}}=2\beta_p^{(2)},$$

we find that the relation (ii) in Theorem 6. 1 can be reduced to (6. 6) for $\boldsymbol{X}_p$.

Finally, since $\gamma_p^{(2)}=0$, we see from (6.18) that the relation (iii) in Theorem 6.1 coincides with the classical Einstein relation

$$(6.19)\qquad D=\frac{R_p(0)}{2\beta_p^{(1)}}=\frac{R_p(0)}{2\beta_p^{(2)}}.$$

By taking the above Example 6.1 into account, we will introduce the following

DEFINITION 6.1. We call the relation (i), (ii) and (iii) in Theorem 6.1 the **generalized first fluctuation-dissipation theorem,** the **generalized second fluctuation-dissipation theorem** and the **generalized Einstein relation for** the process X **based on equation** (6.3), respectively.

REMARK 6.1. The left hand side in the relation (i) in Theorem 6.1 expresses a **complex mobility** of the system X described by the second KMO-Langevin equation (6.3). In this sense, our generalized first fluctuation-dissipation theorem provides a justification of the unfamiliar nomenclature of complex mobility function given in Definition 2.1.

REMARK 6.2. We note that the classical Einstein relation (6.19) for the simplest system described by equation (6.13) (or(6.14)) does not hold for the general system described not only by the first KMO-Langevin equation (6.1), but also by the second one (6.3). In the next section we will investigate these deviations from the classical Einstein relation.

EXAMPLE 6.2. Let us consider a real stationary Gaussian process X treated in Example 4.2. By using (3.17) and (6.11), we can see that the diffusion constant D of X becomes

$$(6.20)\qquad D=\frac{1}{2}\left(\sigma_1\frac{1+p_1}{1-p_1}+\sigma_2\frac{1+p_2}{1-p_2}\right).$$

It would be troublesome to check the generalized Einstein relation via a direct method of substituting the second KMO-Langevin data (3.17).

§7. The deviation from the Einstein relation

We will begin with the discussions of several relations between two KMO-Langevin data $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$ which appear in our KMO-Langevin equations (6.1) and (6.3), respectively.

PROPOSITION 7.1.

(i) $R(0)=\dfrac{\sqrt{2\pi}\,\alpha_2}{1+\beta_2+\gamma_2(0)}$

(ii) $R(0)=\sqrt{2\pi}\,\alpha_2-\frac{\alpha_1^2}{4}$

(iii) $R(0)=\sqrt{2\pi}\,\frac{\alpha_2}{\beta_2}-\frac{\alpha_1^2}{4\beta_1^2}$.

PROOF. We have already shown (i) in (4.18). By Theorem 5.2 (i) in [8], we have

$$\frac{\alpha_1^2}{4}=\int_{-1}^{1}\frac{1-t}{1+t}\sigma(dt). \tag{7.1}$$

Hence, (ii) follows from Theorem 3.2(i). Furthermore, by Theorems 5.2(i) and 5.2(ii) in [8], we have

$$\frac{\alpha_1^2}{4\beta_1^2}=\int_{-1}^{1}\frac{1+t}{1-t}\sigma(dt). \tag{7.2}$$

On the other hand, Theorems 3.2(i) and 3.2(ii) tell us that

$$\sqrt{\frac{\pi}{2}}\,\frac{\alpha_2}{\beta_2}=\int_{-1}^{1}\frac{1}{1-t}\sigma(dt). \tag{7.3}$$

Therefore, (iii) follows from (7.2) and (7.3). (Q.E.D.)

PROPOSITION 7.2.

$$C_{\beta_1,\gamma_1}=2(\beta_2+\gamma_2(0)).$$

PROOF. By using (6.6), Propositions 2.1(i) and 2.1(ii), we get

$$R(0)=R(0)(1+\beta_2+\gamma_2(0))-\frac{R(0)}{2}C_{\beta_1,\gamma_1},$$

which yields Proposition 7.2. (Q.E.D.)

PROPOSITION 7.3.

(i) $\alpha_1^2=4\sqrt{2\pi}\,\alpha_2\frac{\beta_2+\gamma_2(0)}{1+\beta_2+\gamma_2(0)}$

(ii) $\beta_1^2=\beta_2\frac{\beta_2+\gamma_2(0)}{1+\gamma_2(0)}$.

PROOF. (i) follows from (6.6), Propositions 7.1(i) and 7.2. By using Proposition 7.2 again, (ii) follows from (6.8) and Theorem 6.1 (iii). (Q.E.D.)

As straightforward consequences of Proposition 7.3(i), we obtain

PROPOSITION 7.4.

(i) $\alpha_1^2<4\sqrt{2\pi}\,\alpha_2$

(ii) $\gamma_2(0)=\dfrac{\alpha_1^2}{4\sqrt{2\pi}\,\alpha_2-\alpha_1^2}-\beta_2$

(iii) $\beta_2\le\dfrac{\alpha_1^2}{4\sqrt{2\pi}\,\alpha_2-\alpha_1^2}.$

Now we are in a position to prove

THEOREM 7.1.
(i) *If* $\rho_1=0$, *then* $\rho_2=0$ *and* $\beta_1=\beta_2$.
(ii) *If* $\rho_2=0$, *then* $\rho_1=0$ *and* $\beta_1=\beta_2$.

PROOF. Suppose that $\rho_1=0$. Then it follows from Theorem 7.1 in [8] and Proposition 7.2 that

$$\beta_1=\beta_2+\gamma_2(0). \tag{7.4}$$

By substituting (7.4) into Proposition 7.3(ii), we see that

$$\beta_2=\beta_1(1+\gamma_2(0)),$$

which, together with (7.4), implies that $\gamma_2(0)=0$ and $\beta_1=\beta_2$ and so $\rho_2=0$.
Conversely we suppose that $\rho_2=0$. It then follows from Proposition 7.3 (ii) that $\beta_1=\beta_2$. Therefore, by Proposition 7.2, we see that

$$C_{\beta_1,\gamma_1}=2\beta_1.$$

By using Theorem 7.1 in [8] again, we find that $\rho_1=0$. (Q. E. D.)
Furthermore we can show

THEOREM 7.2. *If* $\rho_2\neq 0$, *then*

$$\begin{cases}\beta_1^2<\beta_2<\beta_1<1 & \text{if } \beta_2<1\\ \beta_1=1 & \text{if } \beta_2=1\\ \beta_1^2>\beta_2>\beta_1>1 & \text{if } \beta_2>1.\end{cases}$$

PROOF. By Proposition 7.3(ii), we see that if $\beta_2=1$, then $\beta_1=1$. We suppose that $\beta_2<1$. By Proposition 7.3(ii), we see that $\beta_1^2<\beta_2$. Then we make use of Proposition 7.3(ii) again to obtain

$$\gamma_2(0)=\frac{(\beta_2-\beta_1)(\beta_2+\beta_1)}{\beta_1^2-\beta_2}>0, \tag{7.5}$$

which implies that $\beta_2<\beta_1$. We have thus proved that $\beta_1^2<\beta_2<\beta_1<1$. The case $\beta_2>1$ can be proved similarly. (Q. E. D)

REMARK 7.1. In particular, Theorem 7.2 tells us the following

$$\beta_2\begin{Bmatrix}<\\=\\>\end{Bmatrix}1 \textit{ if and only if } \beta_1\begin{Bmatrix}<\\=\\>\end{Bmatrix}1.$$

