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Representations of chordal subalgebras of von Neumann algebras

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§ 1. Introduction

Let M be a von Neumann algebra and $\mathfrak{T}$ a σ-weakly closed subalgebra of M containing the identity of M. A σ-weakly continuous, contractive representation of $\mathfrak{T}$ is a homomorphism ρ of $\mathfrak{T}$ into the algebra of bounded linear operators $B(H)$ on a Hilbert space H such that $\rho(1)=I$ and $\|\rho(t)\|\leq\|t\|$ for all $t\in\mathfrak{T}$. Thus as an operator from $\mathfrak{T}$ to $B(H)$, $\|\rho\|=1$. In recent years the following question has attracted considerable interest: Given such a representation ρ of $\mathfrak{T}$, when is it possible to find a triple (π, V, K) where π is a (normal) $*$-representation of M on the Hilbert space K and V is an isometry mapping H into K such that

$$\rho(t)=V^{*}\pi(t)V$$

for all $t\in\mathfrak{T}$? Such a triple, should it exist, is called a *W^{*}-dilation*, or simply a *dilation*, for ρ. It was Arveson [A] who found the fundamental criterion for deciding if ρ has a dilation. To state it, let $\mathfrak{T}\otimes M_{n}$ be viewed as the $n\times n$ matrices over $\mathfrak{T}$ endowed with the norm inherited from $M\otimes M_{n}$ and let ρ_{n} be the obvious extension of ρ to $\mathfrak{T}\otimes M_{n}$, mapping into $B(H)\otimes M_{n}=B(H\otimes \boldsymbol{C}^{n})$. Then ρ is called *completely contractive* if and only if $\|\rho_{n}\|=1$ for all n. Arveson's dilation theorem asserts that ρ has a dilation if and only if ρ is completely contractive. In a recent paper [PPS], Paulsen, Power and Smith showed that if $\mathfrak{T}$ is a subalgebra of M_{n} that is linearly spanned by the matrix units it contains and if the support of $\mathfrak{T}$, which is the set of (i, j) such that matrix unit e_{ij} lies in $\mathfrak{T}$, satisfies a certain graph-theoretic property which they call "chordal", then every contractive representation of $\mathfrak{T}$ is completely contractive and so admits a dilation. Our objective in this note is to generalize this notion of "chordal" to the context of von Neumann algebras and to show that if $\mathfrak{T}$ is a chordal, triangular subalgebra of M in a sense to be defined in a minute, and if M is hyperfinite, then every σ-weakly contractive representation of $\mathfrak{T}$ is completely contractive.

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§ 2. Chordal operator systems

Recall that an *operator system* in a von Neumann algebra M is simply a σ-weakly closed, self-adjoint linear subspace of M that contains the identity. Our objective in the section is to define and develop the notion of a " chordal operator system ".

Throughout this paper, we assume that our von Neumann algebra M contains a Cartan subalgebra A in the sense that A is a masa in M that is the range of a faithful normal expectation on M and $\{u \in M | u\text{-unitary}, uAu^* = A\}$ generates M as a von Neumann algebra. On the basis of this assumption, we may use the work of Feldman and Moore [FM1, 2] to realize M as an algebra of matrices indexed by an equivalence relation. We shall freely use the notation, terminology and results from [FM1, 2] and [MSS] in what follows. We fix a standard Borel space X and an equivalence relation $R \subseteq X \times X := X^2$ whose equivalence classes are countable. We also fix a probability measure μ on X that is quasi-invariant for R and we fix a 2-cocycle σ on R with values in the circle $\boldsymbol{T}$. By our assumption that M has a Cartan subalgebra, we may choose these things in such a way that M is realized as matrices indexed by R and endowed with the product

$$a*b(x, y) = \sum_z a(x, z) b(z, y) \sigma(x, z, y), \tag{2.1}$$

where the sum runs over the (countable) set of all z equivalent to x (and y). With its elements parameterized this way, M is represented on $L^2(R, \nu)$, where ν is a certain measure on R built from μ and counting measure on the equivalence classes, by the formula (2.1) where it is assumed that $a \in M$ and $b \in L^2(R, \nu)$. We write $M = M(R, \sigma)$ to indicate the dependence of M on R and σ. The algebra A may then be identified with $L^\infty(X, \mu)$ viewed as functions living on the diagonal of $X \times X$, which we denote by Δ. It should be noted that if M is hyperfinite, then $\sigma = 1$ [FM1, Theorem 6] and we simply write $M = M(R)$.

