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p-uniform measures on linear spaces ($0\leqq p<\infty$)

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§ 1. Introduction and preliminaries

Let E be a locally convex Hausdorff space and E' be the topological dual space. We denote by $C(E, E')$ the minimal σ-algebra of subsets of E that makes all elements x' in E' measurable. Let μ be a probability measure on $C(E, E')$. Then $L^{0}(\mu)$ denotes the set of all μ-measurable functions on E; here two measurable functions are identified if they agree μ-almost everywhere (μ-a. e.). As usual we put on $L^{0}(\mu)$ the topology of convergence in measure, and then $L^{0}(\mu)$ is a complete linear metric space. Let $0<p<\infty$. We also denote by $L^{p}(\mu)$ the set of all measurable functions on E having p-integrable absolute value; $L^{p}(\mu)$ is a complete quasi-normed space (a Banach space for $p\geqq 1$), and the identity map: $L^{p}(\mu)\rightarrow L^{0}(\mu)$ is continuous.

Let $R_{\mu}: E'\rightarrow L^{0}(\mu)$ be the canonical map defined by $(R_{\mu}x')(x)=\langle x, x'\rangle\in L^{0}(\mu)$. If μ is of weak r-th order ($0\leqq r<\infty$), that is, $R_{\mu}(E')\subset L^{r}(\mu)$, then the r-measurable dual $\Lambda_{r}(\mu)$ of E is defined as the closure of $R_{\mu}(E')$ in $L^{r}(\mu)$, and the vector topology τ_{μ}^{r} on E' is also defined as the inverse image of the $L^{r}(\mu)$-topology under R_{μ}. Since $\Lambda_{r}(\mu)$ is a closed linear subspace of $L^{r}(\mu)$, it is a complete linear metric space (a Banach space for $r\geqq 1$).

Let μ and ν be two probability measures on $C(E, E')$ of weak p-th order, $0\leqq p<\infty$. After Dudley [7], we say μ is p-subordinate to ν (denoted by $\mu\ s_{p}\ \nu$) if the identity $(E', \tau_{\nu}^{p})\rightarrow(E', \tau_{\mu}^{p})$ is continuous, that is, τ_{ν}^{p} is finer than τ_{μ}^{p}.

For each A in $C(E, E')$ with $\mu(A)>0$, μ_{A} denotes a probability measure on $C(E, E')$ defined by $\mu_{A}(B)=\mu(A\cap B)/\mu(A)$. Let μ be of weak p-th order, $0\leqq p<\infty$. We say μ is p-uniform if $\mu\ s_{p}\ \mu_{A}$ whenever $\mu(A)>0$.

The notion of p-uniformness was first introduced by Dudley [7], and 0-uniform measures (called simply uniform measures) were mainly investigated connecting with the absolute continuity of measures. But the case $p>0$ was not considered further since the p-uniformness is not compatible with the absolute continuity.

In [32], the authors studied 0-uniform measures in terms of 0-1 laws. In this paper, we shall study the p-uniformness for $0<p<\infty$, and give characterizations of the p-uniformness.

In Secticn 2, we characterize the p-uniformness by a 0-1 law. It is shown that μ is p-uniform if and only if for each sequence $\{x'_n\}$ in E', $\mu(x; \sum_n|x'_n(x)|^p<\infty)=1$ or 0 according as the series $\sum_n\|R_\mu x'_n\|_p^p$ converges or not; here $\|\cdot\|_p$ denotes the $L^p(\mu)$-norm (quasi-norm for $p<1$). The p-uniformness is also characterized as follows; μ is p-uniform if and only if μ is 0-uniform and $\Lambda_0(\mu)\subset L^p(\mu)$. In particular, Gaussian measures are p-uniform for every $p\geqq 0$, r-stable measures are p-uniform $(0\leqq p<r\leqq 2)$, and s-convex measures are p-uniform $(0\leqq p<-1/s,\ -\infty<s\leqq 0)$.

In section 3, we investigate the Banach spaces $\Lambda_p(\mu)$, $1\leqq p<\infty$. Suppose that μ is 0-uniform and of weak p-th order. Then it is shown that μ is p-uniform $(p\neq 2)$ if and only if $\Lambda_p(\mu)$ contains no subspace isomorphic to l^p. The p-uniformness is also characterized as follows; for $1\leqq p<2$, μ is p-uniform if and only if $\Lambda_p(\mu)$ is of type p-stable, and for $p>2$, μ is p-uniform if and only if $\Lambda_p(\mu)$ is isomorphic to a Hilbert space.

In Section 4, we generalize a result of Chevet [3], Chobanjan and Tarieladze [4] and Maurey [15] on the structure of Gaussian measures on cotype 2 spaces. Suppose that E is a Banach space of cotype 2. It is shown that if E has the G. L. P. (Gordon-Lewis property), then every 2-uniform Radon probability measure on E has a Hilbertian support; the same is true if E is an S-space, as remarked by Mushtari [18]. Let us mention that every S-space imbeds in some L^0-space, but there are cotype 2 spaces not contained in any L^0-space, see [18]. Let $1\leqq p<2$ and suppose that E is of type p-stable. Then it is also shown that every p-uniform Radon probability measure on E has a Hilbertian support if and only if E is isomorphic to a Hilbert space. Remark that if every Radon probability measure on E has a Hilbertian support, then E is isomorphic to a Hilbert space, see Sato [27].

