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Convolution semigroups of local type on a commutative hypergroup

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§ 1. Introduction

Hypergroups are locally compact spaces with a group-like structure on which the bounded measures admit a convolution similar to that on a locally compact group. Important examples of hypergroups are double coset spaces, conjugacy spaces and duals of compact groups and also orbit spaces of certain locally compact groups. Moreover, the sets $\boldsymbol{Z}_+$, $\boldsymbol{R}_+$ of non-negative integers and reals respectively, the unit interval I and the unit disk $\boldsymbol{D}$ are also hypergroups with special operations different from the usual semigroup operations. In fact, a hypergroup K can be viewed as a probabilistic structure in the sense that to each pair x, y of points in K there exists a probability measure $\varepsilon_x * \varepsilon_y$ on K with compact support such that $(x, y) \to \operatorname{supp}(\varepsilon_x * \varepsilon_y)$ is a continuous mapping from $K \times K$ into the space of compact subsets of K. The convolution $*$ between Dirac measures extends to all bounded measures on K and transplants the algebraic-topological analysis from the sparcely structured basic space K to the generalized measure algebra of K.

In this paper we continue studying convolution semigroups of measures on K in terms of their generators. Our discussion is based on previous work on the subject as f. e. the contributions [4], [5], [6] of W. R. Bloom and the author, and the article [14] of R. Lasser.

For the full axiomatic of a hypergroup we refer to the paper [12] of R. I. Jewett. It is essentially Jewett's terminology that we adopt. To review some notation will prove useful. By K we denote a commutative hypergroup with involution $\cdot^-$ and neutral element e. Occasionally we need to deal with the pointed hypergroup $K^\times := K \setminus \{e\}$. For every $x \in K$ the symbol $\mathfrak{V}_x(K)$ stands for the system of open neighborhoods of x (in K). It is known that K has a Haar measure ω_K and a Plancherel measure π on the dual space $K^\wedge$ of K. For a complex-valued function f on K the function f^- is defined by $f^-(x) := f(x^-)$ for all $x \in K$. The *translate* by $x \in K$ of an admissible function f on K is given by

$$T^x f(y) := \int f(z)\varepsilon_x * \varepsilon_y(dz)$$

which can be written as $f_x(y)$ or $f(x*y)$ for all $y \in K$. Given a bounded nonnegative measure μ and a function f on K we shall agree on the notation

$$\mu * f(x) := \int f_x^- d\mu = \int f(x*y^-)\mu(dy)$$

whenever $x \in K$. The symbols $\hat{\mu}$ and $\check{\mu}$ for the Fourier transform of a bounded measure and its inverse transform of $K^\wedge$ and K respectively are chosen in accordance with [12].

It is the purpose of our contribution to initiate the analysis of local convolution semigroups on a hypergroup K and the corresponding diffusion processes with K as their state space. Much of the basic theory can be developed as in the group case; the appropriate reference is the book [3] by C. Berg and G. Forst. There are, however, significant limitations of the translation procedure. Some of these points will be prepared in Section 2. In Section 3 we study the generators of convolution semigroups of measures on various function spaces and give a first characterization of locality (Theorem 3. 3). A construction due to C. Berg [2] of the Lévy measure of a convolution semigroup is extended to hypergroups in Section 4. Theorem 4. 1 is slightly more general than the corresponding result of R. Lasser in [14]. In Section 5 we prove a characterization of local convolution semigroups in terms of their Lévy measures and, under additional assumptions, also in terms of their Lévy-Khintchine representations (Theorem 5. 3). Some applications to transient convolution semigroups follow. The paper ends with a discussion of examples, in which local convolution semigroups are exhibited.

§ 2. Preparations.

Given a locally compact space K we will write $\mathscr{C}(K)$ for the space of continuous functions on K. The inclusion $\mathscr{K}(K) \subset \mathscr{C}^0(K) \subset \mathscr{C}^b(K)$ contains the subspaces of functions in $\mathscr{C}(K)$ that are of compact support, vanish at infinity or are just bounded respectively. Analoguously there is the inclusion $\mathscr{M}^1(K) \subset \mathscr{M}^{(1)}(K) \subset \mathscr{M}^b(K)$ between the spaces of probability measures, (nonnegative) contraction measures and arbitrary bounded measures on K, respectively. $\mathscr{C}(K)$ will be furnished with the compact-open topology $\mathscr{T}_{co}$, $\mathscr{C}^b(K)$ with the topology induced by the uniform morm $\|\cdot\|$. In $\mathscr{M}^b(K)$ we shall consider the norm topology and also the weak topology $\mathscr{T}_w$ according to our particular demands. In the space $\mathscr{M}_+(K)$ of all nonnegative (not necessarily bounded Radon) measures on K we are given the vague topology $\mathscr{T}_v$.

From now on we assume that K is a commutative hypergroup. For any $p\in[1, \infty]$ the spaces $L^p(K, \omega_K)$ are defined as in the group case. There is also the space $\mathscr{C}_u(K)$ of bounded uniformly continuous functions on K. Here, a function f on K is said to be *uniformly continuous* if for given $\varepsilon>0$ and any $x_0\in K$ there exists a $U\in\mathfrak{V}_{x_0}(K)$ such that $\|f_{x_0}-f_x\|<\varepsilon$ for all $x\in U$.

