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On equivalence of Hadamard matrices

Dedicated to Professor Nagayoshi IWAHORI on his 60th birthday

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1. Introduction

An n-dimensional Hadamard matrix is an n by n matrix of 1's and -1's with $HH^{T}=nI$. In such matrix, n is necessarily 2 or a multiple of 4. An automorphism of H is a signed permutation g on the set of rows and columns such that $H^{g}=H$. The set of automorphisms forms a group under composition called the automorphism group of H. Two Hadamard matrices H_1 and H_2 are equivalent if there exists a signed permutation g of rows and columns with $H_1^{g}=H_2$. An Hadamard matrix is normalized if its first row and column consist entirely of 1's.

Let H_1 and H_2 be two Hadamard matrices of order n. It is difficult to know whether they are equivalent or not. But we know some invariants for equivalence classes. For instance if automorphism groups of H_1 and H_2 are nonisomorphic as permutation groups, they are not equivalent. For $n=28$ we defined the K-matrix $K(H)$ associated with H depending only on equivalence class containing H ([2]). In the case $n=28$ it seems that $K(H)$ is very useful. By computing K-matrices we obtained Hadamard matrices with trivial automorphism groups. Furthermore we constructed about four hundred new Hadamard matrices of order 28 ([2], [3] and [4]).

In § 2 we discuss K-matrices in general cases. In § 3 we introduce a new invariant called K-boxes. For $n=28$ we give examples with same K-matrix such that they have different K-boxes and therefore they are not equivalent.

2. On K-matrices

Let $H=(h_{ij})$ be an Hadamard matrix of order n with $0\leq i,\ j\leq n-1$. H is equivalent to $H'=(h'_{i,j})$ with $h'_{i,0}=h'_{0,j}=1$, $0\leq i,\ j\leq n-1$. From H' we have an incidence matrix $D(H)$ of a symmetric $2-(v, k, \lambda)$ design associated with H, where $v=n-1$, $k=(v-1)/2$, $\lambda=(k-1)/2$:

$$D(H)=d_{i,j}),\ i,\ j=1,\dots,\ n-1$$

$$\text{where } d_{i,j}=\begin{cases}1, & \text{if } h'_{i,j}=1\\ 0, & \text{if } h'_{i,j}=-1\end{cases}$$

For any different four rows i. j, k and m of H, we define a_{ijkm} as follows:

$$a_{ijkm}(r)=\begin{cases}1, & \text{if } h_{ir}h_{jr}h_{kr}h_{mr}=1\\ 0, & \text{if } h_{ir}h_{jr}h_{kr}h_{mr}=-1\end{cases}$$

Then $a_{ijkm}(0)+\ldots+a_{ijkm}(n-1)$ is divisible by 4 ([2] or [5]). Let x be an integer with $0\leq x\leq n/4$. For fixed i and j, let $\kappa'_{ij}(x)$ be a number of pairs k and m of rows such that $a_{ijkm}(0)+\ldots+a_{ijkm}(n-1)=4x$. For $0\leq x\leq n/8$, put

$$\kappa_{ij}(x)=\begin{cases}\kappa'_{ij}(x)+\kappa'_{ij}(n/4-x), & \text{if } x\neq n/4-x\\ \kappa'_{ij}(x), & \text{if } x=n/8.\end{cases}$$

Then $\kappa_{ij}(x)$ does not change by multiplication of rows i or j by -1. By a permutation of coordinates we assume that $\kappa_{ij}(x)\leq\kappa_{ik}(x)$ if $j<k$. Put

$$K_{ij}(x)=\begin{cases}\kappa_{ij}(x), & \text{if } i>j\\ \kappa_{ij+1}(x), & \text{if } i\leq j.\end{cases}$$

Furthermore the rows of the $n\times(n-1)$ matrix $(K_{ij}(x))$ are ordered lexicographically, that is, if $i<i'$, then $K_{ij}(x)=K_{ij}(x)$ for $j=1,\ldots,n-1$, or there exists an integer j such that $K_{i,j}(x)=K_{i',j'}(x)$ for $j'<j$ and $K_{i,j}(x)<K_{i',j}(x)$. We call the matrix $K_x(H)=(K_{ij}(x))$ an associated x-th K-matrix of H.