With the help of the above equalities, we can now calculate the ratio r of the deviations of the two generalized Einstein relations (6.8) and Theorem 6.1(iii) from the classical Einstein relation; by using Propositions 7.2 and 7.3(ii), we have

$$r=\frac{C_{\beta_1,\gamma_1}(2\beta_1)^{-1}}{1+\gamma_2(0)}=\frac{\beta_1}{\beta_2}, \tag{7.6}$$

which concludes from Theorems 7.1, 7.2 and Remark 7.1 that

$$\begin{cases} \textit{if } \rho_1=0, \textit{ then } r=1, \textit{ while} \\ \textit{if } \rho_1\neq 0, \textit{ then } r \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 1 \textit{ according as } \beta_1 \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} 1. \end{cases} \tag{7.7}$$

§ 8. The Kubo noise (2)

This section is devoted to the further study of the Kubo noise $\boldsymbol{I}$ derived from the original process $\boldsymbol{X}$ via equation (4.3). Since $\boldsymbol{X}$ has the canonical representation (4.1) in terms of a normalized Gaussian white noise $\boldsymbol{\xi}$, we are interested in the explicit relation between $\boldsymbol{I}$ and $\boldsymbol{\xi}$ other than (4.10). In fact, from our point of view of KMO-Langevin equations, we will give two different interesting descriptions of the time evolution of the Kubo noise $\boldsymbol{I}$. One is the stochastic difference equation (8.1) below, which says that $\boldsymbol{I}$ and $\boldsymbol{\xi}$ can play the same role as a random force; the other is equation (8.5) of $\boldsymbol{I}$ with random force $\boldsymbol{\xi}$, which is nothing but the first KMO-Langevin equation for $\boldsymbol{I}$.

Let $(\alpha_1, \beta_1, \rho_1)$ (or$(\alpha_1, \beta_1, \gamma_1)$) and $(\alpha_2, \beta_2, \rho_2)$ (or$(\alpha_2, \beta_2, \gamma_2)$) be the first and second KMO-Langevin data associated with σ in the covariance function (2.1) of $\boldsymbol{X}$, respectively.

THEOREM 8.1.

$$\begin{aligned} &\alpha_2\{I(n)-I(n-1)+\beta_1(I(n)+I(n-1))+(\gamma_1 * I)(n)\} \\ &=\alpha_1\{\xi(n)-\xi(n-1)+\beta_2(\xi(n)+\xi(n-1))+(\gamma_2 * \xi)(n)\} \text{ a.s. } (n\in\boldsymbol{Z}). \end{aligned} \tag{8.1}$$

PROOF. From Theorem 5.1 in [8], Theorems 3.1 and 4.2(ii), we see that the outer function h_I of $\boldsymbol{I}$ in (4.6) is expressed in the form

$$h_I(z)=\frac{\alpha_1\alpha_2^{-1}}{\sqrt{2\pi}}\,\frac{\beta_2(1+z)+1-z+(1-z^2)\int_{-1}^{1}\frac{1}{1-tz}\rho_2(dt)}{\beta_1(1+z)+1-z+(1-z^2)\int_{-1}^{1}\frac{1}{1-tz}\rho_1(dt)},\quad z\in\overline{U_1(0)}. \tag{8.2}$$

Furthermore, by noting (4.10) and then taking the same consideration as in the proof of Theorem 5.1, we see that for any $n\in\boldsymbol{Z}$,

$$I(n)-I(n-1)+\beta_1(I(n)+I(n-1))+(\gamma_1*I)(n)$$
$$=\frac{1}{\sqrt{2\pi}}\sum_{m=-\infty}^{\infty}\{\int_{-\pi}^{\pi}e^{-i(n-m)\theta}(1-e^{i\theta}+\beta_1(1+e^{i\theta})$$
$$+(1-e^{2i\theta})\int_{-1}^{1}\frac{1}{1-te^{i\theta}}\rho_1(dt))h_I(e^{i\theta})d\theta\}\xi(m) \qquad \text{a. s.}$$

and

$$\xi(n)-\xi(n-1)+\beta_2(\xi(n)+\xi(n-1))+(\gamma_2*\xi)(n)$$
$$=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\{\int_{-\pi}^{\pi}e^{-i(n-m)\theta}(1-e^{i\theta}+\beta_2(1+e^{i\theta})$$
$$+(1-e^{2i\theta})\int_{-1}^{1}\frac{1}{1-te^{i\theta}}\rho_2(dt))d\theta\}\xi(m) \qquad \text{a. s.}$$

Therefore, by (8.2), we have (8.1). (Q. E. D.)

REMARK 8.1. The stochastic difference equation (8.1) tells us that the Kubo noise $\boldsymbol{I}$ and the white noise $\boldsymbol{\xi}$ can play equivalent roles in various descriptions of the time evolution of discrete time series. Such a situation is also observed in the theory of the autoregressive-moving average model (ARMA model) with rational spectral densities (see [1]). It deserves mention that the spectral density of the Kubo noise $\boldsymbol{I}$ is not always rational.

We will now turn to the second description of the time evolution of the Kubo noise $\boldsymbol{I}$ from the point of view of the first KMO-Langevin equation for $\boldsymbol{I}$. To derive such an equation, there arises the difficulty that the results in [8] cannot be applied directry to the Kubo noise $\boldsymbol{I}$, because it does not possess reflection positivity except the trivial case where $\boldsymbol{I}$ becomes a white noise.

We start with the following new expression of the outer function h_I of $\boldsymbol{I}$.

LEMMA 8.1.

$$h_I(z)=\frac{2\alpha_1\alpha_2^{-1}}{\sqrt{2\pi}}\frac{1}{\beta_1\beta_2^{-1}(1+z)+1-z+F_3(z)} \qquad (z\in U_1(0)),$$

where

$$F_3(z)=\sqrt{\frac{2}{\pi}}\alpha_2^{-1}(1-z^2)\{\frac{\sqrt{\pi}\alpha_2}{2\sqrt{2}}(1-\beta_1)(\beta_2^{-1}-1)$$
$$-\beta_1(1+z)\int_{-1}^{1}\frac{1}{1-tz}\frac{t^2}{1-t^2}\sigma(dt)$$
$$-(1-z)\int_{-1}^{1}\frac{1}{1-tz}\frac{t^2}{1-t^2}\sigma(dt)$$
$$+\int_{-1}^{1}\frac{1}{1-tz}\rho_1(dt)\int_{-1}^{1}\frac{1}{1-tz}\sigma(dt)\}. \tag{8.3}$$

PROOF. By Theorem 4.1 in [8], (2.7) and (4.6), we have, for any $z \in U_1(0)$,

$$\alpha_1/h_I(z)=(\beta_1(1+z)+1-z+(1-z^2)\int_{-1}^{1}\frac{1}{1-tz}\rho_1(dt))\int_{-1}^{1}\frac{1}{1-tz}\sigma(dt). \tag{8.4}$$

Since for any $t\in(-1,1)$ and $z\in U_1(0)$,

$$\frac{1}{1-tz}=\frac{1}{1-t}+(z-1)\frac{1}{1-tz}\,\frac{t}{1-t}=\frac{1}{1+t}+(z+1)\frac{1}{1-tz}\,\frac{t}{1+t},$$

we have

$$\begin{aligned}\frac{1}{1-tz}&=\frac{1}{1-t}+(z-1)\frac{t}{1-t^2}+(z^2-1)\frac{1}{1-tz}\,\frac{t^2}{1-t^2}\\&=\frac{1}{1+t}+(z+1)\frac{t}{1-t^2}+(z^2-1)\frac{1}{1-tz}\,\frac{t^2}{1-t^2}.\end{aligned}$$