It is important to keep in mind that a large part of this coordinatization theory of von Neuumann algebra is analogous to the theory of direct integrals in the sense that one should seek global formulation of one's results, i. e. formulations that are coordinate free, and use the coordinates only where it is necessary in the proofs. This is the philosophy that underlies the main result of this section, Theorem 2. 4.

If v is a partial isometry in M, then we shall write $e(v) := v^*v$ and $f(v) := vv^*$ for the projections onto the initial and final spaces of v, respectively. We shall also write $N(A)$ for the collection of all partial isometries $v \in M$ such that $e(v), f(v) \in A$ and $vAv^*, v^*Av \subseteq A$. We call $N(A)$ the *normalizer* of A in M. Note that $N(A)$ is an inverse semigroup of partial isometries that generates M as a von Neumann algebra and contains all the partial isometries in A. When M is realized as $M(R, \sigma)$, each $v\in N(A)$ may be written in terms of a partial Borel isomorphism $g=g(v)$, whose graph, $\Gamma(g)$, lies in R, according to the formula $v\xi(x, y)=1_{\Gamma(g^{-1})}*\xi(x, y)=\xi(g^{-1}x, y)\sigma(x, g^{-1}x, y)$, $\xi\in L^2(R, \nu)$. Given a partial Borel isomorphism g such that $\Gamma(g)\subseteq R$, we shall write L_g for the element of $N(A)$ it determines through this formula. We then have $L_{g_1}L_{g_2}=\Theta(g_1, g_2)L_{g_1g_2}$ where $\Theta(g_1, g_2)$ is the element of A which, when viewed as a function on X, is given by the formula $\Theta(g_1, g_2)(x)=\sigma(x, g_1^{-1}(x), g_2^{-1}\circ g_1^{-1}(x))$, when x lies in the domain of $g_2^{-1}\circ g_1^{-1}$ and is set equal to 1 otherwise. Also, $(L_g)^*=L_{g^{-1}}$, so that $L_gL_g^*=L_{r(g)}$ and $L_g^*L_g=L_{d(g)}$, where $d(g)$ and $r(g)$ denote, respectively, either the identity transformations on the domain and range of g or the sets themselves; i. e., $(L_{d(g)}\xi)(x, y)=1_{d(g)}(x)\xi(x, y)$ and similarly for $L_{r(g)}$.

A key result from [MSS], the Spectral Theorem for Bimodules, Theorem 2. 1, asserts that every σ-weakly closed subspace $\mathfrak{S}$ of $M(R, \sigma)$ that is a bimodule over A, i. e., $A\mathfrak{S}A\subseteq\mathfrak{S}$, is completely determined by the common support of the elements it contains. That is, there is a Borel set $P\subseteq R$, which is unique up to a set of ν-measure zero, such that $\mathfrak{S}=\{a\in M(R, \sigma)| a(x, y)=0,\ (x, y)\notin P\}$. Following [MSS], we write $\mathfrak{S}=\mathfrak{I}(P)$. As a corollary of the Spectral Theorem for Bimodules it is shown in [MSS] that $\mathfrak{S}$ is the σ-weakly closed, linear span of $\mathfrak{S}\cap N(A)$. Note that in this case, $\mathfrak{S}$ is self-adjoint if and only if $P=\theta(P)$ a. e. ν where $\theta(x, y)=(y, x)$, $(x, y)\in R$. Note, too, that $\mathfrak{S}=\mathfrak{I}(P)$ contains the identity operator if and only if $A\subseteq\mathfrak{S}$, if and only if $\Delta\subseteq P$.