Throughout the paper, we assume that all linear spaces are with real coefficients.

§ 2. Characterization of p-uniformness $(0<p<\infty)$

Let E be a locally convex Hausdorff space, μ be a probability measure on $C(E, E')$ of weak p-th order, and R_μ: $E'\to L^p(\mu)$ be the canonical map. First we characterize the p-uniformness by the 0-1 laws.

THEOREM 1. *Let $0<p<\infty$. Then the following conditions are equivalent.*

(1) *μ is p-uniform.*

(2) *For each sequence $\{x'_n\}$ in E', $\mu(x; \sum_n|x'_n(x)|^p<\infty)=0$ or 1 according as $\sum_n\|R_\mu x'_n\|_p^p=\infty$ or $<\infty$.*

(3) *For each sequence* $\{x'_n\}$ *in* E', $\mu(x\,;\,x'_n(x)\to 0)>0$ *implies that* $\sum_j\|R_\mu x'_{n_j}\|_p^p<\infty$ *for a suitable subsequence* $\{x'_{n_j}\}$ *of* $\{x'_n\}$.

(4) *For each sequence* $\{x'_n\}$ *in* E', $\mu(x\,;\,x'_n(x)\to 0)>0$ *implies that* $x'_{n_j}\to 0$ *in* τ_μ^p *for a suitable subsequence* $\{x'_{n_j}\}$.

PROOF. (1)$\Longrightarrow$(2): Suppose that μ is p-uniform and let $\mu(x\,;\,\sum_n|x'_n(x)|^p<\infty)>0$. Then there exists $C>0$ such that $\mu(x\,;\,\sum_n|x'_n(x)|^p\leqq C)>0$. We set $A=\{x\,;\,\sum_n|x'_n(x)|^p\leqq C\}$. Since μ is p-uniform, $\tau_{\mu_A}^p$ is finer than τ_μ^p, and so there exists $D>0$ such that

$$\int_E|x'(x)|^p d\mu(x)\leqq D\int_A|x'(x)|^p d\mu(x) \quad \text{for all } x'\in E'.$$

Thus we have

$$\begin{aligned}\sum_n\|R_\mu x'_n\|_p^p&=\int_E\left(\sum_n|x'_n(x)|^p\right)d\mu(x)\\&\leqq D\int_A\left(\sum_n|x'_n(x)|^p\right)d\mu(x)\\&\leqq D\cdot C<\infty,\end{aligned}$$

which proves (2).

(2)$\Longrightarrow$(3): Suppose that (2) holds and let $\mu(x\,;\,x'_n(x)\to 0)>0$. If we set $A=\{x\,;\,x'_n(x)\to 0\}$, then by the Egorov's theorem, there exist a subset B of A with $\mu(B)>0$, and a subsequence $\{x'_{n_j}\}$ such that for each j, $|x'_{n_j}(x)|^p\leqq 2^{-j}$ for all $x\in B$. Since $\mu(x\,;\,\sum_j|x'_{n_j}(x)|^p<\infty)>0$, by assumption (2) we have $\sum_j\|R_\mu x'_{n_j}\|_p^p<\infty$, proving (3).

(3)$\Longrightarrow$(4) is clear, and (4)$\Longrightarrow$(1) follows from the fact that for each A in $C(E, E')$ with $\mu(A)>0$, $x'_n\to 0$ in $\tau_{\mu_A}^p$ implis $x'_{n_j}(x)\to 0$ μ_A-a.e. for a suitable subsequence $\{x'_{n_j}\}$.

This completes the proof.

COROLLARY 1. *If* μ *is* p*-uniform, then for each sequence* $\{x'_n\}$ *in* E', $\mu(x\,;\,\sum_n|x'_n(x)|^p<\infty)=0$ *or* 1. *In particular, every* p*-uniform measure is* 0*-uniform.*

REMARK 1. The measure μ is 0-uniform if and only if for every fixed $p\in(0,\infty)$, $\mu(x\,;\,x'_n(x)\to 0)>0$ implies that $\mu(x\,;\,\sum_j|x'_{n_j}(x)|^p<\infty)=1$ for a suitable subsequence $\{x'_{n_j}\}$, see Takahashi and Okazaki [32].

COROLLARY 2. *Let* $\{a_n\}$ *be a real sequence such that* $\sum_n|a_n|^p<\infty$. *If* μ *is* p*-uniform, then for each sequence* $\{x'_n\}$ *in* E', $\mu(x\,;\,a_n^{-1}x'_n(x)\to 0)>0$ *implies* $\mu(x\,;\,x'_n(x)\to 0)=1$.

PROOF. Let us set $A=\{x\,;\,a_n^{-1}x'_n(x)\to 0\}$. Then we have $\mu(x\,;$

$\sum_n|x'_n(x)|^p<\infty)\geqq\mu(A)>0$. Thus the assertion follows from Corollary 1.

Now we shall give examples of p-uniform measures. Let μ be a Radon probability measure on E. For a real number α $(\alpha\neq 0)$, $T_\alpha\mu$ denotes the measure on E defined by $T_\alpha\mu(A)=\mu(\alpha^{-1}A)$ for all Borel sets A of E.