Therefore, by noting condition (2.3), we see that for any $z\in U_1(0)$,

$$\begin{aligned}\int_{-1}^{1}\frac{1}{1-tz}\sigma(dt)&=\int_{-1}^{1}\frac{1}{1-t}\sigma(dt)+(z-1)\int_{-1}^{1}\frac{t}{1-t^2}\sigma(dt)\\&\quad+(z^2-1)\int_{-1}^{1}\frac{1}{1-tz}\,\frac{t^2}{1-t^2}\sigma(dt)\end{aligned} \tag{8.5}$$

$$\begin{aligned}\int_{-1}^{1}\frac{1}{1-tz}\sigma(dt)&=\int_{-1}^{1}\frac{1}{1+t}\sigma(dt)+(z+1)\int_{-1}^{1}\frac{t}{1-t^2}\sigma(dt)\\&\quad+(z^2-1)\int_{-1}^{1}\frac{1}{1-tz}\,\frac{t^2}{1-t^2}\sigma(dt).\end{aligned} \tag{8.6}$$

We use (8.5) and (8.6) in the terms $\beta_1(1+z)\int_{-1}^{1}\frac{1}{1-tz}\sigma(dt)$ and $(1-z)\cdot\int_{-1}^{1}\frac{1}{1-tz}\sigma(dt)$ of (8.4), respectively, to find that for any $z\in U_1(0)$,

$$\begin{aligned}(\alpha_1^{-1}h_I(z))^{-1}&=\beta_1\int_{-1}^{1}\frac{1}{1-t}\sigma(dt)(1+z)+\int_{-1}^{1}\frac{1}{1+t}\sigma(dt)(1-z)\\&\quad+(1-z^2)(1-\beta_1)\int_{-1}^{1}\frac{t}{1-t^2}\sigma(dt)\\&\quad-(1+z)^2(1-z)\beta_1\int_{-1}^{1}\frac{1}{1-tz}\,\frac{t^2}{1-t^2}\sigma(dt)\\&\quad-(1+z)(1-z)^2\int_{-1}^{1}\frac{1}{1-tz}\,\frac{t^2}{1-t^2}\sigma(dt)\\&\quad+(1-z^2)\int_{-1}^{1}\frac{1}{1-tz}\rho_1(dt)\int_{-1}^{1}\frac{1}{1-tz}\sigma(dt).\end{aligned} \tag{8.7}$$

On the other hand, it follows from Theorem 3.2 that

$$(8.8)\qquad \begin{cases} \int_{-1}^{1}\frac{1}{1+t}\sigma(dt)=\sqrt{\frac{\pi}{2}}\alpha_2,\ \int_{-1}^{1}\frac{1}{1-t}\sigma(dt)=\sqrt{\frac{\pi}{2}}\alpha_2\beta_2^{-1} \quad \text{and} \\ \int_{-1}^{1}\frac{t}{1-t^2}\sigma(dt)=\frac{\sqrt{\pi}}{2\sqrt{2}}\alpha_2(\beta_2^{-1}-1). \end{cases}$$

Therefore, by (8.7) and (8.8), we have proved Lemma 8.1. (Q.E.D.)

It follows from (2.3) and the estimate (5.11) in [8] that $F_3(e^{i\theta})\in L^1((-\pi,\pi))$, which enables us to define a function γ_3 on $\boldsymbol{Z}$ by

$$(8.9)\qquad \gamma_3=\frac{1}{2\pi}(F_3(e^{i\cdot}))^{\wedge}.$$

For any bounded Borel measure μ on $[-1,1]$, we denote by $M_\mu(n)$ the moment function of μ:

$$(8.10)\qquad M_\mu(n)=\int_{[-1,1]}t^n\mu(dt)\qquad (n\in N^*).$$

Furthermore we define a bounded Borel measure $\check{\mu}$ on $[-1.1]$ by

$$(8.11)\qquad \check{\mu}(dt)=\mu(-dt).$$

The above γ_3 possesses the following properties:

LEMMA 8.2.

(i)

$$\gamma_3(n)=\begin{cases} 0 & \text{for } n\leq -1 \\ (\sqrt{\frac{\pi}{2}}\alpha_2)^{-1}\{\frac{\sqrt{\pi}}{2\sqrt{2}}\alpha_2(1-\beta_1)(\beta_2^{-1}-1)-(1+\beta_1)\sum_{m=0}^{\infty}M_\sigma(2m+2) & \\ \qquad +\gamma_1(0)M_\sigma(0)\} & \text{for } n=0 \\ (\sqrt{\frac{\pi}{2}}\alpha_2)^{-1}\{-\beta_1\sum_{m=0}^{\infty}M_\sigma(m+2)+\sum_{m=0}^{\infty}M_{\check{\sigma}}(m+2) & \\ \qquad +\sum_{m=0}^{1}\gamma_1(1-m)M_\sigma(m)\} & \text{for } n=1 \\ (\sqrt{\frac{\pi}{2}}\alpha_2)^{-1}\{-\frac{\sqrt{\pi}}{2\sqrt{2}}\alpha_2(1-\beta_1)(\beta_2^{-1}-1)+\beta_1(M_\sigma(2) & \\ \qquad -\sum_{m=0}^{\infty}M_\sigma(2m+3))+M_\sigma(2)+\sum_{m=0}^{\infty}M_\sigma(2m+3) & \\ \qquad +\sum_{m=0}^{2}\gamma_1(2-m)M_\sigma(m)\} & \text{for } n=2 \\ (\sqrt{\frac{\pi}{2}}\alpha_2)^{-1}\{\beta_1(M_\sigma(n-1)+M_\sigma(n))-(M_\sigma(n-1)-M_\sigma(n)) & \\ \qquad +\sum_{m=0}^{n}\gamma_1(n-m)M_\sigma(m)\} & \text{for } n\geq 3 \end{cases}$$

(ii) $\gamma_3\in l^1(\boldsymbol{Z})$

(iii) $\sum_{n=0}^{\infty}\gamma_3(n)=0$

(iv) $\sum_{n=0}^{\infty}(-1)^n\gamma_3(n)=0.$

PROOF. If follows from (8.3) that for any $n\in\boldsymbol{Z}$,

$$\sqrt{\frac{\pi}{2}}\alpha_2\gamma_3(n)=\mathrm{I}_n+\mathrm{II}_n+\mathrm{III}_n+\mathrm{IV}_n, \tag{8.12}$$

where

$$\mathrm{I}_n=\frac{\sqrt{\pi}}{2\sqrt{2}}\alpha_2(1-\beta_1)(\beta_2^{-1}-1)\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-in\theta}(1-e^{2i\theta})\,d\theta$$

$$\mathrm{II}_n=-\beta_1\int_{-1}^{1}\Big(\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-in\theta}\frac{(1+e^{i\theta})^2(1-e^{i\theta})}{1-te^{i\theta}}d\theta\Big)\frac{t^2}{1-t^2}\sigma(dt)$$

$$\mathrm{III}_n=-\int_{-1}^{1}\Big(\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-in\theta}\frac{(1+e^{i\theta})(1-e^{i\theta})^2}{1-te^{i\theta}}d\theta\Big)\frac{t^2}{1-t^2}\sigma(dt)$$

$$\mathrm{IV}_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-in\theta}\Big((1-e^{2i\theta})\int_{-1}^{1}\frac{1}{1-te^{i\theta}}\rho_1(d\theta)\Big)\int_{-1}^{1}\frac{1}{1-te^{i\theta}}\sigma(dt)\Big)d\theta.$$