2.1 THEOREM. ([6], 2.7). $\mathscr{K}(K)\subset\mathscr{C}_u(K)$

Now let $(\mu_t)_{t\geq 0}$ denote a continuous convolution semigroup of measures in $\mathscr{M}^{(1)}$ where continuity is understood in the sense of $\mathscr{T}_v\text{-}\lim_{t\to 0}\mu_t=\varepsilon_e$. For any of the Banach spaces $E=\mathscr{C}^0(K)$, $\mathscr{C}_u(K)$ and $L^2(K, \omega_K)$ $(\mu_t)_{t\geq 0}$ induces a strongly continuous contraction semigroup $(P_t)_{t\geq 0}$ of operators on E defined by

(C) $P_tf:=\mu_t*f$

for all $f\in E$, $t\geq 0$. Clearly, $(P_t)_{t\geq 0}$ is *translation invariant* in the sense that

(a) $P_tE\subset E$ and
(b) $T^xP_t=P_tT^x$

hold for all $x\in K$ $(t\geq 0)$. One easily verifies that if $(\mu_t)_{t\geq 0}$ is a convolution semigroup in $\mathscr{M}^1(K)$, then $(P_t)_{t\geq 0}$ is *Markovian* in the sense of the property

(M) $\sup\{P_tf : f\in E,\ 0\leq f\leq 1\}=1$ for all $t\geq 0$.

If, moreover, $(\mu_t)_{t\geq 0}$ is symmetric and $E=L^2(K, \omega_K)$ then $(P_t)_{t\geq 0}$ is selfadjoint which means that P_t is a selfadjoint operator for every $t\geq 0$. The converse of these statements is contained in the following

2.2 THEOREM. ([10], 1.7 *of Chapitre III*). *There is a one-to-one correspondence between continuous convolution semigroups* $(\mu_t)_{t\geq 0}$ *in* $\mathscr{M}^{(1)}(K)$ *and translation invariant, strongly continuous semigroups* $(P_t)_{t\geq 0}$ *of positive contraction operators on E which is given by* (C). *For this correspondence we have that*

(i) $(\mu_t)_{t\geq 0}$ *is in* $\mathscr{M}^1(K)$ *iff* $(P_t)_{t\geq 0}$ *is Markovian,*

and in the case of $E=L^2(K, \omega_K)$ *that*

(ii) $(\mu_t)_{t\geq 0}$ *is symmetric iff* $(P_t)_{t\geq 0}$ *is selfadjoint.*

2.3 REMARK. The Markovian property (M) of $(P_t)_{t\geq 0}$ on $E=L^2(K, \omega_K)$ *can generally not be replaced by the property*

(M') $\|P_t\|=1$ for all $t\geq 0$.

In fact, if we wish to preserve the statements of the theorem with (M) being replaced by (M') we have to make the additional (Godement) assumption that the unit character **1** of K belongs to the support of the Plancherel measure π.

Let $(\rho_\lambda)_{\lambda>0}$ denote the resolvent family associated with $(\mu_t)_{t\geqq 0}$ given by

$$\rho_\lambda(f) := \int_0^\infty e^{-\lambda t}\mu_t(f)\,dt$$

for all $f\in\mathscr{C}^b(K)$. There always exists the extended real number

$$\lim_{\lambda\to 0}\rho_\lambda(f) = \int_0^\infty \mu_t(f)\,dt$$

for all $f\in\mathscr{C}^b(K)$. If this limit is finite for all $f\in\mathscr{K}(K)$ then

$$\varkappa := \mathscr{T}_v - \lim_{\lambda\to 0}\rho_\lambda$$

defines the *potential kernel* $\varkappa$ of $(\mu_t)_{t\geqq 0}$ as a measure in $\mathscr{M}_+(K)$. In this case $(\mu_t)_{t\geqq 0}$ is called a *transient* convolution semigroup.

2.4 THEOREM. ([6], 5.3). *A measure $\varkappa\in\mathscr{M}_+(K)$ is the potential kernel of a transient convolution semigroup in $\mathscr{M}^{(1)}(K)$ iff $\varkappa$ admits a fundamental family $(\sigma_V)_{V\in\mathfrak{B}}$ of measures in $\mathscr{M}^{(1)}(K)$ indexed by a base $\mathfrak{B}$ of compact open neighborhoods of e, which has the following properties valid for all $V\in\mathfrak{B}$:*

(a) $\sigma_V*\varkappa\leqq\varkappa,\ \sigma_V*\varkappa\neq\varkappa$.

(b) $\sigma_V*\varkappa=\varkappa$ on $\complement V$.

(c) $\mathscr{T}_v-\lim_{n\to\infty}\sigma_V^n*\varkappa=0$.

§ 3. Generators.

The *generator* of a convolution semigroup $(\mu_t)_{t\geqq 0}$ in $\mathscr{M}^{(1)}(K)$ can be introduced as the infinitesimal generator $(A, D(A))$ of the contraction semigroup $(P_t)_{t\geqq 0}$ on E which corresponds to $(\mu_t)_{t\geqq 0}$ by Theorem 2.2. More explicitly we have

$$Af: \lim_{t\to 0}\frac{1}{t}(P_tf-f)$$

for all

$$f\in D(A): \{h\in E: \lim_{t\to 0}\frac{1}{t}(P_th-h) \text{ exists}\}.$$

Let $(R_\lambda)_{\lambda>0}$ denote the resolvent of $(P_t)_{t\geqq 0}$ which admits the representation

$$R_\lambda f=\rho_\lambda*f$$

for all $f\in E$, $\lambda>0$. As in the case of an Abelian group K ([3], 12.11) one shows

(1) $R_\lambda(E)\subset D(A)$ for all $\lambda>0$.