The measure μ is said to be stable if for each $\alpha>0$, $\beta>0$, there exist $\gamma>0$ and $x\in E$ such that $(T_\alpha\mu)*(T_\beta\mu)=(T_\gamma\mu)*\delta_x$, where δ_x denotes the Dirac measure concentrated at x. In particular, μ is said to be p-stable $(0<p\leqq 2)$ if for each $\alpha>0$, $\beta>0$, the choice $\gamma=(\alpha^p+\beta^p)^{1/p}$ is possible. Of course, every Gaussian measure is 2-stable.

Let ψ be a μ-measurable seminorm on E (not necessarily everywhere finite) and suppose that $\mu(x;\psi(x)<\infty)>0$. As well known, if μ is a Gaussian measure, then $\psi\in L^p(\mu)$ for every $p>0$; and if μ is an r-stable measure, $0<r\leqq 2$, then $\psi\in L^p(\mu)$ for every $p\in(0, r)$, see Acosta [1]. It is also known that if μ is an s-convex measure, $-\infty<s\leqq 0$, then $\psi\in L^p(\mu)$ for every $p\in(0, -1/s)$, see Borell [2]. We shall see that these measures are p-uniform.

Suppose that μ is a Radon probability measure on E such that for each measurable seminorm ψ on E, $\mu(x;\psi(x)<\infty)>0$ implies $\psi\in L^p(\mu)$. Then μ is p-uniform. In fact, let $\{x'_n\}$ be any sequence in E' such that $\mu(x; x'_n(x)\to 0)>0$. Taking a subsequence, we may assume $\mu(x; \sum_n|x'_n(x)|<\infty)>0$. If we put $\psi(x)=\sum_n|x'_n(x)|$ for $x\in E$, then ψ is a measurable seminorm on E, and so by assumption, it follows that $\psi\in L^p(\mu)$. But this implies $\psi<\infty$ μ-a. e., and in particular, $x'_n\to 0$ μ-a. e. From the Lebesgue's dominated convergence theorem we deduce that $x'_n\to 0$ in τ_μ^p. Hence μ is p-uniform, see Theorem 1, (4).

Thus we have the following examples:

EXAMPLE 1. Every Gaussian measure is p-uniform for all $p>0$.

EXAMPLE 2. Every r-stable measure, $0<r\leqq 2$, is p-uniform for all $p\in(0, r)$.

EXAMPLE 3. Every s-convex measure, $-\infty<s\leqq 0$, is p-uniform for all $p\in(0, -1/s)$.

The p-uniformness is also characterized by the equivalence of the topologies τ_μ^0 and τ_μ^p. By the same way as in the proof of Theorem 1, we have

THEOREM 2. *Let $0<p<\infty$ and suppose that μ is of weak p-th order. Then μ is p-uniform if and only if μ is 0-uniform and the topologies τ_μ^0 and τ_μ^p on E' are equivalent.*

COROLLARY 3. *Let $0\leqq q\leqq p<\infty$. Then every p-uniform measure is* *q-uniform.*

As mentioned before, $\Lambda_0(\mu)$ is the closure of $R_\mu(E')$ in $L^0(\mu)$. Each element in $\Lambda_0(\mu)$ is called a μ-measurable linear functional. The p-uniformness is characterized by the integrability of μ-measurable linear functionals.

THEOREM 3. *Let* $0<\mathrm{p}<\infty$. Then μ is p-uniform if and only if μ is 0-*uniform and* $\Lambda_0(\mu) \subset L^p(\mu)$.

PROOF. By Theorem 2, it suffices to show that the condition $\Lambda_0(\mu) \subset L^p(\mu)$ implies that τ_μ^0 is finer than τ_μ^p. Suppose that the inclusion $\Lambda_0(\mu) \subset L^p(\mu)$ holds. Since the identity map $L^p(\mu) \to L^0(\mu)$ is continuous and $\Lambda_0(\mu)$ is a closed subspace of $L^0(\mu)$, the inclusion map $\Lambda_0(\mu) \to L^p(\mu)$ has the closed graph, and so it is continuous by the closed graph theorem, see Yosida [35, Ch. II, 6]. Thus we have the assertion.

Using the above theorem, we shall give examples of p-uniform product measures on R^∞. Let R^∞ be the countable product of the real numbers R with the product topology. Let $\{\mu_n\}$ be a sequence of probability measures on R and $\mu = \otimes \mu_n$ be the product measure on R^∞. In the following, we assume that for each n, μ_n is symmetric and has no atom. It is well known that μ satisfies the 0-1 law for measurable subspaces, see Hoffmann-Jϕrgensen [12, Theorem 3. 1], and hence μ is 0-uniform, see [32]. Thus μ is p-uniform if and only if $\Lambda_0(\mu) \subset L^p(\mu)$, see Theorem 3.

EXAMPLE 4. Suppose that each μ_n is symmetric and has no atom. If $\sup_n \int |t|^2 d\mu_n(t) < \infty$, then $\mu = \otimes \mu_n$ is 2-uniform. In fact, by [12, Theorem 4. 9], $\Lambda_0(\mu) \subset L^2(\mu)$ holds.

EXAMPLE 5. Suppose that each μ_n is symmetric and has no atom. Moreover, if we assume that the following conditions

(1) $\sup_n \mu_n([-a, a]) < 1$ for a suitable $a > 0$, and

(2) $\int \|x\|_\infty^p d\mu(x) < \infty$, where $\|x\|_\infty = \sup_n |x_n|$,

hold, then $\mu = \otimes \mu_n$ is p-uniform.