It is easy to see that

$$\mathrm{I}_n=\frac{\sqrt{\pi}}{2\sqrt{2}}\alpha_2(1-\beta_1)(\beta_2^{-1}-1)(\delta_{n,0}-\delta_{n,2}). \tag{8.13}$$

Since for any $t\in(-1,1)$ and $n\in\boldsymbol{Z}$,

$$\int_{-\pi}^{\pi}e^{-in\theta}\frac{(1+e^{i\theta})^2(1-e^{i\theta})}{1-te^{i\theta}}d\theta
=\begin{cases}0 & \text{for } n\le -1\\ 1 & \text{for } n=0\\ 1+t & \text{for } n=1\\ -1+t+t^2 & \text{for } n=2\\ -t^{n-3}(1+t)^2(1-t) & \text{for } n\ge 3,\end{cases}$$

we have

$$\mathrm{II}_n=\begin{cases}0 & \text{for } n\le -1\\ -\beta_1\int_{-1}^{1}\frac{t^2}{1-t^2}\sigma(dt) & \text{for } n=0\\ -\beta_1\int_{-1}^{1}\frac{t^2}{1-t}\sigma(dt) & \text{for } n=1\\ \beta_1\int_{-1}^{1}\frac{(1-t-t^2)t^2}{1-t^2}\sigma(dt) & \text{for } n=2\\ \beta_1\int_{-1}^{1}t^{n-1}(1+t)\sigma(dt) & \text{for } n\ge 3.\end{cases} \tag{8.14}$$

Similarly, we have

$$(8.15)\qquad \mathrm{III}_n=\begin{cases} 0 & \text{for } n\le -1\\ -\int_{-1}^{1}\frac{t^2}{1-t^2}\sigma(dt) & \text{for } n=0\\ \int_{-1}^{1}\frac{t^2}{1+t}\sigma(dt) & \text{for } n=1\\ \int_{-1}^{1}\frac{(1+t-t^2)t^2}{1-t^2}\sigma(dt) & \text{for } n=2\\ -\int_{-1}^{1}t^{n-1}(1-t)\sigma(dt) & \text{for } n\ge 3.\end{cases}$$

Since

$$(1-e^{2i\theta})\int_{-1}^{1}\frac{1}{1-te^{i\theta}}\rho_1(dt)=2\pi\tilde{\gamma}_1(\theta)$$

and

$$\left(\int_{-1}^{1}\frac{1}{1-te^{i\cdot}}\sigma(dt)\right)^{\wedge}(n)=\begin{cases} 0 & \text{for } n\le -1\\ 2\pi\int_{-1}^{1}t^n\sigma(dt) & \text{for } n\ge 0,\end{cases}$$

we see from (6.2) that

$$(8.16)\qquad \mathrm{IV}_n=\begin{cases} 0 & \text{for } n\le -1\\ \sum_{m=0}^{n}\gamma_1(n-m)\int_{-1}^{1}t^m\sigma(dt) & \text{for } n\ge 0.\end{cases}$$

Therefore, (i) can be seen from the above (8.12)~(8.16).

By noting (2.3) and the fact that $\gamma_1\in l^1(\boldsymbol{Z})$, we see that (ii) follows from (i). By virtue of (ii), we can take the Fourier inverse transform of (8.9) in the L^1-sense to obtain

$$2\pi\tilde{\gamma}_3(\theta)=F_3(e^{i\theta})\qquad \text{for any } \theta\in[-\pi,\pi].$$

By substituting $\theta=0$ and $\theta=-\pi$ into the above and then noting (8.3), we have (iii) and (iv), respectively. (Q.E.D.)

By virtue of Lemmas 8.1 and 8.2 (ii), we can prove the following theorem, in a similar manner to Theorem 5.1, by using (4.7) and (4.10).

THEOREM 8.2.

$$(8.17)\qquad I(n)-I(n-1)=-\beta_1\beta_2^{-1}(I(n)+I(n-1))-(\gamma_3*I)(n)+2\alpha_1\alpha_2^{-1}\xi(n)\quad \text{a.s. } (n\in\boldsymbol{Z}).$$

In view of Definition 6.1 in [8], we give the following

DEFINITION 8.1. We call equation (8.17) the **first KMO-Langevin equation for** the Kubo noise $\boldsymbol{I}$, which describes the time evolution of $\boldsymbol{I}$ associated with the original process $\boldsymbol{X}$.

The final topic in this section is an interesting stochastic difference equation obtained by exchanging the role of $\boldsymbol{I}$ and $\boldsymbol{\xi}$ in (8.17): that is, the Kubo noise $\boldsymbol{I}$ is taken as a random force and the time evolution of the white noise $\boldsymbol{\xi}$ is described in the style of the second KMO-Langevin equation.

By Theorem 4.2 in [8], we know that there exists a bounded Borel measure ν on $[-1,1]$ such that

$$\nu(\{-1,1\})=0 \tag{8.18}$$

$$\int_{-1}^{1}\left(\frac{1}{1-t}+\frac{1}{1+t}\right)\nu(dt)<\infty \tag{8.19}$$

$$h(z)=\frac{1}{2\pi}\int_{-1}^{1}\frac{1}{1-tz}\nu(dt) \qquad (z\in U_1(0)). \tag{8.20}$$

We note that (8.18)~(8.20) for the outer function h of $\boldsymbol{X}$ correspond to (2.2), (2.3) and (2.6) for the mobility function $[R]$ of $\boldsymbol{X}$, respectively. In addition, there exists a bijective correspondence between σ and ν via the following relation:

$$\sigma(dt)=\frac{1}{2\pi}(\int_{-1}^{1}\frac{1}{1-ts}\nu(ds))\,\nu(dt). \tag{8.21}$$

LEMMA 8.3.

$$(\sqrt{2\pi}\,h_I(z))^{-1}=\frac{2\alpha_1^{-1}\alpha_2}{\beta_1^{-1}\beta_2(1+z)+1-z+F_4(z)} \qquad (z\in U_1(0)),$$

where

$$\begin{aligned}F_4(z)=\sqrt{\frac{2}{\pi}}\alpha_1^{-1}(1-z^2)\{&\frac{\sqrt{\pi}}{2\sqrt{2}}\alpha_1(\beta_1^{-1}-1)(1-\beta_2)\\ &-\beta_2(1+z)\int_{-1}^{1}\frac{1}{1-tz}\frac{t^2}{1-t^2}\nu(dt)\\ &-(1-z)\int_{-1}^{1}\frac{1}{1-tz}\frac{t^2}{1-t^2}\nu(dt)\\ &+\int_{-1}^{1}\frac{1}{1-tz}\rho_2(dt)\int_{-1}^{1}\frac{1}{1-tz}\nu(dt)\}.\end{aligned} \tag{8.22}$$

PROOF. With the proof of Lemma 8.1 in mind, we need the following two facts:

(i) $(\sqrt{2\pi}\,h_I(z))^{-1}$

$$=\frac{\sqrt{2\pi}\,\alpha_2}{(\beta_2(1+z)+1-z+(1-z^2)\int_{-1}^{1}\frac{1}{1-tz}\rho_2(dt))\int_{-1}^{1}\frac{1}{1-tz}\nu(dt)},$$

$$z\in U_1(0)$$

$$\text{(ii)}\quad \begin{cases} \int_{-1}^{1}\frac{1}{1+t}\nu(dt)=\sqrt{\frac{\pi}{2}}\alpha_1,\ \int_{-1}^{1}\frac{1}{1-t}\nu(dt)=\sqrt{\frac{\pi}{2}}\alpha_1\beta_1^{-1} \quad \text{and} \\ \int_{-1}^{1}\frac{t}{1-t^2}\nu(dt)=\frac{\sqrt{\pi}}{2\sqrt{2}}\alpha_1(\beta_1^{-1}-1). \end{cases}$$

(i) corresponds to (8.4) and follows from Theorem 2.1, (4.6) and (8.18). By substituting $\theta=0$ and $-\pi$ into Corollary 4.1 (i) in [8], we get (ii), which corresponds to (8.8).