By the construction of $K_x(H)$ we have the following:

THEOREM 1. *Let H_1 and H_2 be Hadamard matrices of order n which are equivalent, then $K_x(H_1)=K_x(H_2)$ for all $0\leq x\leq n/8$.*

REMARK 1. *By considering the relation of three rows of $D(H)$ it is trivial that $K_0(H)$ is the zero matrix in the case $n\equiv 4$ (8).*

REMARK 2. *$K(H)$ in [2] is $K_1(H)$ for $n=28$.*

Next we prove the following theorems.

THEOREM 2. *Assume $n\equiv 4$ (8). Let a and b be two integers with $1\leq a, b\leq(n-4)/8$. If we know $K_m(H)$ for all $m\neq a, b$, $K_a(H)$ and $K_b(H)$ can be obtained.*

THEOREM 3. *Assume $n\equiv 0$ (8). Let a and b be two integers with $0\leq a, b\leq n/8$. If we know $K_m(H)$ for all $m\neq a, b$, then $K_a(H)$ and $K_b(H)$ can be obtained.*

(1) Case $n\equiv 4$ (8). We arrange the first three rows of $D(H)$ in the following form:

$$\begin{array}{|c|c|c|c|c|c|c|c|}
1\ldots1 & 1\ldots1 & 1\ldots1 & & 1\ldots1 & & & \\
1\ldots1 & 1\ldots1 & & 1\ldots1 & & 1\ldots1 & & \\
1\ldots1 & & 1\ldots1 & 1\ldots1 & & & 1\ldots1 & \\
\multicolumn{1}{c}{\lambda'} & \multicolumn{1}{c}{\lambda-\lambda'} & \multicolumn{1}{c}{\lambda-\lambda'} & \multicolumn{1}{c}{\lambda-\lambda'} & \multicolumn{1}{c}{k+\lambda'-2\lambda} & \multicolumn{1}{c}{k+\lambda'-2\lambda} & \multicolumn{1}{c}{k+\lambda'-2\lambda} & \multicolumn{1}{c}{\lambda-\lambda'}
\end{array}$$

where λ' is a number of columns j such that $d_{i,j}=d_{2,j}=d_{3,j}=1$. Then $a_{0123}(0)+\ldots+a_{0123}(n-1)=4\lambda'+4$.

For $0\leq i\leq\lambda-1$, let α_i be a number of rows j of $D(H)$ such that $\sum_{k=1}\lambda d_{jk}=i$, where $3\leq j\leq n-1$. Since $D(H)^T$ is also an incidence matrix of $2-(v, k, \lambda)$ design, we have the following:

$$\alpha_0+\alpha_1+\ \alpha_2+\ldots+\qquad \alpha_{\lambda-1}=n-3 \qquad (1)$$

$$\alpha_1+2\alpha_2+\ldots+(\lambda-1)\alpha_{\lambda-1}=\lambda(k-2) \qquad (2)$$

$$\alpha_2+\ldots+\ {}_{\lambda-1}C_2\alpha_{\lambda-2}={}_{\lambda}C_2(\lambda-2) \qquad (3)$$

Let a be an integer with $\lambda/2\leq a\leq\lambda-2$. From equalities (1) and (2) we have

$$\sum_{i\neq a}(a-i)\alpha_i=a(n-3)-\lambda(\mathrm{k}-2) \qquad (4)$$

From (2) and (3) we have

$$\sum_{i\neq a} i(a-i)\alpha_i/2=\lambda(k-2)(a-1)/2-(\lambda-2){}_{\lambda}C_2 \qquad (5)$$

From (4) and (5) we eliminate $\alpha_{\lambda-1-a}$. Let A_i be coefficient of new equality $(i\neq a,\ \lambda-1-a)$. Then

$$A_j=(i-\lambda+1+a)(a-i)/2$$

Therefore $A_j=A_i$ if and only if $j=i$ or $j=\lambda-1-i$. Put $\beta_i=\alpha_i+\alpha_{\lambda-i-1}$ for $0\leq i\leq\lambda/2-1$. Then we have the following:

$$\sum_{i\neq a} A_i\beta_i=B' \qquad (6)$$

where B' is a constant. By (1)

$$\sum_{i=0}^{\lambda/2-1}\beta_i=n-3 \qquad (7)$$

By (6) and (7) we can obtain β_a and β_b. Put $\beta_a=\sum_{i\neq a,b}B_i\beta_i+B$, $\beta_b=\sum_{i\neq a,b}F_i\beta_i+F$, where B and F are constants.