PROOF. If suffices to show that each ξ in $\Lambda_0(\mu)$ is p-integrable. Since each μ_n is symmetric and has no atom, each ξ in $\Lambda_0(\mu)$ is represented as

$$\xi(x) = \textstyle\sum_n a_n x_n, \qquad x = (x_n) \in R^\infty,$$

where the infinite sum converges μ-a. e., see [12, Theorem 4. 3]. By the three series theorem of Kolmogorov, it follows that

$$\sum_n \mu(x \in R^\infty ; |a_n x_n| \geqq 1) < \infty,$$

and in particular, $\mu_n(t ; |a_n t| \geqq 1) \to 0$. Hence the sequence $\{a_n\}$ must be bounded by assumption (1). It follows from assumption (3) that $\sup_n |a_n x_n| \in L^p(\mu)$, and so we get $\xi \in L^p(\mu)$, see Hoffmann-Jørgensen [11, Corollary 3.3]. Thus μ is p-uniform.

Finally we characterize the p-uniformness by the absolute continuity or the non-singularity.

Let μ and ν be two probability measures on $C(E, E')$, μ is said to be absolutely continuous with respect to ν ($\mu < \nu$) if for each A in $C(E, E')$, $\nu(A)=0$ implies $\mu(A)=0$. If μ and ν are mutually absolutely continuous, then we say μ and ν are equivalent ($\mu \sim \nu$). The measures μ and ν are said to be singular if there exists a measurable set A such that $\nu(A)=1$ and $\mu(A)=0$.

THEOREM 4. *Let $0<p<\infty$ and suppose that μ is of weak p-th order. Then the following conditions are equivalent.*

(1) *μ is p-uniform.*

(2) *For each probability measure ν such that μ and ν are not singular, τ_ν^0 is finer than τ_μ^p.*

(3) *For each probability measure $\nu < \mu$, τ_ν^0 and τ_μ^p are equivalent.*

PROOF. (1)$\Longrightarrow$(2): Suppose that μ is p-uniform, and the measures μ and ν are not singular. Let $\{x_n'\}$ be a sequence in E' such that $x_n' \to 0$ in τ_ν^0. Taking a subsequence, we may assume $x_n'(x) \to 0$ ν-a.e. Since μ and ν are not singular, it follows that $\mu(x ; x_n'(x) \to 0) > 0$, and hence $x_n' \to 0$ in τ_μ^p by the p-uniformness, see Theorem 1.

(2)$\Longrightarrow$(3) and (3)$\Longrightarrow$(1) are clear.

This completes the proof.

COROLLARY 4. *Let μ and ν be p-uniform measures on $C(E, E')$. Then μ and ν are singular, or τ_μ^p and τ_ν^p are equivalent.*

§ 3. The spaces $\Lambda_p(\mu)$

Let E be a locally convex Hausdorff space and μ be a probability measure on $C(E, E')$ of weak p-th order, $0<p<\infty$. As mentiond in Section 1, the p-measurable dual $\Lambda_p(\mu)$ is the closure of $R_\mu(E')$ in $L^p(\mu)$, where $R_\mu: E' \to L^p(\mu)$ is the canonical map. Since $\Lambda_p(\mu)$ is a closed subspace of $L^p(\mu)$, it is a complete quasi-normed space (Banach space for $p \geqq 1$) with the quasi-norm $\|\cdot\|_p$. If $0 \leqq q < p$, then we have the inclusion $\Lambda_p(\mu) \subset \Lambda_q(\mu)$, but in general, the converse inclusion is not valid.

THEOREM 5. Let $0<p<\infty$ *and suppose that* μ *is of weak p-th order. Then* μ *is p-unifom if and only if* μ *is* 0*-uniform and* $\Lambda_p(\mu)=\Lambda_q(\mu)$ *for some* $q\in[0, p)$.

PROOF. If $\Lambda_p(\mu)=\Lambda_q(\mu)$ for some $q<p$, then by the closed graph theorem, the topologies L^p and L^q on $R_\mu(E')$ are equivalent, and hence the topologies L^p and L^0 on $R_\mu(E')$ are also equivalent, see Schwartz [29, Lemma 15. 1]. But this means that τ_μ^p and τ_μ^0 on E' are equivalent. Thus the assertion follows from Theorem 2.

COROLLARY 5. *Let* $0<p<\infty$ *and suppose that* $R_\mu(E')$ *is of finite dimension. Then* μ *is p-uniform if and only if* μ *is* 0*-uniform and of weak p-th order.*

Of course if $R_\mu(E')$ is of infinite dimension, then the above result is false. In the following we shall consider such cases.

From now on we assume that μ is of weak p-th order and $\Lambda_p(\mu)$ is an infinite dimensional Banach space, where $1\leqq p<\infty$. Let us denote by $\|\cdot\|_p$ the usual L^p-norm.

Following Kadec and Pelczyński [13], for each $\varepsilon>0$, we set

$$M_\varepsilon^p=\{f\in L^p(\mu)\ ;\ \mu(x\ ;\ |f(x)|\geqq\varepsilon\|f\|_p)\geqq\varepsilon\}.$$

LEMMA 1. *The following conditions are equivalent.*

(1) *The topologies* L^p *and* L^0 *on* $\Lambda_p(\mu)$ *are equivalent.*

(2) $\Lambda_p(\mu)\subset M_\varepsilon^p$ *for some* $\varepsilon>0$.