Then, we can take the same procedure as in the proof of Lemma 8.1 to obtain Lemma 8.3. (Q. E. D.)

Recalling that the explicit form of γ_3 in (8.9) was derived in Lemma 8.2 from F_3 in (8.3), we define, by using F_4 in (8.22), a function γ_4 on $\boldsymbol{Z}$ by

$$(8.23)\qquad \gamma_4=\frac{1}{2\pi}(F_4(e^{i\cdot}))^{\wedge}$$

to get

LEMMA 8.4.

(i)

$$\gamma_4(n)=\begin{cases} 0 & \text{for } n\le -1 \\ (\sqrt{\frac{\pi}{2}}\alpha_1)^{-1}\{\frac{\sqrt{\pi}}{2\sqrt{2}}\alpha_1(\beta_1^{-1}-1)(1-\beta_2)-(1+\beta_2)\sum_{m=0}^{\infty}M_\nu(2n+2) & \\ \qquad +\gamma_2(0)M_\nu(0)\} & \text{for } n=0 \\ (\sqrt{\frac{\pi}{2}}\alpha_1)^{-1}\{-\beta_2\sum_{n=0}^{\infty}M_\nu(n+2)+\sum_{n=0}^{\infty}M_\nu(n+2) & \\ \qquad +\sum_{m=0}^{1}\gamma_2(1-m)M\nu(m)\} & \text{for } n=1 \\ (\sqrt{\frac{\pi}{2}}\alpha_1)^{-1}\{-\frac{\sqrt{\pi}}{2\sqrt{2}}\alpha_1(\beta_1^{-1}-1)(1-\beta_2)+\beta_2(M_\nu(2) & \\ \qquad -\sum_{m=0}^{\infty}M_\nu(2m+3))+M_\nu(2)+\sum_{m=0}^{\infty}M_\nu(2m+3) & \\ \qquad +\sum_{m=0}^{2}\gamma_2(2-m)M_\nu(m)\} & \text{for } n=2 \\ (\sqrt{\frac{\pi}{2}}\alpha_1)^{-1}\{\beta_2(M_\nu(n-1)+M_\nu(n))-(M_\nu(n-1)-M_\nu(n)) & \\ \qquad +\sum_{m=0}^{n}\gamma_1(n-m)M_\nu(m)\} & \text{for } n\ge 3 \end{cases}$$

(ii) $\gamma_4 \in l^1(\boldsymbol{Z})$

(iii) $\sum_{n=0}^{\infty} \gamma_4(n) = 0$

(iv) $\sum_{n=0}^{\infty} (-1)^n \gamma_4(n) = 0.$

With the help of the above lemmas, we show

THEOREM 8.3.

(8.24) $\xi(n) - \xi(n-1) = -\beta_1^{-1}\beta_2(\xi(n) + \xi(n-1)) - (\gamma_4 * \xi)(n) + 2\alpha_1^{-1}\alpha_2 I(n)$ a. s. $(n \in \boldsymbol{Z})$.

PROOF. By Lemma 8.4(ii), we see that the following two random series are absolutely convergent (a. s.):

$$\xi(n) - \xi(n-1) + \beta_1^{-1}\beta_2(\xi(n) + \xi(n-1)) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_{-\pi}^{\pi} e^{-i(n-m)\theta} \{1 - e^{i\theta} + \beta_1^{-1}\beta_2(1 + e^{i\theta})\} d\theta \xi(m)$$

and

$$(\gamma_4 * \xi)(n) = \sum_{m=-\infty}^{\infty} \int_{-\pi}^{\pi} e^{-i(n-m)\theta} \tilde{\gamma}_4(\theta) d\theta \xi(m).$$

Since $\tilde{\gamma}_4(\theta) = \frac{1}{2\pi} F_4(e^{i\theta})$, it follows from Lemma 8.3 that

$$\begin{aligned} &\xi(n) - \xi(n-1) + \beta_1^{-1}\beta_2(\xi(n) + \xi(n-1)) + (\gamma_4 * \xi)(n) \\ &= \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_{-\pi}^{\pi} e^{-i(n-m)\theta} \{1 - e^{i\theta} + \beta_1^{-1}\beta_2(1 + e^{i\theta}) + F_4(e^{i\theta})\} d\theta \xi(m) \\ &= 2\alpha_1^{-1}\alpha_2 \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} \int_{-\pi}^{\pi} e^{-i(n-m)\theta} h_I(e^{i\theta}) d\theta \xi(m) \qquad \text{a. s.} \end{aligned}$$

Therefore, by (4.7) and (4.10), we have (8.24). (Q. E. D.)

REMARK 8.2. Similarly to equation (8.1), we note that the white noise $\boldsymbol{\xi}$ and the Kubo noise $\boldsymbol{I}$ associated with the process $\boldsymbol{X}$ can play an exchangeable role through equations (8.17) and (8.24).

REMARK 8.3. We have seen in Example 4.1 that $\boldsymbol{X}_0$ is nothing but one realization of the white noise $\boldsymbol{\xi}$ and its associated Kubo noise $\boldsymbol{I}_0$ becomes $\sqrt{2\pi}\,\boldsymbol{\xi}$. Since the second KMO-Langevin data $(\alpha_0^{(2)}, \beta_0^{(2)}, \rho_0^{(2)})$ associated with $\boldsymbol{X}_0$ becomes $(\sqrt{2/\pi}, 1, 0)$, we can see from Theorem 5.1 that $\boldsymbol{\xi}$ is governed by the following first KMO-Langevin equation:

(8.25) $\xi(n) - \xi(n-1) = -(\xi(n) + \xi(n-1)) + 2\xi(n)$ a. s. $(n \in \boldsymbol{Z})$.

We note that equation (8.25) coincides with the second KMO-Langevin equations associated with the white noise $\boldsymbol{\xi}$, but equation (8.24) is not the second KMO-Langevin equation associated with $\boldsymbol{\xi}$, because the noise process $\boldsymbol{I}$ is not the Kubo noise associated with $\boldsymbol{\xi}$.

§ 9. Generalized fluctuation-dissipation theorems (2)

We stated in § 6 the generalized fluctuation-dissipation theorems ((6.5), (6.6) and (6.8) with (6.9)) based on the first KMO-Langevin equation (6.1) of the original process $\boldsymbol{X}$, and in § 8 we derived equation (8.17) of the same type describing the time evolution of the Kubo noise $\boldsymbol{I}$ associated with $\boldsymbol{X}$. We will in this section obtain analogous results for $\boldsymbol{I}$ based on (8.17).