DEFINITION 2. 1. Let $\mathfrak{S}$ be an operator system in M that is a bimodule over A.

i) A *k-cycle* in $\mathfrak{S}$ is a family of k partial isometries, $\{v_1, v_2,\cdots, v_k\}$, in $\mathfrak{S}\cap \mathrm{N(A)}$ such that

a) $v_kv_{k-1}\cdots v_1=e(v_1)$,

b) $f(v_iv_{i-1}\cdots v_1)\perp f(v_jv_{j-1}\cdots v_1)$ for all $i\neq j$,

c) $f(v_i)=f(v_iv_{i-1}\cdots v_1)$, and

d) $f(v_k)=e(v_1)$ while $f(v_{i-1})=e(v_i)$ for $i=2, 3,\cdots k$.

ii) *A chord* for a k-cycle $\{v_1,\cdots, v_k\}$ $(k\geq 4)$ is a triple $\{e_i', i, j\}$ where e_i' is a projection in A with $e_i'\leq e(v_i)$ and where i and j are indices satisfying: $(i, j)\neq(1, k)$, $2\leq j-i$, and $(v_{j-i}v_{j-2}\cdots v_i)e_i'\in\mathfrak{S}$.

iii) We say that $\mathfrak{S}$ is *chordal* if for each $k\geq 4$, every k-cycle in $\mathfrak{S}$ has a chord.

In the definition of k-cycle, conditions a) and b) are the essential ones.

If a k-tuple of partial isometries in $N(A)$ satisfies a) and b), then by pre- and post-multiplying them with suitable projetion in A, conditions c) and d) can be obtained. The following definition, although not word for word the same as that given in [PPS, § 2], is easily seen to be equivalent to it.

DEFINITION 2.2. Assume that X is a countable set and that P is a symmetric subset of $X \times X$ containing Δ.

i) A finite set $\{x_1, x_2, \cdots, x_k\}$ in X is a *k-cycle* in X if $(x_i, x_{i+1}) \in P$, $1 \leq i < k$, and $(x_k, x_1) \in P$.

ii) A *chord* for the k-cycle $\{x_1, x_2, \cdots, x_k\}$ $(k \geq 4)$ is a pair (i, j) with $1 \leq i, j \leq k$, $j - i \geq 2$, $(i, j) \neq (1, k)$ and $(x_i, x_j) \in P$.

iii) The set P is *chordal* if each k-cycle, $k \geq 4$, has a chord.

To connect Definitions 2.1 and 2.2, we require one more notion of "chordal", an intermediate concept.

DEFINITION 2.3. Assume that P is a symmetric Borel subset of R containing Δ.

i) A k-cycle for P is a family $\{g_1, g_2, \cdots, g_k\}$ of partial Borel isomorphisms whose graphs are contained in P and whose domains (and ranges) have positive μ-measure such that

a) If $C_1 = d(g_1)$ and if for $2 \leq i \leq k$, $C_i = g_{i-1} \circ g_{i-2} \circ \cdots \circ g_1(C_1)$, then $g_k(C_k) = C_1$ and $C_i \cap C_j = \emptyset$ for every $i \neq j$.

b) The composition $g_k \circ g_{k-1} \circ \cdots \circ g_1$ is the identity on C_1.

ii) Given a k-cycle $\{g_1, g_2, \cdots, g_k\}$ $(k \geq 4)$, we say that $\{C_1', i, j\}$ is a *chord* for it if C_1' is a Borel subset of $d(g_1)$ with positive μ-measure, $2 \leq j - i$, $(i, j) \neq (1, k)$, and the graph of $g_{j-1} \circ \cdots \circ g_i$, restricted to $g_{i-1} \circ \cdots \circ g_1(C_1')$, is contained in P. (It is sometimes preferable to think of this restriction as the chord instead of $\{C_1', i, j\}$.)

iii) If, for each $k \geq 4$, every k-cycle in P has a chord, then P is called *chordal*.

The following theorem relates the three notions of "chordal" just defined. In it, we employ the following notation. For $x \in X$, $R(x)$ denotes the equivalence class of x, i.e., $R(x) = \{y \in X \mid (x, y) \in R\}$. We write $P(x)$ for $P \cap (R(x) \times R(x))$, i.e., $P(x) = \{(y, z) \in P \mid (x, y) \in R\}$.

THEOREM 2.4. *Let $\mathfrak{S}$ be an operator system in $M(R, \sigma)$ that is a bimodule over A and realize $\mathfrak{S}$ as $\mathfrak{I}(P)$ for an essentially unique symmetric Borel subset P of R that contains Δ. Then the following assertions are equivalent.*

1) *$\mathfrak{S}$ is chordal as an operator system in the sense of Definition* 2.1.