PROOF. (1)$\Longrightarrow$(2): Suppose that (1) holds. For $\varepsilon>0$, we set

$$V_\varepsilon=\{f\in\Lambda_p(\mu)\ ;\ \mu(x\ ;\ |f(x)|\geqq\varepsilon)<\varepsilon\}.$$

Then by assumption (1), there exists an $\varepsilon>0$ such that $V_\varepsilon\subset\{f\in\Lambda_p(\mu)\ ;\ \|f\|_p<1\}$. We show $\Lambda_p(\mu)\subset M_\varepsilon^p$. In fact, let $f\in\Lambda_p(\mu)$ and put $g=f/\|f\|_p$. Since $\|g\|_p=1$, V_ε does not contain g, that is,

$$\mu(x\ ;\ |g(x)|\geqq\varepsilon)\geqq\varepsilon.$$

But this means that $f\in M_\varepsilon^p$.

(2)$\Longrightarrow$(1): If $\Lambda_p(\mu)\subset M_\varepsilon^p$ for some $\varepsilon>0$, then it is easy to see that $f\in V_\varepsilon$ implies $\|f\|_p\leqq 1$. Hence the topologies L^p and L^0 on $\Lambda_p(\mu)$ are equivalent.

This completes the proof.

PROPOSITION 1. *Let* $1\leqq p<\infty$ $(p\neq 2)$ *and suppose that* μ *is of weak p-th order. If* $\Lambda_p(\mu)$ *contains no subspace isomorphic to* l^p, *then it holds* $\Lambda_p(\mu)=\Lambda_0(\mu)$.

PROOF. If $\Lambda_p(\mu)$ does not contain l^p, then by Kadec and Pelczyński [13, Theorem 2], there exists an $\varepsilon>0$ such that $\Lambda_p(\mu)\subset M_\varepsilon^p$. It follows from Lemma 1 that the topologies L^p and L^0 on $\Lambda_p(\mu)$ are equivalent, and so we have $\Lambda_p(\mu)=\Lambda_0(\mu)$.

REMARK 2. For the case $p=2$, $\Lambda_2(\mu)$ is a Hilbert space, and hence it always contains a subspace isomorphic to l^2. For the case $p>2$, it is known that $\Lambda_p(\mu)$ contains no subspace isomorphic to l^p if and only if it is isomorphic to a Hilbert space; and if and only if $\Lambda_p(\mu)\subset M_\varepsilon^p$ for some $\varepsilon>0$, see [13, Theorem 3].

As mentioned above, for the case $p>2$, $\Lambda_p(\mu)$ contains no subspace isomorphic to l^p if and only if $\Lambda_p(\mu)=\Lambda_0(\mu)$. In the following we show this is also true for the case $1\leqq p<2$.

Two Banach spaces X and Y are said to be λ-isomorphic, $1<\lambda<\infty$, if there exists an isomorphism $S: X\rightarrow Y$ such that $\|S\|\cdot\|S^{-1}\|\leqq\lambda$. We say that Y is finitely representable in X if for some $\lambda>1$, each finite dimensional subspace of Y is λ-isomorphic to a suitable subspace of X.

Let $0<p\leqq 2$ and denote by $\{\theta_n^{(p)}\}$ a sequence of independent identically distributed real random variables with the characteristic function $\exp(-|t|^p)$, $t\in R$. We say that a Banach space X is of type p-stable if for each sequence $\{x_n\}$ in X such that $\sum_n\|x_n\|^p<\infty$, the series $\sum_n x_n\theta_n^{(p)}$ converges almost surely (a. s.). Let $\{\varepsilon_n\}$ be the Bernoulli sequence. We say that X is of cotype q, $2\leqq q<\infty$, if the a. s. convergence of $\sum_n x_n\varepsilon_n$ implies $\sum_n\|x_n\|^q<\infty$. Here we list the well-known facts which are used in the ensuing discussion: For $1\leqq p<2$, X is of type p-stable if and only if l^p is not finitely representable in X; in particular l^p is not of type p-stable (this is false for $p=2$). Type interval is open, that is, type p-stable with $p<2$ implies type r-stable for some $r>p$. For the duality of type and cotype, type p-stable implies cotype q, where $1/p+1/q=1$; the converse is false (l^q has cotype q, but not cotype r if $2\leqq r<q$). For more information on type and cotype, we refer to [15], [16], [17], [22] and [29].

THEOREM 6. *Let $1\leqq p<2$ and suppose that μ is 0-uniform and of weak p-th order. Then the following conditions are equivalent.*

(1) *μ is p-uniform.*

(2) *$\Lambda_p(\mu)$ is of type p-stable.*

(3) *There exist a $q\in(p, 2]$ and an integrable function ϕ on E with $\phi(x)>0$ (μ-a. e.) such that*

$$\int_E|f(x)|^q\phi(x)^{1-q/p}d\mu(x)<\infty \text{ for all } f\in\Lambda_p(\mu).$$

(4) *There exist a* $q \in (p, 2]$ *and a q-uniform measure* ν *on* $C(E, E')$ *such that* $\Lambda_p(\mu)$ *is isomorphic to a subspace of* $L^q(\nu)$.