By (3.8) and Theorem 4.2(iii), we have

(9.1) $\quad R_I \in l^1(\boldsymbol{Z}).$

So, as in the definition (6.4) of the diffusion constant D_X of $\boldsymbol{X}$, we can define a diffusion constant D_I of $\boldsymbol{I}$ by

(9.2) $$D_I = \lim_{N\to\infty} \frac{1}{2N} E\Big(\big(\sum_{n=0}^{N} I(n)\big)^2\Big)$$

to get

(9.3) $$D_I = \sum_{n=1}^{\infty} R_I(n) + \frac{R_I(0)}{2}.$$

THEOREM 9.1. (i) *For any* $\theta \in (-\pi, \pi)$

$$\frac{1}{\beta_1\beta_2^{-1}(1+e^{i\theta})+1-e^{i\theta}+2\pi\tilde{\gamma}_3(\theta)} = \frac{R_I(e^{i\theta})}{2\lim_{\tau\downarrow\pi} h_I(e^{i\tau})}.$$

(ii) $$\frac{(2\alpha_1\alpha_2^{-1})^2}{2} = R_I(0)\, C_{\beta_1\beta_2^{-1},\gamma_3},$$

where

(9.4) $$C_{\beta_1\beta_2^{-1},\gamma_3} = \pi\Big(\int_{-\pi}^{\pi} |\beta_1\beta_2^{-1}(1+e^{i\theta})+1-e^{i\theta}+2\pi\tilde{\gamma}_3(\theta)|^{-2}d\theta\Big)^{-1}.$$

(iii) $$D_I = \frac{R_I(0)}{2\beta_1\beta_2^{-1}} C_I,$$

where

(9.5) $$C_I = \frac{2\sqrt{\beta_2(\beta_2+\gamma_2(0))(1+\gamma_2(0))}}{\int_{-1}^{1}(1+t)\rho_2(dt)+\beta_2\big(2+\int_{-1}^{1}(1-t)\rho_2(dt)\big)+\frac{1}{2}\int_{-1}^{1}\int_{-1}^{1}\frac{(t-s)^2}{1-ts}\rho_2(dt)\rho_2(ds)}.$$

PROOF. Since Lemma 8.1 holds for $z=e^{i\theta}$ and $\lim_{\theta\downarrow-\pi}F_3(e^{i\theta})=0$, we have

$$(9.6)\qquad 2\lim_{\theta\downarrow-\pi}h_I(e^{i\theta})=\frac{2\alpha_1\alpha_2^{-1}}{\sqrt{2\pi}}.$$

By combining (9.6) with Lemma 8.1 again, we have (i). The proof of (ii) is similar to that of (6.6) (given in [8], Theorem 7.1(ii)).

We proceed to the proof of (iii). By Theorem 4.2(iii), we have

$$(9.7)\qquad R_I(0)=2c_6\{\int_{-1}^{1}(1+t)\rho_2(dt)+\beta_2(2+\int_{-1}^{1}(1-t)\rho_2(dt)) +\frac{1}{2}\int_{-1}^{1}\int_{-1}^{1}\frac{(t-s)^2}{1-ts}\rho_2(dt)\rho_2(ds)\}$$

$$(9.8)\qquad R_I(n)=c_6\{(1-\beta_2)\gamma_2(n+1)-\sum_{l=1}^{\infty}\gamma_2(l)\gamma_2(n+l)\}\qquad(n\in\boldsymbol{N}),$$

where $c_6=\sqrt{2\pi}(\alpha_2(1+\beta_2+\gamma_2(0)))^{-1}$.

By (3.9), we see that

$$(9.9)\qquad \sum_{n=1}^{\infty}R_I(n)=c_6\{(\beta_2-1)\int_{-1}^{1}(1+t)\rho_2(dt)+\sum_{l=1}^{\infty}\gamma_2(l)(\sum_{n=0}^{l}\gamma_2(n))\}.$$

We claim

$$(9.10)\qquad \sum_{l=1}^{\infty}\gamma_2(l)(\sum_{n=0}^{l}\gamma_2(n))=-\frac{1}{2}\int_{-1}^{1}\int_{-1}^{1}\frac{(t-s)^2}{1-ts}\rho_2(dt)\rho_2(ds).$$

By (3.7) and (3.9),

the left hand side of (9.10)

$$=-\gamma_2(0)(\gamma_2(0)+\gamma_2(1))+\sum_{n=2}^{\infty}\gamma_2(n)(\sum_{l=n}^{\infty}\gamma_2(l))$$
$$=-\gamma_2(0)\int_{-1}^{1}(1+t)\rho_2(dt)-\sum_{n=2}^{\infty}\int_{-1}^{1}t^{n-2}(t^2-1)\rho_2(dt)\int_{-1}^{1}s^{n-2}(1+s)\rho_2(ds)$$
$$=-\gamma_2(0)\int_{-1}^{1}(1+t)\rho_2(dt)-\int_{-1}^{1}\int_{-1}^{1}\frac{(t^2-1)(1+s)}{1-ts}\rho_2(dt)\rho_2(ds)$$
$$=-\frac{1}{2}\int_{-1}^{1}\int_{-1}^{1}\{(1+t)+(1+s)+\frac{(t^2-1)(1+s)+(s^2-1)(1+t)}{1-ts}\}\rho_2(dt)\rho_2(ds)$$

= the right hand side of (9.10).

Therefore, by substituting (9.7) and (9.9) with (9.10) into (9.3),

$$D_I=2c_6\beta_2(1+\gamma_2(0)),$$

which, with Proposition 7.3(ii) and (9.7), yields (iii). (Q.E.D.)

DEFINITION 9.1. We call (i), (ii) and (iii) in Theorem 9.1 the **generalized first fluctuation-dissipation theorem**, the **generalized second**

fluctuation-dissipation theorem and the **generalized Einstein relation for** the Kubo noise $\boldsymbol{I}$ **based on equation** (8.17), respectively.

The final problem is to estimate the constant C_I in (9.5) and see how large the deviation from the classical Einstein relation is.

THEOREM 9.2.

(i) *The case where* $\rho_2=0$: $C_I=1$.

(ii) *The case where* $\rho_2\neq 0$:

$$\begin{cases} 0<C_I<\beta_2^{-1/2} & \text{if } 0<\beta_2<1 \\ 0<C_I\leq 1 & \text{if } \beta_2=1 \\ 0<C_I<\beta_2 & \text{if } \beta_2>1. \end{cases}$$

(iii) $0<C_I\leq 1 \qquad$ if $(1-\beta_2)\int_{-1}^{1} t\rho_2(dt)\geq 0$.

PROOF. (i) is an immediate consequence of (9.5). Since, by (9.5),

$$0<C_I\leq\frac{2\sqrt{\beta_2(\beta_2+\gamma_2(0))(1+\gamma_2(0))}}{\int_{-1}^{1}(1+t)\rho_2(dt)+\beta_2(2+\int_{-1}^{1}(1-t)\rho_2(dt))},$$

we see that (ii) holds.