(2) A is *translation invariant* in the sense that

(a) $T^x D(A)\subset D(A)$ and

(b) $T^x A = A T^x$

hold for all $x\in K$.

(3) For any $f\in D(A)$ and $g\in\mathscr{K}(K)$ the function $f*g\in D(A)$, and

$$A(f*g)=(Af)*g.$$

For $i=0$, u, 2 the pair (A_i, D_i) denotes the infinitesimal generator of the semigroup $(P_t)_{t\geqq 0}$ considered as a translation invariant, strongly continuous contraction semigroup on $\mathscr{C}^0(K)$, $\mathscr{C}_u(K)$ and $L^2(K, \omega_K)$, respectively.

From now on we shall assume that the dual $K^\wedge$ of K is a hypergroup (with respect to pointwise multiplication of characters). In this case $\pi=\omega_{K^\wedge}$.

The proofs of the following results are carried out in analogy to the group case treated in [9] or [3], § 18. One just has to apply Theorems 2.1 and 2.4.

3.1 THEOREM. *$\mathscr{K}(K)\cap D_0\cap D_2$ is a dense subspace of $\mathscr{K}(K)$, $\mathscr{C}_u(K)$ and $L^2(K, \omega_K)$.*

PROOF. The measure ρ_1 is the potential kernel of the convolution semigroup $(e^{-t}\mu_t)_{t\geqq 0}$ which is obviously transient. Then by Theorem 2.4 for every compact $V\in\mathfrak{V}_e(K)$ there exists a measure $\sigma_V\in\mathscr{M}^{(1)}(K)$ satisfying the inequalities

(a) $\rho_1*\sigma_V\leqq\rho_1$, $\rho_1*\sigma_V\neq\rho_1$.

(b) $\rho_1*\sigma_V=\rho_1$ on $\complement V$.

After appropriate norming by numbers $a_V>0$ the measures

$$\eta_V := a_V(\rho_1-\rho_1*\sigma_V)$$

are in $\mathscr{M}^1(K)$ and have supp $\mu_V\subset V$. Now we take a function $f\in\mathscr{K}_+(K)$ and an $\varepsilon>0$. We want to show that for every $U\in\mathfrak{V}_e(K)$ there is a function $g\in\mathscr{K}_+(K)\cap D_0\cap D_2$ satisfying supp $g\subset(\text{supp} f)*U$ such that

$$(*)\quad \|f-g\|<\varepsilon$$

holds. From Theorem 2.1 we infer that for given f there exists a compact $V\in\mathfrak{V}_e(K)$, $V\subset U$ such that

$$\|f_{x^-}-f\|<\varepsilon$$

for all $x \in V$. The function

$$g = \mu_V * f$$
$$= \rho_1 * (a_V f) - \rho_1 * \sigma_V * (a_V f)$$

belongs to $\mathscr{K}(K)$ and satisfies (*). Since the measure σ_V is bounded, $\sigma_V * (a_V f) \in \mathscr{C}^0(K) \cap L^2(K, \omega_K)$ and hence

$$g \in \rho_1 * (\mathscr{C}^0(K) \cap L^2(K, \omega_K)) \subset D_0 \cap D_2.$$

3.2 COROLLARY. *Let U and V be relatively compact open subsets of K such that $\bar{U} \subset V$. There exists a function $f \in D_0 \cap D_2$ satisfying*

$$\begin{cases} 0 \leqq f \leqq 1, \\ f = 1 \text{ on } U, \text{ and} \\ f = 0 \text{ on } \complement V. \end{cases}$$

PROOF. For the given sets U and V there is a relatively compact $W \in \mathfrak{V}_e(K)$ such that

$$(\bar{W} * \bar{U}) \cap (\bar{W} * \complement V) = \emptyset.$$

But then there is a function $g \in \mathscr{K}_+(K)$ satisfying

$$\begin{cases} 0 \leqq g \leqq 1, \\ g = 1 \text{ on } W * U, \text{ and} \\ g = 0 \text{ on } W * \complement V. \end{cases}$$

It follows from the proof of the theorem that we can find a function $h \in \mathscr{K}_+(K) \cap D_0 \cap D_2$ such that supp $h \subset W$ and $\int h d\omega_K = 1$. From (3) we infer that $f := g * h \in D_0 \cap D_2$, and by construction f has the required properties.

3.3 THEOREM. *The following statements are equivalent*:
(i) supp $(A_0 f) \subset$ supp f *for all* $f \in D_0$.
(ii) supp $(A_u f) \subset$ supp f *for all* $f \in D_u$.
(iii) supp $(A_2 f) \subset$ supp f *for all* $f \in D_2$.

PROOF. The implication (ii)$\Longrightarrow$(i) is clear. Since the remaining implications (i)$\Longrightarrow$(iii) and (iii)$\Longrightarrow$(ii) are shown similarly, we restrict ourselves to the proof of (i)$\Longrightarrow$(iii).