(5) $\Lambda_p(\mu)$ *contains no subspace isomorphic to* l^p.

PROOF. (1)$\Longrightarrow$(2) : Suppose that μ is p-uniform. Then by Theorem 2, the topologies L^p and L^0 on $\Lambda_p(\mu)$ are equivalent. Hence $\Lambda_p(\mu)$ is of type p-stable, see Maurey [16, Théorème 98].

(2)$\Longrightarrow$(3) : Suppose that $\Lambda_p(\mu)$ is of type p-stable. Then $\Lambda_p(\mu)$ does not contain l^p.
By Rosenthal [26, Theorem 8], there exists an $r > p$ such that $\Lambda_p(\mu)$ is isomorphic to a subspace of $L^r(\mu)$. Let $p < q < r$. We use the Maurey's factorization theorem for the identity map: $\Lambda_p(\mu) \rightarrow L^p(\mu)$. By Maurey [16, Théorèmes 8 and 50], there exists a measurable function g on E with $g \in L^s(\mu)$ such that

$$\int_E |f(x)/g(x)|^q d\mu(x) < \infty \text{ for all } f \in \Lambda_p(\mu).$$

where $1/p = 1/q + 1/s$. Here we may assume $|g(x)| > 0$ μ-a. e. If we put $\phi = |g|^s$, then ϕ satisfies the condition (3).

(3)$\Longrightarrow$(4) : Suppose that (3) holds. We may assume $\int \phi d\mu = 1$. Let $d\sigma = \phi d\mu$, and define a linear isometry $V : \Lambda_p(\mu) \rightarrow L^p(\mu)$ by $V(f) = f/\phi^{1/p}$. Since

$$\int |Vf|^q d\sigma = \int |f|^q \phi^{1-q/p} d\mu < \infty \text{ for all } f \in \Lambda_p(\mu),$$

we have $V(\Lambda_p(\mu)) \subset L^q(\sigma)$, and hence by the closed graph theorem, V : $\Lambda_p(\mu) \rightarrow L^q(\sigma)$ is continuous. Thus the topologies $L^q(\sigma)$ and $L^p(\sigma)$ are equivalent on $V(\Lambda_p(\mu))$, and so the topologies $L^q(\sigma)$ and $L^0(\sigma)$ are also equivalent on $V(\Lambda_p(\mu))$, see [29, Lemma 15, 1]. Let $\psi = \min(1, \phi^{1-q/p})$, and define a probability measure ν on $C(E, E')$ by $d\nu = C\psi d\mu$, where C is a normalized constant. Then μ and ν are clearly equivalent. We show ν is q-uniform. In fact, ν is 0-uniform since μ is 0-uniform. To prove the q-uniformness, it suffices to show that the topologies $L^0(\nu)$ and $L^q(\nu)$ are equivalent on $\Lambda_p(\mu)$. Let $\{f_n\}$ be a sequence in $\Lambda_p(\mu)$ such that $f_n \rightarrow 0$ in $L^0(\nu)$. Then it clearly holds that $V(f_n) \rightarrow 0$ in $L^0(\sigma)$. Since the topologies $L^0(\sigma)$ and $L^q(\sigma)$ are equivalent on $V(\Lambda_p(\mu))$, we have $V(f_n) \rightarrow 0$ in $L^q(\sigma)$. But the inequality

$$\|f\|_{L^q(\nu)} \leqq C \|V(f)\|_{L^q(\sigma)} \qquad \text{for all } f \in \Lambda_p(\mu)$$

clearly holds, and hence $f_n \rightarrow 0$ in $L^q(\nu)$, which proves the q-uniformness.

By the same way as above, we can show that the topologies $L^p(\mu)$ and $L^q(\nu)$ are equivalent on $\Lambda_p(\mu)$, and so $\Lambda_p(\mu)$ is isomorphic to a subspace of $L^q(\nu)$.

Since L^q does not contain l^p with $1 \leqq p < q \leqq 2$, (4)$\Longrightarrow$(5) holds, and (5)$\Longrightarrow$(1) follows from Theorem 5 and Proposition 1.

This completes the proof.

By the same way as in the proof of Theorem 6, we have

THEOREM 7. *Let $1 \leqq p < q < 2$ and suppose that μ is p-uniform. Then the following conditions are equivalent.*

(1) *$\Lambda_p(\mu)$ is of type q-stable.*

(2) *There exists a q-uniform measure on $C(E, E')$ which is equivalent to μ.*

REMARK 3. In Theorem 7, if $\Lambda_p(\mu)$ is not of type q-stable, then for each q-uniform measure ν, μ and ν are singular, see Corollay 4.

Finally we shall consider the case $2 < p < \infty$.

THEOREM 8. *Let $2 < p < \infty$ and suppose that μ is 0-uniform and of weak p-th order. Then the following conditions are equivalent.*

(1) *μ is p-uniform.*

(2) *The dual space $\Lambda_p(\mu)'$ is of type p'-stable, where $1/p + 1/p' = 1$.*

(3) *$\Lambda_p(\mu)$ contains no subspace isomorphic to l^p.*

PROOF. (1)$\Longrightarrow$(2): If μ is p-uniform, then by Theorem 5, $\Lambda_p(\mu)$ is isomorphic to a Hilbert space $\Lambda_2(\mu)$. Hence (2) clearly holds.