For the proof of (iii), we put

$e_I=$ (denominator of the right hand side of $(9.5))^2$
$\qquad -$(numerator of the right hand side of $(9.5))^2$

and

$$c_7=\int_{-1}^{1}(1+t)\rho_2(dt)+\frac{1}{2}\int_{-1}^{1}\int_{-1}^{1}\frac{(t-s)^2}{1-ts}\rho_2(dt)\rho_2(ds).$$

Then we have

$$\begin{aligned}(9.11)\qquad e_I&=\{(2+\int_{-1}^{1}(1-t)\rho_2(dt))^2-4(1+\gamma_2(0)\}\beta_2^2\\ &\qquad+\{2(2+\int_{-1}^{1}(1-t)\rho_2(dt))c_7-4\gamma_2(0)(1+\gamma_2(0)\}\beta_2+c_7^2\\ &=\{(\int_{-1}^{1}(1+t)\rho_2(dt))^2-4(1+\gamma_2(0))\int_{-1}^{1}t\rho_2(dt)\}\beta_2^2\\ &\qquad+\{-2\int_{-1}^{1}(1+t)\rho_2(dt))c_7+4(1+\gamma_2(0))(c_7-\gamma_2(0)\}\beta_2+c_7^2\\ &=(\int_{-1}^{1}(1+t)\rho_2(dt))\beta_2-c_7)^2+4(1+\gamma_2(0))\beta_2\{(1-\beta_2)\int_{-1}^{1}t\rho_2(dt)\\ &\qquad+\frac{1}{2}\int_{-1}^{1}\int_{-1}^{1}\frac{(t-s)^2}{1-ts}\rho_2(dt)\rho_2(ds)\}.\end{aligned}$$

Hence, if $(1-\beta_2)\int_{-1}^{1} t\rho_2(dt) \geq 0$, then $e_I \geq 0$, and so $0 < C_I \leq 1$. (Q. E. D.)

REMARK 9.1. Let us consider the following simple example:

(9.12) $\rho(\mathrm{dt}) = \rho\delta_{\{\frac{1}{2}\}}(\mathrm{dt})$ with $\rho > 0$.

Then it follows from (9.11) that

$$e_I = \frac{\rho}{4}(\beta_2 - 1)((\rho - 8)\beta_2 - 9\rho),$$

which leads us to state

(a) the case where $\rho \leq 8$:

$$\begin{cases} 0 < C_I < 1 \\ C_I = 1 \\ C_I > 1 \end{cases} \quad \text{according as} \quad \begin{array}{l} 0 < \beta_2 < 1 \\ \beta_2 = 1 \\ \beta_2 > 1. \end{array}$$

(b) the case where $\rho > 8$:

$$\begin{cases} 0 < C_I < 1 \\ C_I = 1 \\ C_I > 1 \end{cases} \quad \text{according as} \quad \begin{array}{l} \beta_2 \in (0, 1) \cup (\frac{9\rho}{\rho - 8}, \infty) \\ \beta_2 = 1 \text{ or } \frac{9\rho}{\rho - 8} \\ \beta_2 \in (1, \frac{9\rho}{\rho - 8}). \end{array}$$

REMARK 9.2. Consider the analogous example:

(9.13) $\rho(dt) = \rho\delta_{\{-\frac{1}{2}\}}$ with $\rho > 0$.

Then

$$e_I = \frac{\rho(8 + 9\rho)}{4}(\beta_2 - 1)(\beta_2 - \frac{\rho}{8 + 9\rho})$$

and so

$$\begin{cases} 0 < C_I < 1 \\ C_I = 1 \\ C_I > 1 \end{cases} \quad \text{according as} \quad \begin{array}{l} \beta_2 \in (0, \frac{\rho}{8 + 9\rho}) \cup (1, \infty) \\ \beta_2 = \frac{\rho}{8 + 9\rho} \text{ or } 1 \\ \beta_2 \in (\frac{\rho}{8 + 9\rho}, 1). \end{array}$$

APPENDIX.

Let ν be a bounded Borel measure on $[0, \infty)$ such that

(A.1) $\quad \nu(\{0\})=0$ and $0<\int_0^\infty \lambda^{-1}\nu(d\lambda)<\infty.$

Then we define a bounded Borel measure σ on $[0,\infty)$ by

(A.2) $\quad \sigma(d\lambda)=\frac{1}{2\pi}(\int_0^\infty \frac{1}{\lambda+\lambda'}\nu(d\lambda'))\nu(d\lambda),$

which satisfies (c.f. Lemma 2.6 in [6])

(A.3) $\quad \sigma(\{0\})=0$ and $0<\int_0^\infty (\lambda+\lambda^{-1})\sigma(d\lambda)<\infty.$

Furthermore, we define a non-negative L^1-function Δ by

(A.4) $\quad \Delta(\xi)=\frac{1}{\pi}\int_0^\infty \frac{\lambda}{\lambda^2+\xi^2}\sigma(d\lambda),\ \xi\in \boldsymbol{R}.$

Noting by Theorem 2.1 in [3] that

(A.5) $\quad \frac{\log \Delta(\xi)}{1+\xi^2}\in L^1(\boldsymbol{R}),$

we get the outer function h of Δ defined by

(A.6) $\quad h(\zeta)=\exp(\frac{1}{2\pi i}\int_{\boldsymbol{R}} \frac{1+\lambda\zeta}{\lambda-\zeta}\frac{\log \Delta(\lambda)}{1+\lambda^2}d\lambda),\ \zeta\in \boldsymbol{C}^+.$

This outer function h can be expressed in a simple form in terms of the original measure ν.

THEOREM A.

$$h(\zeta)=\frac{1}{2\pi}\int_0^\infty \frac{1}{\lambda-i\zeta}\nu(d\lambda) \qquad (\zeta\in \boldsymbol{C}^+).$$

PROOF. Put

(A.7) $\quad F(\zeta)=\frac{1}{2\pi}\int_0^\infty \frac{1}{\lambda-i\zeta}\nu(d\lambda),\ \zeta\in \boldsymbol{C}^+\cup \boldsymbol{R}.$

(STEP 1) We show

(i) $F\in \mathcal{O}(\boldsymbol{C}^+)\cap C(\boldsymbol{C}^+\cup \boldsymbol{R})$

(ii) $\overline{F(\xi)}=F(-\xi) \qquad (\xi\in \boldsymbol{R})$

(iii) $|F(\xi)|^2=\Delta(\xi) \qquad (\xi\in \boldsymbol{R})$

(iv) $\mathrm{Re}\, F(\zeta)>0 \qquad (\zeta\in \boldsymbol{C}^+\cup \boldsymbol{R})$

(v) $|\mathrm{Log}\, F(Re^{i\theta})|\leq|\log\frac{1}{2\pi}\int_0^\infty \lambda^{-1}\nu(d\lambda)|+|\log\frac{1}{2\pi}\int_0^\infty \frac{R}{(\lambda+R)^2}\nu(d\lambda)|$
$+\pi \qquad (R>0,\ \theta\in(0,\pi)).$

(i) and (ii) are clear. (iii) follows from the following computation: by (A. 2), (A. 4) and (A. 7),

$$
\begin{aligned}
|F(\xi)|^2 &= \frac{1}{(2\pi)^2}\int_0^\infty\int_0^\infty \frac{\lambda\lambda'+\xi^2}{(\lambda^2+\xi^2)(\lambda'^2+\xi^2)}\nu(d\lambda)\nu(d\lambda') \\
&= \frac{1}{(2\pi)^2}\int_0^\infty\int_0^\infty\left(\frac{\lambda}{\lambda^2+\xi^2}+\frac{\lambda'}{\lambda'^2+\xi^2}\right)\frac{1}{\lambda+\lambda'}\nu(d\lambda)\nu(d\lambda') \\
&= \frac{1}{\pi}\int_0^\infty \frac{\lambda}{\lambda^2+\xi^2}\sigma(d\lambda)=\Delta(\xi), \quad \xi\in\boldsymbol{R}.
\end{aligned}
$$

Since

$$
\mathrm{Re}(F(\xi+i\eta))=\frac{1}{2\pi}\int_0^\infty\frac{\lambda+\eta}{(\lambda+\eta)^2+\xi^2}\nu(d\lambda)>0 \qquad \text{for all } \eta>0,
$$

we have (iv) by noting (A. 1).