Let $f \in D_2$. In view of Theorem 3.1 it suffices to show that $\langle A_2 f, g \rangle = 0$ for all $g \in \mathscr{K}(K)$ satisfying

$$\begin{cases} g^- \in D_0 \cap D_2 \quad \text{and} \\ \text{supp } g \cap \text{supp } f = \emptyset. \end{cases}$$

For such functions we get

$$\begin{aligned}\langle A_2 f, g\rangle &= \lim_{t\to 0}\frac{1}{t}\langle \mu_t * f - f, g\rangle \\ &= \lim_{t\to 0}\frac{1}{t}\langle f, (\mu_t * g^- - g^-)^-\rangle \\ &= \langle f, (A_0 g^-)^-\rangle = 0,\end{aligned}$$

the latter equality following from $\operatorname{supp}(A_0 g^-)^- \subset \operatorname{supp} g$ which is available by hypothesis.

3.4 COROLLARY. *Let A_0 satisfy* (i) *of the theorem. Suppose that for $f \in \mathscr{C}^b(K)$ the limit*

$$g := \mathscr{T}_{co} - \lim_{t\to 0}\frac{1}{t}(\mu_t * f - f)$$

exists ($\in \mathscr{C}(K)$). *Then* $\operatorname{supp} g \subset \operatorname{supp} f$.

The proof runs as in the group case.

From [4] we recall that there is a one-to-one correspondence between (continuous) convolution semigroups $(\mu_t)_{t\geq 0}$ in $\mathscr{M}^1(K)$, (strongly) negative definite functions ψ on $K^\wedge$, and resolvent families $(\rho_\lambda)_{\lambda>0}$ in $\mathscr{M}_+^b(K)$ given by

$$\hat{\mu}_t = \exp(-t\psi)$$

on $K^\wedge (t \geqq 0)$ and

$$\rho_\lambda = \int_0^\infty e^{-\lambda t}\mu_t dt$$

on $\mathscr{C}^b(K)$ $(\lambda > 0)$, respectively. In [5] it was shown that the domain D_2 of the generator A_2 can be described as the set

$$D_2 = \{f \in L^2(K, \omega_K) : \hat{f}\psi \in L^2(K^\wedge, \omega_{K^\wedge})\},$$

and that

$$(A_2 f)^\wedge = -\hat{f}\psi$$

whenever $f \in D_2$. We shall apply this fact in the following section.

§ 4. Lévy measures.

Let $(\mu_t)_{t\geq 0}$ be a convolution semigroup in $\mathscr{M}^{(1)}(K)$ with corresponding negative definite function ψ on $K^\wedge$. The following result is a slight extension of Proposition 3.3 of [14]. See also [2] for the group case.

4.1 THEOREM. *There exists a measure* $\eta\in\mathscr{M}_+(K^\times)$ *satisfying*

$$\lim_{t\to 0}\frac{1}{t}\int fd\mu_t=\int fd\eta$$

for every $f\in\mathscr{C}^b(K)$ *such that* $\operatorname{supp} f\subset K^\times$.

PROOF. Let

$$\mathscr{S}:=\{\mu\in\mathscr{M}^1(K^\wedge): \mu \text{ is symmetric and } \operatorname{supp}\mu \text{ is compact}\}.$$

Then for given $\sigma\in\mathscr{S}$ and all $t>0$ we obtain

$$[\frac{1}{t}(1-\check{\sigma})\cdot\mu_t]^\wedge=\frac{1}{t}[1-\exp(-t\psi)]*(\sigma-\varepsilon_1).$$

It can be easily shown that

$$\mathscr{T}_{co}-\lim_{t\to 0}[\frac{1}{t}(1-\check{\sigma})\cdot\mu_t]^\wedge=\psi*\sigma-\psi,$$

and hence $\psi*\sigma-\psi$ is a (strongly) positive definite function on $K^\wedge$ in the sense of [4]. This means that there exists a measure $\eta_\sigma\in\mathscr{M}^b_+(K)$ satisfying

$$\hat{\eta}_\sigma=\psi*\sigma-\psi.$$

Applying the (continuity) Theorem 6.5 of [4] or Satz 2.1.5 of [15] we obtain

$$\mathscr{T}_w-\lim_{t\to 0}\frac{1}{t}(1-\check{\sigma})\cdot\mu_t=\eta_\sigma.$$

Now let $f\in\mathscr{C}^b(K)$ with $\operatorname{supp} f\subset K^\times$. By Lemma 3.1 of [14] there exists a measure $\sigma\in\mathscr{S}$ such that $\check{\sigma}\leqq\frac{1}{2}$ on $\operatorname{supp} f$. Consequently the funtion f_σ defined by

$$f_\sigma(x):=\begin{cases}\dfrac{f(x)}{1-\check{\sigma}(x)} & \text{if } x\in\operatorname{supp} f\\ 0 & \text{if } x\notin\operatorname{supp} f\end{cases}$$

is an element of $\mathscr{C}^b(K)$, and we obtain

$$\begin{aligned}\lim_{t\to 0}\frac{1}{t}\int fd\mu_t&=\lim_{t\to 0}\int f_\sigma d[\frac{1}{t}(1-\check{\sigma})\cdot\mu_t]\\&=\int f_\sigma d\eta_\sigma.\end{aligned}$$