(2)$\Longrightarrow$(3): If $\Lambda_p(\mu)'$ is of type p'-stable, then it is of type r-stable for some $r > p'$, since type interval is open, see [29, Theorem 12.7]. By the duality of type and cotype, it follows that $\Lambda_p(\mu)$ is of cotype r', where $1/r + 1/r' = 1$. Since l^p is not of cotype r' with $r' < p$, (3) holds.

(3)$\Longrightarrow$(1) follows from Theorem 5 and Proposition 1.

This completes the proof.

By Theorems 6 and 8, we have

THEOREM 9. *Let $1 \leqq p < \infty$ $(p \neq 2)$ and suppose that μ is 0-uniform and of weak p-th order. Then μ is p-uniform if and only if $\Lambda_p(\mu)$ contains no subspace isomorphic to l^p.*

§ 4. Hilbertian support

Throughout this section, we assume that E is a Banach space and μ is a Radon probability measure on E; i.e. for each $\varepsilon > 0$, there exists a compact subset K of E such that $\mu(K) > 1 - \varepsilon$.

We say that μ has a Hilbertian support if there exists a continuous linear injective map T from a Hilbert space H into E such that $\mu(T(H))=1$. It is well-known that E is of cotype 2 if and only if every Gaussian Radon measure on E has a Hilbertian support, see Chevet [3], Chobanjan and Tarieladze [4] and Maurey [15]. We shall extend this result to p-uniform measures $(2 \leqq p < \infty)$.

Let ν be a cylindrical measure on E and $L_\nu : E' \to L^0(\Omega, P)$ be a random linear functional (r. l. f.) associated with ν, where (Ω, P) is a probability space. It is well-known that if L is any linear map from E' into $L^0(\Omega, P)$, then there exists a cylindrical measure ν on E such that $L_\nu = L$; ν is uniquely determined, see Dudley [8]. The cylindrical measure ν is said to be of type p, $0 \leqq p < \infty$, if ν is of weak p-th order and an r. l. f. $L_\nu : E' \to L^p(\Omega, P)$ is continuous ; here we regard E' a Banach space.

Let T be a continuous linear operator from E into a Banach space F. Following Schwartz [28], T is p-radonifying if for each cylindrical measure ν on E of type p, the image $T(\nu)$ is a Radon measure on F ; in this case $\mu = T(\nu)$ is of order p, i. e. $\int \|x\|^p d\mu(x) < \infty$.

The operator $T : E \to F$ is said to be p-absolutely summing (p-summing), $0 < p < \infty$, if for each sequence $\{x_n\}$ in E such that $\sum_n |\langle x_n, x' \rangle|^p < \infty$ for all $x' \in E'$, $\sum_n \|Tx_n\|^p < \infty$. For $p=1$, we say "absolutely summing" instead of "1-absolutely summing". For the details of p-summing operators, we refer to Pietsch [20].

The relationship between p-radonifying and p-summing operators was studied by Schwartz [28], [29]. We only mention that every p-radonifying operator is p-summing, and the converse is true if $p > 1$.

After Gordon and Lewis [10], we say that E has G. L. P. (Gordon-Lewis property) if every absolutely summing operator from E into any Banach space factors through some L^1-space. Gordon and Lewis [10] proved that if E has local unconditional structure, then E has G. L. P. It is known that E has G. L. P. if and only if E' has it ; and if E is of cotype 2 and has G. L. P., then every closed subspace of E has G. L. P., see Pisier [21]. In particular, every closed subspace of L^1 has G. L. P.

Let μ be a Radon probability measure on E of weak second order, and $R_\mu : E' \to L^2(\mu)$ be the canonical map. As mentioned before, $\Lambda_2(\mu)$ is the closure of $R_\mu(E')$ in $L^2(\mu)$. Now suppose that the topologies L^2 and L^0 are equivalent on $\Lambda_2(\mu)$, that is, $\Lambda_2(\mu) = \Lambda_0(\mu)$. Since μ is Radon, $R_\mu : E' \to \Lambda_2(\mu)$ is continuous with respect to the Mackey-topology $\tau_k(E', E)$. Hence the dual map R'_μ is a continuous linear map from $\Lambda_2(\mu)'$ into E.

LEMMA 2. *The canonical map R_μ: $E'\to\Lambda_2(\mu)$ is p-summing for all* $p>0$.

PROOF. Since $\Lambda_2(\mu)=\Lambda_0(\mu)$, the assertion follows from Okazaki and Takahashi [19, Lemma 1].

THEOREM 10. *Let μ be a Radon probability measure on E such that $\Lambda_2(\mu)=\Lambda_0(\mu)$. If E is of cotype 2 and has G. L. P., then μ has a Hilbertian support.*

PROOF. Let J : $\Lambda_2(\mu)\to L^0(\mu)$ be the identity map and ν be a cylindrical measure on $\Lambda_2(\mu)'$ such that $L_\nu=J$. Then ν is clearly of type 2, and $\mu=R'_\mu(\nu)$. Since E' has G. L. P., by Lemma 2, the map R_μ: $E'\to\Lambda_2(\mu)$ is factorized by continuous linear operators S: $E'\to L^1$ and T : $L^1\to\Lambda_2(\mu)$. Since E is of cotype 2, by [6, Proposition 2. 1] and [16, Corollaire 75], S': $L^\infty\to E''$ is 2-summing, and so is R'_μ: $\Lambda_2(\mu)'\to E$. By the factorization theorem of Pietsch [20], there exist a Hilbert space H, a 2-summing map V : $\Lambda_2(\mu)'\to H$ and a continuous linear map W : $H\to E$ such that $R'_\mu=WV$. Since the cylindrical measure ν on $\Lambda_2(\mu)'$ is of type 2, the image $V(\nu)$ is a Radon measure on a Hilbert space H, see Schwartz [28]. Thus we have $\mu(W(H))=1$.