For the proof of (v), for any fixed $R\in(0,\infty)$ and $\theta\in(0,\pi)$, we note that

$$
|\mathrm{Log}\, F(Re^{i\theta})|\le|\log|F(Re^{i\theta})||+\pi.
$$

Furthermore, since $0\le\sin\theta\le1$, we see from (A. 7) that

$$
\begin{aligned}
|F(Re^{i\theta})| &\le \frac{1}{2\pi}\int_0^\infty\frac{1}{\sqrt{(\lambda+R\sin\theta)^2+(R\cos\theta)^2}}\nu(d\lambda) \\
&\le \frac{1}{2\pi}\int_0^\infty \lambda^{-1}\nu(d\lambda)
\end{aligned}
$$

and

$$
\begin{aligned}
&|F(Re^{i\theta})|^2 \\
&= \frac{1}{(2\pi)^2}\int_0^\infty\int_0^\infty\frac{\lambda\lambda'+R(\lambda+\lambda')\sin\theta+R^2}{(\lambda^2+2\lambda R\sin\theta+R^2)(\lambda'^2+2\lambda' R\sin\theta+R^2)}\nu(d\lambda)\nu(d\lambda') \\
&\ge \frac{1}{(2\pi)^2}\int_0^\infty\int_0^\infty\frac{R^2}{(\lambda+R)^2(\lambda'+R)^2}\nu(d\lambda)\nu(d\lambda') \\
&= \left(\frac{1}{2\pi}\int_0^\infty\frac{R}{(\lambda+R)^2}\nu(d\lambda)\right)^2.
\end{aligned}
$$

Therefore, we have (v).

(STEP 2) We claim that for any $\zeta\in\boldsymbol{C}^+$,

$$
\begin{aligned}
&\frac{1}{2\pi i}\int_{\boldsymbol{R}}\frac{1+\lambda\zeta}{\lambda-\zeta}\,\frac{\log\Delta(\lambda)}{1+\lambda^2}d\lambda \\
&= \frac{1}{2\pi i}\int_{\boldsymbol{R}}\frac{1+\lambda\zeta}{\lambda-\zeta}\,\frac{\mathrm{Log}\, F(\lambda)}{1+\lambda^2}d\lambda+\frac{1}{2\pi i}\int_{\boldsymbol{R}}\frac{1+\lambda\zeta}{\lambda-\zeta}\,\frac{\mathrm{Log}\, F(-\lambda)}{1+\lambda^2}d\lambda.
\end{aligned}
$$

By (ii) and (iii) in Step 1,

$$\log \Delta(\lambda)=\log|F(\lambda)|+\log|F(-\lambda)| \qquad (\lambda\in \boldsymbol{R}).$$

On the other hand, by (ii) in Step 1,

$$\begin{aligned}\operatorname{Log} F(\lambda)&=\log|F(\lambda)|+i \operatorname{Arg} F(\lambda)\\ \operatorname{Log} F(-\lambda)&=\log|F(-\lambda)|-i \operatorname{Arg} F(\lambda) \qquad (\lambda\in \boldsymbol{R}).\end{aligned}$$

Hence, we see that

$$\log \Delta(\lambda)=\operatorname{Log} F(\lambda)+\operatorname{Log} F(-\lambda) \qquad (\lambda\in \boldsymbol{R}),$$

which yields the desired equality.

(STEP 3) We claim that for any $\xi\in \boldsymbol{C}^{+}$,

(i) $\displaystyle \frac{1}{2\pi i}\int_{\boldsymbol{R}}\frac{1+\lambda\xi}{\lambda-\xi}\,\frac{\operatorname{Log} F(\lambda)}{1+\lambda^{2}}d\lambda=\operatorname{Log} F(\xi)-\frac{\operatorname{Log} F(i)}{2}.$

(ii) $\displaystyle \frac{1}{2\pi i}\int_{\boldsymbol{R}}\frac{1+\lambda\xi}{\lambda-\xi}\,\frac{\operatorname{Log} F(-\lambda)}{1+\lambda^{2}}d\lambda=\frac{\operatorname{Log} F(i)}{2}.$

Since $\dfrac{1+z\xi}{z-\xi}\,\dfrac{\operatorname{Log} F(z)}{1+z^{2}}$ has two simple poles ξ and i on $\boldsymbol{C}^{+}$, we see that for any $R\in(|\xi|+1, \infty)$,

$$\begin{aligned}&\frac{1}{2\pi i}\int_{-R}^{R}\frac{1+\lambda\xi}{\lambda-\xi}\,\frac{\operatorname{Log} F(\lambda)}{1+\lambda^{2}}d\lambda\\ &=-\frac{1}{2\pi}\int_{0}^{\pi}\frac{1+Re^{i\theta}\xi}{Re^{i\theta}-\xi}\,\frac{\operatorname{Log} F(Re^{i\theta})}{1+(Re^{i\theta})^{2}}Re^{i\theta}d\lambda+\operatorname{Log} F(z)-\frac{\operatorname{Log} F(i)}{2}.\end{aligned}$$

Since

$$\left|\frac{1+Re^{i\theta}\xi}{Re^{i\theta}-\xi}\,\frac{\operatorname{Log} F(Re^{i\theta})}{1+(Re^{i\theta})^{2}}Re^{i\theta}\right|\leq\frac{1+R|\xi|}{R-|\xi|}\,\frac{|\operatorname{Log} F(Re^{i\theta})|}{R^{2}-1}R,$$

we use (v) in Step 1 and apply Lebesgue's convergence theorem to get

$$\lim_{R\to\infty}\int_{0}^{\pi}\frac{1+Re^{i\theta}\xi}{Re^{i\theta}-\xi}\,\frac{\operatorname{Log} F(Re^{i\theta})}{1+(Re^{i\theta})^{2}}Re^{i\theta}d\theta=0.$$

Therefore, we have (i).

For the proof of (ii), we see that $\dfrac{1+z\xi}{z-\xi}\,\dfrac{\operatorname{Log} F(-z)}{1+z^{2}}$ has a simple pole $-i$ on $\boldsymbol{C}^{-}$, and so that

$$\frac{1}{2\pi i}\int_{-R}^{R}\frac{1+\lambda\xi}{\lambda-\xi}\,\frac{\operatorname{Log} F(-\lambda)}{1+\lambda^{2}}d\lambda$$

$$=\frac{1}{2\pi}\int_{-\pi}^{0}\frac{1+Re^{i\theta}\zeta}{Re^{i\theta}-\zeta}\,\frac{\operatorname{Log}F(-Re^{i\theta})}{1+(Re^{i\theta})^{2}}Re^{i\theta}d\lambda+\frac{\operatorname{Log}F(i)}{2}.$$

Since, for any $\theta\in(-\pi, 0)$ $-e^{i\theta}=e^{i(\theta+\pi)}$ and $\theta+\pi\in(0, \pi)$, we can take the same procedure as in (i) to obtain (ii).

Thus, by combining Step 2 with Step 3, we conclude from (A. 6) that Theorem A is proved. (Q. E. D.)

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