PROOF THEOREM 2. Fix $0\leq i<j\leq n-1$. We assume that $h_{ik}=1$ and $h_{k0}=1$ for all $0\leq k\leq n-1$. By the definition of $\kappa_{ij}(x)$

$$\sum_{x=1}^{\lambda/2}\kappa_{ij}(x)={}_{n-2}C_2 \qquad (8)$$

For $k \neq j$, j let $\delta_x(k)$ be a number of rows m such that $a_{ijkm}(0)+ \dots +a_{ijkm}(n-1)=4x$ or $n-4x$. we consider $\varkappa_{ij}(a)$. By the above discussion

$$\delta_a(k)=\sum_{x\neq a,b} B_{x-1}\ \delta_x(k)+B$$
$$\varkappa_{ij}(x)=(\sum_{k\neq i,j}\delta_x(k))/2$$

Therefore

$$\begin{aligned}\varkappa_{ij}(a)&=(\sum_{k\neq i,j}\delta_a(k))/2\\ &=(\sum_{x\neq a,b}B_{x-1}\sum_{k\neq i,j}\delta_x(k))/2+{}_{n-2}C_1B/2\\ &=\sum_{x\neq a,b}B_{x-1}\ \varkappa_{i,j}(x)+(n-2)B/2 \qquad (9)\end{aligned}$$

Form (8) and (9) Theorem 2 is proved.

(2) Case $n\equiv 0$ (8). We arrange the first three rows of $D(H)$ in the following form

$$\begin{array}{|c|c|c|c|c|c|c|c|} 1\dots 1 & 1\dots 1 & 1\dots 1 & & 1\dots 1 & & & \\ 1\dots 1 & 1\dots 1 & & 1\dots 1 & & 1\dots 1 & & \\ 1\dots 1 & & 1\dots 1 & 1\dots 1 & & & 1\dots 1 & \\ \multicolumn{1}{c}{\lambda'} & \multicolumn{1}{c}{\lambda-\lambda'} & \multicolumn{1}{c}{\lambda-\lambda'} & \multicolumn{1}{c}{\lambda-\lambda'} & \multicolumn{1}{c}{k+\lambda'-2\lambda} & \multicolumn{1}{c}{k+\lambda'-2\lambda} & \multicolumn{1}{c}{k+\lambda'-2\lambda} & \multicolumn{1}{c}{\lambda-\lambda'} \end{array}$$

where λ' is a number of columns j such that $d_{1,j}=d_{2,j}=d_{3,j}=1$. Then $a_{0123}(0)+ \dots +a_{0123}(n-1)=4\lambda'+4$.

For $0\leq i\leq\lambda$, let α_i be as in the case $n\equiv 4$ (8). Then we have the following:

$$\alpha_0+\alpha_1+\ \alpha_2+\dots+\quad\alpha_\lambda=n-3 \qquad (10)$$
$$\alpha_1+2\alpha_2+\dots+\ {}_\lambda\alpha_\lambda=\lambda(k-2) \qquad (11)$$
$$\alpha_2+\dots+{}_\lambda C_2\alpha_\lambda={}_\lambda C_2(\lambda-2) \qquad (12)$$

Let a be an integer with $(\lambda+1)/2\leq a\leq\lambda-2$. From equalities (10) and (11) we have

$$\sum_{i\neq a}(a-i)\alpha_i=a(n-3)-\lambda(\mathrm{k}-2) \qquad (13)$$

From (11) and (12) we have

$$\sum_{i\neq a}i(a-i)\alpha_i/2=\lambda(k-2)(a-1)/2-(\lambda-2){}_\lambda C_2 \qquad (14)$$

From (13) and (14) we eliminate $\alpha_{\lambda-1-a}$. Let A_1 be coefficient of new equality $(i\neq a,\ \lambda-1-a)$. Then

$$A_i=(i-\lambda+1+a)(a-i)/2$$

For j, $j\neq(\lambda-1)/2$ or λ, $A_j=A_i$ if and only if $j=i$ or $j=\lambda-1-i$. For $i\neq(\lambda-1)/2$ or λ, $\beta_i=A_i+A_{\lambda-1-i}, \beta_{(\lambda-1)/2}=A_{(\lambda-1)/2}$, and $\beta_{-1}=A_\lambda$. Then we have the following:

$$\sum_{i\neq a} A_i\beta_i=B'$$

where B' is a constant. By the similar way in the case $n\equiv 4$ (8), we can prove Theorem 3.