This completes the proof.

REMARK 4. Theorem 10 was proved by Diallo [5] for the case $E=l^p$, $1\leqq p<2$.

COROLLARY 6. *Let $2\leqq p<\infty$ and suppose that E is of cotype 2 and has G. L. P. Then every p-uniform Radon probability measure on E has a Hilbertian support. In particular, every s-convex Radon probability measure on E has a Hilbertian support, where* $-1/2<s\leqq 0$.

REMARK 5. It is clear that Theorem 10 holds for a Banach space E having the following property ; (*) every absolutely summing operator from E' into a Hilbert space is dual 2-summing. As shown in the proof of Theorem 10, if E is of cotype 2 and has G. L. P., then E has the property (*). We note that there is a cotype 2 space E which does not have the property (*). In fact, suppose that every cotype 2 space E satisfies (*). Then it can be proved that E is isomorphic to a Hilbert space if and only if both E and E' are of cotype 2 ; but this is false, as shown by Pisier [24].

REMARK 6. A Banach space E is said to have the Grothendieck property (G. P.) if every continuous linear operator from L^∞ into E is 2-summing. Grothendieck proved that L^1 has G. P. In the proof of Theorem 10, we used the fact that every cotype 2 space has G. P., see Maurey [16]. It is known that if E has both G. P. and G. L. P., then it is of cotype 2, see Reisner [25].

Finally we shall consider the case $1 \leqq p < 2$. Let $L^p = L^p[0, 1]$. Then $L^{p'}$ is the dual of L^p, where $1/p + 1/p' = 1$. We denote by $\|\cdot\|_p$ the usual L^p-norm. Let γ_p be a cylindrical measure on $L^{p'}$ whose characteristic functional $\hat{\gamma}_p(x) = \exp(-\|x\|_p^p)$, $x \in L^p$. We say that a linear operator $T: L^{p'} \to E$ is γ_p-Radonifying if the image $\mu = T(\gamma_p)$ is a Radon measure on E; in this case μ is symmetric p-stable. It is well-known that if μ is a symmetric p-stable Radon probability measure on E, then there exists a γ_p-Radonifying operator $T: L^{p'} \to E$ such that $\mu = T(\gamma_p)$, see Linde [14]. It is clear that if $\mu = T(\gamma_p)$ has a Hilbertian support, then $T: L^{p'} \to E$ factors through a Hilbert-Schmidt operator, that is, T is factorized by the bounded linear operators $S: L^{p'} \to H$, $V: H \to G$ and $W: G \to E$, where H, G are Hilbert spaces and V is of Hilbert-Schmidt type. Since every Hilbert-Schmidt operator is p-integral $(p > 1)$ in the sense of Pietsch [20], we have the following:

LEMMA 3. *Let $1 < p < 2$ and suppose that every p-stable Radon probability measure on E has a Hilbertian support. Then every γ_p-Radonifying operator from $L^{p'}$ into E is p-integral.*

THEOREM 11. *Let $1 < p < 2$ and suppose that E is of type p-stable. Then the following conditions are equivalent.*

(1) *E is isomorphic to a Hilbert space.*

(2) *Every p-stable Radon probability measure on E has a Hilbertian support.*

PROOF. (1)$\Longrightarrow$(2) is clear. Suppose that (2) holds. Then by Lemma 3, every γ_p-Radonifying operator from $L^{p'}$ into E is p-integral. Since E is of type p-stable, by Takahashi and Okazaki [33, Theorem 5.1], E is isomorphic to a quotient of some L^p-space (called Q_p-type). Since type interval is open, E is of type r-stable for some $r \in (p, 2)$. Thus E' is of cotype r' and isomorphic to a subspace of $L^{p'}$. If E' is not isomorphic to a Hilbert space, then Kadec and Pelczyński [13, Theorem 3], E' contains $l^{p'}$. But this is impossible because $l^{p'}$ is not of cotype r' with $r' < p'$. Hence E' is isomorphic to a Hilbert space, proving (1).

This completes the proof.

THEOREM 12. *Let $1 < p < 2$ and suppose that E is of type p-stable. Then the following conditions are equivalent.*

(1) *E is isomorphic to a Hilbert space.*

(2) *Every p-uniform Radon probability measure on E has a Hilbertian support.*

PROOF. (1)$\Longrightarrow$(2) is clear. Suppose that (2) holds. Since E is of type p-stable, it is of type r-stable for some $r\in(p, 2)$. Since every r-stable measure is p-uniform with $p<r$, see Example 2, it follows from assumption (2) that every r-stable Radon probability measure on E has a Hilbertian support. Thus the assertion follows from Theorem 11.

REMARK 7. Theorems 11 and 12 are true for the case $p=2$; this follows from the well-known fact that if a Banach space is of both type 2 and cotype 2, then it is isomorphic to a Hilbert space.

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