2) *P is a chordal subset of R in the sense of Definition* 2.3.

3) *For μ-almost all $x\in X$, the subset $P(x)$ of $R(x)\times R(x)$ is chordal in $R(x)$ in the sense of Definition* 2.2.

PROOF. Because of the representation of elements in $N(A)$ in terms of partial Borel isomorphisms whose graphs are contained in R, it is an easy matter to see that assertions 1) and 2) are equivalent. We shall prove that assertions 2) and 3) are equivalent. Suppose, then, that for μ-almost all x, $P(x)$ is chordal in $R(x)$ and let $\{g_1, g_2, \cdots, g_k\}$ be a k-cycle in P. For each $x \in C_1: =d(g_1)$, let $x_1=x$, $x_2=g_1(x_1)$, $x_3=g_2(x_2), \cdots$, and $x_k=g_{k-1}(x_{k-1})$. Then $\{x=x_1, x_2, \cdots, x_k\}$ is a k-cycle in the equivalence class of x because $g_k(x_k)=x$ and $(x_i, x_{i+1})=(x_i, g_i(x_i))\in P(x)$. By hypothesis for μ-almost all x, $\{x=x_1, x_2, \cdots, x_k\}$ has a chord, say, $(i(x), j(x))$. Note that it is an easy matter to choose $i(\cdot)$ and $j(\cdot)$ to be Borel functions. For each pair (i, j), $1\leq i, j\leq k$, write $C_{ij}=\{x\in C_1|(i(x), j(x))=(\mathrm{i}, \mathrm{j})\}$. Then the C_{ij} are Borel subsets of C_1 and their union is C_1, except, possibly, for a set of measure zero. It follows that at least one of the C_{ij} has positive measure and satisfies $(i, j)\neq(1, k)$ and $j-i\geq 2$. Hence $\{C_{ij}, i, j\}$ is a chord for $\{g_1, g_2, \cdots, g_k\}$.

For the converse, suppose that there is a Borel subset $C\subseteq X$ of positive measure such that for every $x\in C$, $P(x)$ is *not* chordal. Recall that there is a countable family $\{\varphi_i\}_{i=1}^{\infty}$ of partial Borel isomorphisms of X with the property that their graphs are pairwise disjoint and cover R; i.e., for every $(x, y)\in R$, there is a unique m such that $y=\varphi_m(x)$. Since $P(x)$ is not chordal for all $x\in C$, there are positive integers $p(x)$, and $k_i(x)$, $1\leq i\leq p(x)$, such that $\{\varphi_{k_1(x)}(x), \cdots, \varphi_{k_{p(x)}(x)}(x)\}$ is a $p(x)$-cycle in $P(x)$ with no chord. The set $\{k_1(x), \cdots, k_{p(x)}(x)\}$ is a finite ordered subset of natural numbers. Since there are only countably many such subsets, there is at least one, $\{k_1, k_2, \cdots, k_p\}$ such that $C_1: =\{x\in C|p(x)=p$ and $k_i(x)=k_i\}$ has positive μ-measure. Then clearly, $\{C_1, \varphi_{k_1}, \cdots, \varphi_{k_p}\}$ is a P-cycle with no chord. Hence P is not chordal. ■

EXAMPLE 2.5. Recall that a real-valued, Borel function d on R is called a 1-cocycle, if for almost all triples (x, y, z) with (x, y), $(y, z)\in R$, we have

$$d(x, z)=d(x, y)+d(y, z).$$

Then for every $a\geq 0$, the set

$$P=\{(x, y)\in R||d(x, y)|\leq a\}$$

is chordal. Indeed, using Theorem 2.4, we may restrict our attention to each equivalence class and there the chordality condition is easily checked. This set P is the analogue of a band, because if $R=\{1, 2, \cdots, n\}^2=X\times X$, $a=$

$k<n$ and if $d(i,j)=j-i$, then set $P=\{(i,j)\,\|\,d(i,j)|\leq k\}$ is a band which is symmetric with respect to the main diagonal.

DEFINITION 2.6. Let $\mathfrak{J}$ be a σ-weakly closed subalgebra of $M(R,\sigma)$ containing A. Then $\mathfrak{J}$ is called a *chordal subalgebra* of $M(R,\sigma)$ if the operator system generated by $\mathfrak{J}$, namely, the σ-weakly closure of $\mathfrak{J}+\mathfrak{J}^*=\{t+s^*|t, s\in\mathfrak{J}\}$, is chordal.