Case $n=28$. By Theorem 2 we consider only $K_1(H)$. K-matrices are seemed to be useful for a classification of Hadamard matrices. Actually we obtained 476 non-equivalent Hadamard matrices of order 28 ([3], [4]).

3. On K-boxes

In § 2 we defined k-matrices. But we have some examples of Hadamard matrices of order 28 with same K-matrix. We consider another method of classification of Hadamard matrices. Let H be an Hadamard matrix of order n.

For any different six rows i, j, k, i', j' and k' of H, we define $a_{ijki'j'k'}$ as follow:

$$a_{ijki'j'k'}(r)=\begin{cases}1, & \text{if } h_{ir}h_{jr}h_{kr}h_{i'r}h_{j'r}h_{k'r}=1\\ 0, & \text{if } h_{ir}h_{jr}h_{kr}h_{i'r}h_{j'r}h_{k'r}=-1\end{cases}$$

Let x be an integer with $0\leq x\leq n$. For fixed i, j and k, let $\kappa'_{ijk}(x)$ be a number of triples i', j' and k' of rows such that $a_{ijki'j'k'}(0)+\ldots+a_{ijki'j'k'}(n-1)=x$. For $0\leq x\leq n/2$, put

$$\begin{aligned}\kappa_{ijk}(x)=\kappa'_{ijk}(x)+\kappa'_{ijk}(n-x), &\quad \text{if } x\neq n-x\\ \kappa'_{ijk}(x), &\quad \text{if } x=n/2.\end{aligned}$$

Then $\kappa_{ijk}(x)$ does not change by multiplication of rows i, j or k by -1. By a permutation of coordinates we assume that $\kappa_{ijk}(x)\leq\kappa_{ijk'}(x)$ if $k<k'$. Put

$$K'_{ijk}(x)=\begin{cases}\kappa_{ijk}(x), & \text{if } i>j\\ \kappa_{ij+1k}(x), & \text{if } i\leq j.\end{cases}$$

Next put

$$K_{ijk}(x)=\begin{cases}K'_{ijk}(x), & \text{if } j,\ j>k\\ K'_{ijk+1}(x), & \text{if } i>k\geq j, \text{ or } i\leq k<j\\ K'_{ijk+2}(x), & \text{if } i,\ j\leq k\end{cases}$$

Then, for $0\leq i\leq n-1$, the matrix $K_{i,x}(H)=(K_{ijk}(x))$ is of type $(n-1)\times(n-2)$. For i we rearrange the matrix $K_{i,x}(H)$ as in the case of K-

matrices. Furthermore we rearrange the collection of matrices $K_{i,x}(H)$ with $0 \leqq i \leq n-1$ in the following: if $i<i'$, then matrix $K_{i,x}(H)$ equals the matrix $K_{i',x}(H)$, or there exist integers s and t such that if $j<s$, then $K_{ijk}(x)=K_{i'jk}(x)$ for all k, if $k<t$, then $K_{isk}(x)=K_{i'sk}(x)$ and $K_{ist}(x)=K_{i'st}(x)$. We call this collection $KB_x(H)$ of n matrices $K_{i,x}(H)$ K-box of degree x associated with H.

By the construction of $KB_x(H)$ we have the following:

THEOREM 4. *Let H_1 and H_2 be Hadamard matrices of order n which are equivalent, then $KB_x(H_1)=KB_x(H_2)$ for all $0 \leq x \leq n/2$.*

In the following table we give five matrices of order 28 with same $K(H)$ such that they have different K-boxes of degree 6. All rows of $K(H)$ and $K(H^T)$ are of type (0000000000000000000000000111).

In the following table, for example, the number 32511 becomes 0000000000000111111011111111 in the binary system and the first row of H is (11111111−1111111−1−1−1−1−1−1−1−1−1−1−1−1). In $KB_6(H)$, Symbol '•' represents 0, and also A, B, ... and Z represent 10, 11, ... and 34, respectively. 'Mul=4' means that $KB_6(H)$ contains four matrices as same as a matrix. Moreover, (6 C C G G G G G G I I I I K K K K O O O O O O S S S S) expresses that the multiplicity of rows of type (C C G G G G G G I I I I K K K K O O O O O O S S S S) equals 6.

Acknowledgment. In [8] Professor V. D. Tonchev wrote to us that he knew four nonequivalent Hadamard matrices of order 28 having same K-matrix as $K(H)$.