In particular,

$$\mathscr{T}_v-\lim_{t\to 0}\frac{1}{t}\mathrm{Res}_{K^\times}\mu_t$$

exists as a measure $\eta\in\mathscr{M}_+(K^\times)$, and

$$(1-\check{\sigma})\cdot\eta=\mathrm{Res}_{K^\times}\eta_\sigma$$

holds for all $\sigma\in\mathscr{S}$. Finally for $f\in\mathscr{C}^b(K)$ with $\mathrm{supp}f\in K^\times$ and σ chosen as above we have

$$\begin{aligned}\lim_{t\to 0}\frac{1}{t}\int fd\mu_t&=\int f_\sigma d\eta_\sigma.\\&=\int f_\sigma d(\mathrm{Res}_{K^\times}\eta_\sigma)\\&=\int(1-\check{\sigma})f_\sigma d\eta\\&=\eta(f).\end{aligned}$$

4.2 DEFINITION. The measure $\eta\in\mathscr{M}_+(K^\times)$ constructed in the preceding theorem is said to be the *Lévy measure* of the convolution semigroup $(\mu_t)_{t\geqq 0}$.

4.3 REMARK. The Lévy measure η of $(\mu_t)_{t\geqq 0}$ is uniquely determined by the equality

$$(1-\check{\sigma})\cdot\eta=\mathrm{Res}_{K^\times}\eta_\sigma$$

valid for all $\sigma\in\mathscr{S}$ and coincides with the Lévy measure introduced in [14].

4.4 THEOREM. *Let $(\mu_t)_{t\geqq 0}$ be a convolution semigroup in $\mathscr{M}^{(1)}(K)$ with corresponding resolvent family $(\rho_\lambda)_{\lambda>0}$ and negative definite function ψ. Then for any measure $\eta\in\mathscr{M}_+(K^\times)$, in particular for the Lévy measure η of $(\mu_t)_{t\geqq 0}$, the following statements are equivalent*:

(i) $\eta=\mathscr{T}_v-\lim_{t\to 0}\frac{1}{t}\mathrm{Res}_{K^\times}\eta_t.$

(ii) $\eta=\mathscr{T}_v-\lim_{\lambda\to\infty}\mathrm{Res}_{K^\times}\lambda^2\rho_\lambda.$

(iii) $\eta(f)=A_0f^-(e)$
for all $f\in\mathscr{K}(K)$ *with* $f^-\in D_0$, $\mathrm{supp}f\subset K^\times$.

(iv) $\eta(f)=A_uf^-(e)$
for all $f\in\mathscr{C}_u(K)$ *with* $f^-\in D_u$, $\mathrm{supp}f\subset K^\times$.

(v) $\eta(f^-*\bar{g})=\langle A_2f, g\rangle$
for all $f, g\in\mathscr{K}(K)$, $f\in D_2$ *with* $\mathrm{supp}(f^-*\bar{g})\subset K^\times$.

PROOF. 1) The equivalences (i)$\Longleftrightarrow$(iii)$\Longleftrightarrow$(iv) follow from the equalities

$$\begin{aligned}\eta(f) &= \lim_{t\to 0}\frac{1}{t}\int f d\mu_t \\ &= \lim_{t\to 0}\int fd[\frac{1}{t}(\mu_t-\varepsilon_e)] \\ &= \lim_{t\to 0}\frac{1}{t}(\mu_t-\varepsilon_e)*f^-(e) \\ &= A_i f^-(e)\end{aligned}$$

valid for all $f\in\mathscr{C}^b(K)$, $f^-\in D_i$ such that supp $f\subset K^\times (i=0, u)$.

2) In order to see the equivalences (i)$\Longleftrightarrow$(ii)$\Longleftrightarrow$(v) we deduce the subsequent chain of equalities valid for all f, g $\mathscr{K}(K)$, $f\in D_2$ such that supp $(f^-*\bar{g})\subset K^\times$ and observe that by the proof of Theorem 3.1 the set

$$\mathscr{N}:=\{f^-*\bar{g}: f, g\in\mathscr{K}(K),\ f\in D_2 \text{ supp } (f^-*\bar{g})\subset K^\times\}$$

is total in $\mathscr{K}(K^\times)$:

$$\begin{aligned}\eta(f^-*\bar{g}) &= \lim_{t\to 0}\frac{1}{t}\int f^-*\bar{g}d\mu_t \\ &= \lim_{t\to 0}\langle\frac{1}{t}(\mu_t*f-f), g\rangle \\ &= \langle A_2 f, g\rangle \\ &= -\langle\psi\hat{f}, \hat{g}\rangle \\ &= -\lim_{\lambda\to\infty}\langle\frac{\lambda\psi}{\lambda+\psi}\hat{f}, \hat{g}\rangle \\ &= -\lim_{\lambda\to\infty}(\lambda\langle\hat{f}, \hat{g}\rangle-\lambda^2\langle\frac{\hat{f}}{\lambda+\psi}, \hat{g}\rangle) \\ &= -\lim_{\lambda\to\infty}(\lambda f^-*\bar{g}(e)-\lambda^2\rho_\lambda(f^-*\bar{g}))\end{aligned}$$

[by Plancherel's theoren]

$$= -\lim_{\lambda\to\infty}\lambda^2\rho_\lambda(f^-*\bar{g}).$$

4.5 DISCUSSION. We are now going to look at convolution semigroups in $\mathscr{M}^{(1)}(K)$ whose Lévy measures η admit special properties.