REMARK 2.7. In the definition of chordal operator system it appears that the self-adjointness condition (or, equivalently, the symmetric condition on the underlying support set) does not play an essential role. However, without this assumption there is little chance for there to be any k-cycles with $k\geq 4$. In particular, if $\mathfrak{J}$ is a σ-weakly closed algebra that is triangular with respect to A in the sense that $\mathfrak{J}\cap\mathfrak{J}^*=A$, then there are no k-cycles in $\mathfrak{J}$ for $k\geq 4$. We are interested primarily in σ-weakly continuous, contractive representations of σ-weakly closed algebras $\mathfrak{J}$ containing A and by Proposition 1.2.8 of [A] such representations have unique extensions to σ-weakly continuous positive linear maps on the operator system generated by $\mathfrak{J}$. It is in the analysis of these positive linear maps that the chordality assumption plays a role.

Recall that a σ-weakly closed subalgebra $\mathfrak{J}$ of $M(R,\sigma)$ is called a *σ-Dirichlet algebra* if the operator system generated by $\mathfrak{J}$ is $M(R,\sigma)$. We record the following proposition for later reference. Its proof is trivial, since $M(R,\sigma)$ is obviously a chordal operator system.

PROPOSITION 2.8. *If $\mathfrak{J}$ is σ-Dirichlet algebra in $M(R,\sigma)$ containing A, then $\mathfrak{J}$ is chordal.*

§ 3. Chordal subalgebras of hyperfinite von Neumann algebras

We continue with the notation developed above, but we now restrict our attention to hyperfinite von Neumann algebras. As noted earlier, under this assumption, the 2-cocycle σ is trivial, so we cease to mention it and write $M(R)$ instead of $M(R,1)$. We fix a sequence $\{R_n\}_{n=1}^{\infty}$ of equivalence relations on X satisfying:

i) For each n, the cardinality of every equivalence class $|R_n(x)|$ is finite (however, for n fixed, $|R_n(x)|$ need not be bounded on X);

ii) $R_n\subseteq R_{n+1}$, $n=1, 2,\cdots$; and

iii) $R=\bigcup_{n=1}^{\infty} R_n$.

Then for each n, $M(R_n)$ is a finite type I von Neumann algebra and $\bigcup_{n=1}^{\infty} M(R_n)$ is σ-weakly dense in $M(R)$. Our objective is to prove

THEOREM 3.1. *If $\mathfrak{I}$ is a σ-weakly closed chordal subalgebra of $M(R)$, then every σ-weakly continuous contractive representation of $\mathfrak{I}$ is completely contractive and, so, admits a W^*-dilation.*

Recall that by definition, $\mathfrak{I}$ contains A and so, by the Spectral Theorem for Bimodules, $\mathfrak{I}=\mathfrak{I}(P)$ for a certain Borel set $P\subseteq R$. The assumption that $\mathfrak{I}$ is chordal means that $P':=P\cup\theta(P)$ is chordal in the sense of Definition 2.3. Let $P_n=P\cap R_n$ and $P'_n=P'\cap R_n$; i.e., $P'_n=P_n\cup\theta(P_n)$.

LEMMA 3.2. *The relation P'_n is chordal in R_n.*

PROOF. Let $\{g_1, g_2,\cdots, g_k\}$ be a k-cycle in P'_n. Since P' is chordal in R, there is a chord $\{C'_1, i, j\}$ $\{g_1, g_2,\cdots, g_k\}$ in R. Recall that this means that C'_1 is a subset of $d(g_1)$ with positive measure, $2\leq j-i$, $(i, j)\neq(1, k)$, and the graph of $g_{j-1}\circ\cdots\circ g_i$, restricted to $g_{i-1}\circ\cdots\circ g_1(C'_1)$, is contained in P'. Since the graph of each g_j is contained in R_n, by hypothesis, the graph of $g_{j-1}\circ\cdots\circ g_i$ is contained in R_n, too. Since the same is true of any restriction, we conclude that $\{C'_1, i, j\}$ is a chord for $\{g_1, g_2,\cdots, g_k\}$ in R_n; i.e., P'_n is chordal. ■