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Table of H-Matrices and K-boxes
Blocks of $D(H)$
(1st row—27 th row)

$KB_6(H)$			$KB_6(H^T)$			
3 2 5 1 1	2 0 6 4 7 6 7	1 3 2 1 2 1 0 2 3	1 4 9 1 3 0 7 6 7	1 8 7 4 7 1 4 1 7	2 5 2 9 1 4 2 2 9	2 1 5 3 0 7 3 1 5
1 0 9 2 8 7 3 9 3	2 4 4 4 8 9 7 7	2 2 0 8 4 2 2 1 1	1 6 4 3 1 1 2 5 7	2 4 0 4 2 7 7 2 3	3 0 9 8 0 8 9 9	1 0 5 5 0 8 1 2 1
8 6 4 7 1 4 3 5	1 7 5 2 6 6 1 5 3	2 1 2 1 2 1 9 4 1	2 2 8 0 8 3 5 3 3	7 9 5 7 2 1 3 7	7 5 1 5 8 1 6 5	6 0 1 5 1 9 4 9
1 5 5 6 4 1 2 5 3	1 7 7 0 9 7 6 1 9	1 7 1 5 2 1 4 1 5	4 9 1 1 1 6 5 3	6 2 0 5 5 5 0 7	1 2 2 5 5 9 5 5 9	

Mul=28

6 CCGGGGGGIIIIKKKKOOOOOOSSSS
6 EEGGGGGGIIKKKKMMMMOOOOOOSS
6 EGGIIIIIIIKKKKKKMMMMOOOOOS
6 GGGGIIIIKKKKKKMMMMMMMMOOOO
3 IIIIKKKKKKKKMMMMOOOOOOSSSS

Mul=28

6 EEGGGGGGIIIIKKMMMMMMOOQQUU
6 EEGGGGIIKKKKMMMMMMMMOOOOOO
6 EEGGIIIIIIIKKKKKMMMMMOOOUU
6 GGGGGGIIIIIIKKMMMMOOOOOOQQ
3 IIIIKKKKMMMMMMMMOOOOOOQQQQ

3 2 5 1 1	2 0 6 4 7 6 7	1 3 2 1 2 1 0 2 3	1 4 9 1 3 0 7 6 7	2 5 2 0 2 4 3 7 7	1 5 9 1 3 2 2 1 3	1 7 8 9 7 4 7 7 1
4 0 4 7 8 6 8 9	2 7 6 9 6 0 8 1	2 2 8 1 0 1 3 4 7	1 8 1 2 2 5 6 8 9	2 3 8 6 8 6 9 2 5	1 0 9 9 8 9 6 6 9	7 5 0 2 9 7 8 5
6 0 3 7 3 7 7 1	2 2 0 4 2 4 5 5 3	1 8 9 9 4 1 5 8 7	2 3 9 3 8 4 9 0 3	5 5 8 2 6 5 9 7	1 0 4 2 7 9 1 8 7	8 9 0 5 5 8 8 3
1 4 8 2 5 6 1 6 9	2 1 0 8 8 7 5 7 3	1 5 3 9 6 5 9 5 9	8 3 1 8 6 2 7 5	9 7 2 0 3 2 8 5	6 2 1 8 6 5 7 3	

Mul=4

3 • • OOOOOOOOOOOOOOOOOOOOOOOO
18 EEGGGGHHIIIKMMNNNNOOOOOOOO
6 GGGGGGIIIIIKKKNNNNNNOOOOOO

$KB_6(H)$

Mul=24

6 9FFGGHHJJJJLLLMMNNNNNNNNSS
2 CGGGIIIIKKKKKKMMMNNNNNNOOO
3 EEGGGGHHIIIKMMNNNNOOOOOOOO
3 EEIIIIJJKKKKKKLLLLNNNNNNOO
3 FFFFHHHHJJKKKKLLMMMMOOOOSS
3 FFGGGGIIJJJJJJJJMMNNOOQQSS
3 FFGGIIJJJJKKKKLLMMMMNNNNQQ
1 GGGGGGIIIIIKKKNNNNNNOOOOOO
3 IIIIIIIIKKMMMMNNNNOOQQSSSS