(1) If $(A_i, D_i)(i=0, u, 2)$ are bounded operators, then η is bounded and also the negative definite function ψ corresponding to $(\mu_t)_{t\geq 0}$ is bounded. The boundedness of ψ implies that $(\mu_t)_{t\geq 0}$ is in fact a *Poisson semigroup* of the form

$$\mu_t = e^{-m}\exp(t\mu)$$

for $\mu\in\mathscr{M}_+^b(K)$, $m\geq\|\mu\|(t>0)$, and

$$\psi = m-\hat{\mu}.$$

With an extended definition of negative-definiteness (in the strict sense) a proof of this result has been given recently in [15]. An immediate consequence is the fact that on discrete hypergroups K (with compact dual hypergroup $K^{\wedge}$) any convolution semigroup is a Poisson semigroup.

We note that the Lévy measure of a Poisson semigroup (with defining measure $\mu\in\mathscr{M}_{+}^{b}(K)$) is just $\mathrm{Res}_{K^{\times}}\mu$.

If the convolution semigroup $(\mu_t)_{t\geq 0}$ with bounded Lévy measure μ consists of measures in $\mathscr{M}^1(K)$, then $\psi(1)=0$, whence $m=\|\mu\|$.

(2) Let η be symmetric. In this case the negative definite function ψ corresponding to $(\mu_t)_{t\geq 0}$ admits a Lévy-Khintchine representation provided K satisfies Lasser's property (F). More precisely, in [14] the following structural result has been proved: There exist a number $c\geqq 0$, a homomorphism $l: K^{\wedge}\rightarrow \boldsymbol{R}$ and a nonnegative quadratic form $q: K^{\wedge}\rightarrow\boldsymbol{R}$ such that for all $\chi\in K^{\wedge}$

$$\psi(\chi)=c+il(\chi)+q(\chi)+\int_{K^{\times}}(1-Re\chi(x))\mu(dx).$$

The data c, l and q are in fact uniquely determined by $(\mu_t)_{t\geq 0}$ in terms of $c=\psi(1)$, $l=\mathrm{Im}\psi$, and

$$q(\chi)=\lim_{n\to\infty}\left[\frac{\varepsilon_{\chi}^{n}(\psi)}{n^2}+\frac{\varepsilon_{\chi}^{n}*\varepsilon_{\bar{\chi}}^{n}(\psi)}{2n}\right]$$

are valid for all $\chi\in K^{\wedge}$.

In the next section we shall discuss in more detail the case that η vanishes on $K^{\wedge}$.

§ 5. Locality.

As before we are given a convolution semigroup $(\mu_t)_{t\geq 0}$ in $\mathscr{M}^{(1)}(K)$ with generators (A_i, D_i) $(i=, u, 2)$, Lévy measure η and corresponding negative definite funtion ψ admitting at least under the assumption (2) of Section 3, a Lévy-Khintchine representation with data (c, l, q, η).

5.1 DEFINITION. $(\mu_t)_{t\geq 0}$ is said to be of *local type* if for all $f\in D_0$

$$\mathrm{supp}\,(A_0 f)\subset \mathrm{supp}\, f.$$

5.2 REMARK. We know from Theorem 3.3 that in the definition of locality (A_0, D_0) can be replaced by (A_i, D_i) with $i=u$ or 2.

5.3 THEOREM. *The following statements are equivalent*:

(i) $(\mu_t)_{t\geq 0}$ *is of local type.*

(ii) $\lim_{t\to 0}\frac{1}{t}\mu_t(\complement W)=0$

for all $W \in \mathfrak{V}_e(K)$.

(iii) $\eta \equiv 0$.

If, in addition, K satisfies Property (F) and η *is symmetric, then we have also equivalence to*

(iv) *There exist a triplet* (c, l, q) *consisting of a number* $c \geqq 0$, *a homomorphism* $q: K^\wedge \to \boldsymbol{R}$, *and a nonnegative quadratic form* $q: K^\wedge \to \boldsymbol{R}$ *such that*

$$\psi = c + il + q.$$

The proof follows the lines of [3] with the necessary references to those arguments which are nonroutine for hypergroups.

1) (i)$\Longrightarrow$(ii). For a given $W \in \mathfrak{V}_e(K)$ we choose relatively compact $U, V \in \mathfrak{V}_e(K)$ such that $\bar{U} \subset V \subset W^-$. By Corollary 3.2 there exists a function $h_0 \in D_0$ such that

$$\begin{cases} 0 \leqq h_0 \leqq 1, \\ h_0 = 1 \text{ on } U, \text{ and} \\ h_0 = 0 \text{ on } \complement V. \end{cases}$$

whence a function $h := 1 - h_0$

$$\begin{cases} 0 \leqq h \leqq 1, \\ e \notin \operatorname{supp} h \text{ and} \\ 1_{\complement W} \leqq h^-. \end{cases}$$

Clearly

$$\begin{aligned} \lim_{t \to 0} \frac{1}{t}(\mu_t * 1 - 1) &= \lim_{t \to 0} \frac{1}{t}(\exp(-t\psi(\mathbf{1})) - 1) \\ &= -\psi(\mathbf{1}). \end{aligned}$$