Evidently, $P=\bigcup_{n=1}^{\infty}P_n$, so $\mathfrak{I}=\mathfrak{I}(P)$ is the σ-weak closure of $\bigcup_{n=1}^{\infty}\mathfrak{I}(P_n)$ and, of course, $\mathfrak{I}(P_n)\subseteq\mathfrak{I}(P_{n+1})$, for all n. We fix once and for all a σ-weakly continuous, contractive representation ρ of $\mathfrak{I}(P)$ mapping $\mathfrak{I}(P)$ into $B(H)$. Just as in Lemma 3 of [MS], it suffices to show that for each n, the restriction $\rho|\mathfrak{I}(P_n)$ is completely contractive. Indeed, if $\Phi_n: M(R)\longrightarrow M(R_n)$ is defined by the formula $\Phi_n(a)=a|R_n$, $a\in M(R)$ then by Theorem 3.4 in [MSS], Φ_n is the unique faithful normal expectation from $M(R)$ onto $M(R_n)$ and so is completely contractive by Theorem 1 of [T]. We have that $\{\rho\circ(\Phi_n|\mathfrak{I}(P))\}_{n=1}^{\infty}$ converges to ρ in the topology of simple σ-weak convergence and so ρ is completely contractive if each $\rho\circ(\Phi_n|\mathfrak{I}(P))$ is completely contractive. However, $\rho\circ(\Phi_n|\mathfrak{I}(P))$ is completely contractive if and only if $\rho|\mathfrak{I}(P_n)$ is completely contractive. Hence to prove Theorem 3.1 we may, and will, assume that $|R(x)|<\infty$ for each $x\in X$.

Recall that if R_i is a Borel equivalence relation on a standard Borel space X_i, $i=1, 2$, then we say R_1 is isomorphic to R_2 if and only if there is a $1-1$ Borel map φ from X_1 onto X_2 such that $(\varphi(x), \varphi(y))$ lies in R_2 if and only if (x, y) lies in R_1. The following lemma is essentially Lemma 4 of [MS], so we omit the proof.

LEMMA 3.3. *Let R be a Borel equivalence relation in the standard Borel space X with $|R(x)|<\infty$ for every $x\in X$.*

1) *The sets X_n: $\{x\in X||R(x)|=n\}$ form a disjoint Borel cover of X*

and $R\cap(X_n\times X_n)$ *is an equivalence relation on* X_n *with equivalence classes of cardinality* n.

2) *If* $|R(x)|=n$ *for every* $x\in X$, *then there is a Borel set* $E\subseteq X$ *such that* R *is isomorphic to* $\Delta_E\times\{1,2,\cdots,n\}^2$ *viewed as the equivalence relation in* $(E\times\{1,2,\cdots,n\})^2$ *consisting of all pairs* $((x,i),(y,i))$ *such that* $x=y$.

3) *If* R *is as in* 2) *and if* $P\subseteq R$ *is a Borel set, then* E *can be decomposed as the disjoint union,* $E=\bigcup_{k=1}^{2^{n^2}}E_k$, *where some of the* E'_k *s may be empty, and there are subsets* $P_k\subseteq\{1,2,\cdots,n\}^2$ *such that when* R *is viewed as* $\Delta_E\times\{1,2,\cdots,n\}^2$, *then* $P=\bigcup_{k=1}^{2^{n^2}}\Delta_{E_k}\times P_k$.

On the basis of this lemma, we may relabel the sets produced and assert that we may find a countable disjoint cover of X by Borel sets, $X=\cup X_k$, such that $R\cap(X_k\times X_k)$ is isomorphic to $\Delta_{E_k}\times\{1,2,\cdots,n(k)\}^2$ for a suitable subset E_k of X_k and such that under this isomorphism $P\cap(X_k\times X_k)$ is carried to a set of the form $\Delta_{Ek}\times \mathrm{P}_k$ where P_k is a subset of $\{1,2,\cdots,n(k)\}^2$. The sets X_k are invariant, or saturated, for R and so 1_{X_k} lies in the center of $M(R)$ which, in turn, is contained in A. It follows that each $\rho(1_{X_k})$ is a projection in the commutant of $\rho(\mathfrak{J}(P))$, their (orthogonal) sum is I_H and it is easy to see that ρ is completely contractive if and only if $\rho\,|\mathfrak{J}(P\cap(X_k\times X_k))$ is completely contractive for each k, because $\rho(\mathfrak{J}(P\cap(X_k\times X_k)))=\rho(1_{X_k})\rho(\mathfrak{J}(P))$.