3 2 5 1 1	2 0 6 4 7 6 7	1 3 2 1 2 1 0 2 3	1 4 9 1 3 0 7 6 7	2 5 2 0 2 4 3 7 7	1 5 9 1 3 2 2 1 3	1 7 8 9 7 4 7 7 1
4 4 4 1 1 8 7 3	2 3 7 6 2 8 9 7	1 6 4 9 3 8 9 7 9	2 2 0 8 4 8 3 3 3	2 3 8 4 2 4 7 7 5	8 7 2 5 4 8 1 9	4 1 2 1 9 3 4 1
9 3 1 1 7 1 9 1	2 1 3 3 9 2 7 4 5	1 8 8 9 6 3 1 7 3	2 1 4 3 2 8 1 5 3	1 0 6 6 1 3 4 1 7	8 0 6 8 8 2 9 3	2 8 7 9 2 9 8 5
1 7 7 8 1 7 4 9 3	2 0 6 6 8 2 5 1 5	1 8 5 4 7 8 5 3 9	1 2 7 0 5 2 8 8 5	1 0 7 8 2 6 2 5 9	6 4 5 6 5 8 3 5	

Mul = 24

2 BCEGHHIIIIIKKLMMMNNOPPPQQR
2 BEEGHHIIJKKKKKLMMNNNOOOPQR
2 BEEGHHJJJJJLLLMMNNNNNNNORY
2 CCDEIIJJJKKKKLLMMNNNNNOOPQR
1 CCEGGHHIIIIKLLMMNNNNNNNOQYY
1 DEGGHHHHHJKKKLLLMMOOOOOPQR
1 DHHIIIIJJKKKKLLLLLMMMMNOOR
2 EEFFHHHIIJJJJLLMNNNNOOPPQR
2 EFFFGHIIJJJJKLLLMMMNNOOQQY
1 FFHHIIJJJJKKKKKKLLNNNNOOPP
1 FGGGGHJJJJKKLLLLMMNOOOOOQ
1 FGGHHIIIIJJLLLLMMMMMMNNOPQ
2 FGGHHIIJJJKKKKLLMMMNNNOOOP
2 GGHHIIIIIJJJKLLLLMMNNOOOOP
1 GGHHJJLLLLMMNNNNOOOOPPQQRR
1 HHHHJJJJKKKKKKLLLLLLMMNNOO
1 HHIIIIIIIIIJJKKKKLLNNNNOOPP
1 IIJJJJKKKKMMMMNNOOOOPPQQQ

Mul = 4

6 DEGGHHHHHJKKKLLLMMOOOOOPQR
6 DHHIIIIJJKKKKLLLLLMMMMNOOR
6 FGGGGHJJJJKKLLLLMMNOOOOOQ
6 FGGHHIIIIJJLLLLMMMMMMNNOPQ
3 JJJJLLLLLLLLMMMMOOOOOOPPPP

Mul = 4

6 CDGHHIJJJJJJKLLLLMMMNNQQQR
6 DGGGHIJJJKKKKLLLMMMMNNNOPQ
6 EFGGGHHIJJJJJKKKKLMPPQQQQR
6 EFGHHHIJJJJJJLLMMMMNNNOPQQ
3 FFFFLLLLNNNNNNNNOOOOOOQQQQ

Mul = 24

1 CDGHHIJJJJJJKLLLLMMMNNQQQR
1 CEEHHIIIIJJKKKLLLLNNPPQQQQ
1 DDGGGGHHIIKKLLNNNNNNNNPPQQ
2 DFFGHHIJJJJKKKLMMMMMNPPPQQ
1 DGGGHIJJJKKKKLLLMMMMNNNOPQ
2 DGGHIIIIJJJJKKLMMMMNNNNPPR
2 DHHHIIIIJKKKKKLLMMMNNNNOS
1 EFGGGHHIJJJJJKKKKLMPPQQQQR
1 EFGHHHIJJJJJJLLMMMMNNNOPQQ
2 EGGGGGIJJJKKKLLLLLMMNOPPQR
2 EGHHHIIIIJJKKKKLLMMNNPQQR
2 EGHHIIIJJJKKKKKKKLLMNNPQQQ
2 FGGHHIIJJJJKKKKLLMMMMNOPPS
1 GGIIJJKKLLLMMMMPPPPPPQQSS
2 GIIIIJJJJJKKKKLLLLMMNNPQ
1 HHIIIIJJJJKKKKLLLLMMNNNNNN
1 IIIIIIIIJJKKKKLLLLLLLOOPP
1 IIJJJJKKLLMMMMMMMMNNQQRRRR