From this we deduce that

$$\begin{aligned} \mathcal{T}_{co} - \lim_{t \to 0} \frac{1}{t}(\mu_t * (1 - h_0) - (1 - h_0)) \\ = -\psi(\mathbf{1}) - A_0 h_0. \end{aligned}$$

and so by Corollary 3.4 that

$$\lim_{t \to 0} \frac{1}{t}(\mu_t * h(e) - h(e)) = 0.$$

The desired assertion now follows from

$$0 \leqq \varliminf_{t \to 0} \frac{1}{t} \mu_t(\complement W)$$

$$\leq \overline{\lim_{t\to 0}} \frac{1}{t}\mu_t(\complement W)$$
$$\leq \overline{\lim_{t\to 0}} \int h^- d\left(\frac{1}{t}\mu_t\right)$$
$$= \lim_{t\to 0} \frac{1}{t}(\mu_t * h(e) - h(e)) = 0.$$

2) (ii)$\Longrightarrow$(iii). For $f \in \mathscr{K}_+(K)$ with supp$f \subset K^\times$ we choose a $W \in \mathfrak{V}_e(K)$ such that supp$(f) \cap W = \emptyset$. Then Theorem 4.1 yields

$$0 \leq \int f d\eta$$
$$= \lim_{t\to 0} \int f d\left(\frac{1}{t}\mu_t\right)$$
$$\leq \|f\| \lim_{t\to 0} \frac{1}{t}\mu_t(\complement W) = 0$$

which shows that $\eta = 0$.

3) (iii)$\Longrightarrow$(i). Let $g \in D_0$. We want to show that

$$\operatorname{supp}(A_0 g) \subset \operatorname{supp} g$$

holds. In fact, let $g = 0$ on a neighborhood $V \in \mathfrak{V}_e(K)$. Since $D_0 \subset D_u$, (iv) of Theorem 4.4 is applicable and thus

$$A_0 g(e) = \int g^- d\eta = 0.$$

Moreover, if $g = 0$ on some $V \in \mathfrak{V}_x(K)$, then $g_x \in D_0$ and $g_x = 0$ on some $V' \in \mathfrak{V}_e(K)$, and consequently

$$A_0 g(x) = T^x(A_0 g)(e) = A_0(T^x g)(e) = A_0(g_x)(e) = 0.$$

4) (iii)$\Longleftrightarrow$(iv) is immediate from (2) of Section 4.

The following applications of Theorem 5.3 are proved similarly to the group case (See [3], 18.28 and 18.30). Let $(\mu_t)_{t\geq 0}$ be a transient convolution semigroup in $\mathscr{M}^{(1)}(K)$ with potential kernel $\kappa \in \mathscr{M}_+(K)$.

5.4 Let $(\sigma_V)_{V\in\mathfrak{V}}$ denote a fundamental family associated with κ in the sense of Theorem 2.4. Then

$$A_0 f = \lim_{V\in\mathfrak{V}} a_V(\sigma_V - \varepsilon_e) * f$$

for all $f \in D_0$, and

$$\eta = \mathscr{T}_v - \lim_{V\in\mathfrak{V}} a_V \operatorname{Res}_{K^\times} \sigma_V.$$

Here the $a_V>0$ are chosen such that $a_V(\varkappa-\sigma_V*\varkappa)\in\mathscr{M}^1(K)$ for $V\in\mathfrak{V}$.

5.5 $(\mu_t)_{t\geqq 0}$ is of local type iff there exists a fundamental family $(\sigma_V)_{V\in\mathfrak{V}}$ associated with $\varkappa$ such that supp $\sigma_V\subset V$ for all $V\in\mathfrak{V}$.

5.6 EXAMPLES.

All of the hypergroups appearing in the following examples admit a hypergroup dual and satisfy property (F).

5.6.1 Abelian locally compact groups, in particular the Euclidean groups $\boldsymbol{R}^d$ for $d\geqq 1$.

Convolution semigroups of local type on the Euclidean groups are the Brownian semigroup on $\boldsymbol{R}^d$ which for $d\geqq 3$ is transient and admits a fundamental family associated with the Newton kernel ([3], 17.16), and the heat semigroup on $\boldsymbol{R}^{d+1}$ which is transient for all $d\geqq 1$. In contrast to these examples the symmetric stable semigroup of order $\alpha\in]0, 2[$ on $\boldsymbol{R}^d$ is not of local type; its Lévy measure can be computed to be non-zero ([3], 18.23).

5.6.2 Orbit hypergroups G_B of locally compact groups $G\in[FIA]_B^-$ and (relatively compact) subgroups B of Aut(G) such that $B\supset$Int(G).

We note that this class of hypergroups comprises the orbit hypergroups of compact groups as well as the conjugacy hypergroups (for $B=$Int(G)) of locally compact groups $G\in[FC]^-\cap[SIN]$.

Orbit hypergroups have been discussed in [11]. For the conjugacy hypergroups of compact groups see [12], 8.4B.

In the special case $G:=\boldsymbol{R}^d$ and $B:=SO(d)$ for $d\geqq 1$ we have $G_B\cong G_B^\wedge\cong\boldsymbol{R}_+$. Convolution semigroups on G_B admit Lévy-Khintchine representations which in a more general framework are established in [7]. In [7] also quadratic forms are computed. Convolution semigroups of local type on G_B appear in [13].