Thus we may, and will, assume from now on that $R=\Delta_E\times\{1,2,\cdots,n\}^2$ for a suitable Borel set $E\subseteq X$ and that $P=\Delta_E\times P_0$ for a suitable subset P_0 of $\{1,2,\cdots,n\}^2$. We may assume, also, that the measure μ on X is $\mu_0\times\frac{1}{n}\sum_{k=1}^{n}\delta_k$, where μ_0 is a measure on E and where δ_k is the point mass at k. We then may identify $M(R)$ with $L^\infty(E,\mu_0)\otimes M_n$. Since under this identification, $L^\infty(E,\mu_0)\otimes I$ corresponds to the center of $M(R)$, we shall simply write $M(R)=\mathscr{X}\otimes M_n$. The algebra $\mathfrak{J}(P)$, then, is simply $\mathscr{X}\otimes\mathfrak{J}(P_0)$. Noting that chordality is preserved under restriction and isomorphism, we conclude that $\Delta_E\times P':=\Delta_E\times P_0\cup\theta(\Delta_E\times P_0)=\Delta_E\times(P_0\cup\theta(P_0))$ is a chordal subset of $\Delta_E\times\{1,2,\cdots,n\}^2$, so that P' is a chordal subset of $\{1,2,\cdots,n\}^2$. Thus $\mathfrak{J}(P_0)$ is a chordal subalgebra of M_n.

The proof of Theorem 3.1 is completed now just as is the proof of Theorem 1 in [MS]. Let ρ_0 be the contractive representation of $\mathfrak{J}(P_0)$ on H defined by the formula $\rho_0(t)=\rho(1\otimes t)$ and define the representation θ_0 of $\mathscr{X}$ on H by the formula $\theta_0(a)=\rho(a\otimes 1)$, $a\in\mathscr{X}$. Note that since ρ is contractive and $\mathscr{X}$ is self-adjoint, θ_0 is a $*$-representation of $\mathscr{X}$. Also, θ_0 is normal since ρ is σ-weakly continuous. Thus $\theta_0(\mathscr{X})$ is an abelian von Neumann algebra of operators on H commuting with $\rho_0(\mathfrak{J}(P_0))$. Since $\mathfrak{J}(P_0)$ is a chordal algebra and ρ_0 is a contractive representation of $\mathfrak{J}(P_0)$, we may apply Theorem 5, 2 of [PPS] to conclude that ρ_0 is completely contractive. Let (π_0, V, K) be a W^*-dilation of ρ_0 that is minimal in the sense that $[\pi_0(M_n) V(H)]=K$. Observe that since $\theta_0(\mathscr{X})$ is an abelian von Neumann algebra of operators on H commuting with $\rho_0(\mathfrak{J}(P_0))$, Theorem 1.3.1 in [A] implies that there is a normal $*$-representation θ of $\mathscr{X}$ on K such that $\theta(\mathscr{X})$ is reduced by $V(H)$, $V^*\theta(a)V=\theta_0(a)$, $a\in\mathscr{X}$, and $\theta(\mathscr{X})$ commutes with $\pi_0(M_n)$. If π is defined on elementary tensors $a\otimes t$ in $\mathscr{X}\otimes M_n$ by the formula $\pi(a\otimes t)=\theta(a)\pi_0(t)$ and extended by linearity, then π is a normal $*$-representation of $M(R)=\mathscr{X}\otimes M_n$ that dilates ρ. Thus ρ is completely contractive and the proof of Theorem 3.1 is complete. ■

Combining Proposition 2.8 with Theorem 3.1 yields Theorem 1 of [MS]:

COROLLARY 3.4. *If $\mathfrak{J}$ is a σ-Dirichlet algebra in a hyperfinite von Neumann algebra M and if $\mathfrak{J}$ contains a Cartan subalgebra, then every σ-weakly continuous contractive representation is completely contractive and, so, admits a W^*-dilation.*

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