5.6.3 Bessel-Kingman hypergroups $(\boldsymbol{R}_+, *_\alpha)$ with defining Bessel convolution $*_\alpha$ where $\alpha\geqq-\frac{1}{2}$, have been treated in [8]. For $\alpha:=\frac{d}{2}-1$ $(d\geqq 1)$ these hypergroups reduce to those of the special case above. Convolution semigroups of local type on $(\boldsymbol{R}_+, *_\alpha)$ appear already in [13].

5.6.4 Jacobi hypergroups $(\boldsymbol{Z}_+, *_{(\alpha,\beta)})$ with $\alpha\geqq\beta\geqq-\frac{1}{2}$. On these hypergroups whose dual hypergroups can be identified with $I:=[-1,1]$ any convolution semigroup $(\mu_t)_{t\geqq 0}$ is a Poisson semigroup whose negative definite function has the form

$$\psi=\sum_{n\geqq 1}(1-R_n^{(\alpha,\beta)})\eta(\{n\}),$$

where η is the Lévy measure of $(\mu_t)_{t\geqq 0}$. Here $(R_n^{\alpha,\beta})_{n\geqq 1}$ denotes the sequence of normed Jacobi polynomials on I which defines the convolution $*_{(\alpha,\beta)}$ in $\boldsymbol{Z}_+$. Clearly, there is no convolution semigroup of local type on $(\boldsymbol{Z}_+, *_{(\alpha,\beta)})$. (See [14]).

5.6.5 Dual Jacobi hypergroups $(I, *_{(\alpha,\beta)})$ with $\alpha\geqq\beta\geq-\frac{1}{2}$. The hypergroup duals of these hypergroups can be identified with $\boldsymbol{Z}_+$. Homomorphisms vanish, quadratic forms q are computed for all $n\in\boldsymbol{Z}_+$ as

$$q(n)=a\frac{n(n+\alpha+\beta+1)}{\alpha+\beta+2}$$

with $a\geqq 0$. Given a convolution semigroup $(\mu_t)_{t\geqq 0}$ on $(I, *_{(\alpha,\beta)})$ its corresponding negative definite function ψ is of the form

$$\psi(n)=c+q(n)+\int_{I_\times}(1-R_n^{(\alpha,\beta)})d\eta$$

valid for all $n\in\boldsymbol{Z}_+$, where $c\geqq 0$ and η is the Lévy measure of $(\mu_t)_{t\geqq 0}$. Convolution semigroups of local type on $(I, *_{(\alpha,\beta)})$ are characterized by negative definite functions of the form $\psi=c+q$. (See [14]).

The special case $(I, *_{(\alpha,\beta)})$ with $\alpha=\frac{d-3}{2}$ covers the double coset hypergroups $SO(d)/\!/SO(d-1)$ corresponding to the (spherical) Gelfand pair $(SO(d), SO(d-1))$, $d\geqq 3$. More generally we add to our list of examples

5.6.6 the compact double coset hypergroups $K:=G/\!/H$ arising from symmetric Riemannian pairs (G, H) of compact type. In this case $K^\wedge$ is a countably discrete hypergroup. Every H-biinvariant negative definite function on G is a negative definite function on K, but not every H-biinvariant function on G which is negative definite on the hypergroup K, is negative definite on G. ([7], Théorème 6.4). Since K in general is not hermitian, the Lévy-Khintchine formula of (2) of Section 4 only applies to convolution semigroups with symmetric Lévy measure. It generalizes the representation of H-invariant negative definite functions on G corresponding to symmetric H-invariant convolution semigroups on G. The representation given in [7] can be applied to characterize local convolution semigroups on K.

5.6.7 Two-variable Jacobi hypergroup $(\boldsymbol{D}, *_\alpha)$ with $\alpha>0$. Its dual hypergroup can be identified with $\boldsymbol{Z}_+^2$. In [1] it has been shown that convolution semigroups on $(\boldsymbol{D}, *_\alpha)$ with symmetric Lévy measure η can be characterized in terms of their negative definite functions ψ given by

$$\psi(m, n)=c+a(m-n)^2+b\left(m+n+\frac{2mn}{\alpha+1}\right)$$
$$\times\int_{D^\times}(1-\tilde{R}^{\alpha}_{m,n}(x, y))\eta(d(x, y))$$

for all $(m, n)\in \boldsymbol{Z}^2_+$, where c, a, $b\geqq 0$ and $(\tilde{R}^{\alpha}_{m,n})_{(m,n)\in Z^2_+}$ denotes the sequence of symmetrized two-variable Jacobi polynomials $\tilde{R}^{\alpha}_{m,n}$ defined by

$$\tilde{R}^{(\alpha)}_{m,n}(x, y):=\tilde{R}^{(\alpha,|m-n|)}_{m\wedge n}(2(x^2+y^2)-1)\sum_{j\leqq\frac{|m-n|}{2}}\binom{|m-n|}{2j}(-1)^j x^{|m-n|-2j}y^{2j}$$

for all $(x, y)\in \boldsymbol{D}$. It turns out that a convolution semigroup on $(\boldsymbol{D}, *_\alpha)$ with corresponding negative definite function $\psi(\geqq 0)$ is of local type iff ψ is of the form $\psi=c+q$, where

$$q(m, n):=a(m-n)^2+b\left(m+n+\frac{2mn}{\alpha+1}\right)$$

for all $(m, n)\in \boldsymbol{Z}^2_+$ defines a quadratic form on $\boldsymbol{Z}^2_+$. For an interpretation of the two summands in the representation of q see [16].

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