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On approximation by Riemann sums in generalized Orlicz spaces

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1. 1. In [1], M. Yu. Fominykh has shown that the order of approximation of the integral $\int_0^1 f(x)dx$ of a function $f\in L^p(0.1)$, $1\leq p<\infty$, by means of its Riemann sums with equidistant partitions in space $L^p(0.1)$, is $0\left(\omega_p\left(f, \frac{1}{n}\right)\right)$. This result will be extended to a generalized Orlicz space $L^\varphi(Q)$, where $Q=[0,1]^m$, generated by a function $\varphi: Q\times \boldsymbol{R}_+\longrightarrow \boldsymbol{R}_+$, $m=1, 2, \ldots$, satisfying the following conditions:

1° $\varphi(t, u)$ is a convex function of $u\geq 0$ such that $\varphi(t, 0)=0$ and $\varphi(t, u)>0$ for $u>0$, for a. e. $t\in Q$,

2° $\varphi(t, u)$ is a Lebesgue measurable function of t for every $u>0$,

3° there are a set $A\subset Q$ of measure zero, a positive constant K and a nonnegative, Lebesgue integrable function h on Q such that

$$u\leq\varphi(t, Ku)+h(t)$$

for all $u\geq 0$ and $t\in Q\backslash A$.

Condition 3° is necessary and sufficient in order that $L^\varphi(Q)\subset L^1(Q)$, (see [2] and [3]; it may be found also in [4] th. 1.8). If necessary, we shall extend φ to a function $\varphi: Q\times \boldsymbol{R}\longrightarrow[0, +\infty)$ by the condition $\varphi(t, u)=\varphi(t, -u)$ for $u<0$, or we extend φ periodically with period 1 with respect to the first m variables.

1. 2. Let us recall (see [5], 7.5 and 7.11) that the function φ is called:

(a) integrable, if $\int_Q \varphi(t, u)dt<\infty$ for every $u\geq 0$,

(b) τ-bounded, if there exist a constant $K_1>0$ and a function $F: \boldsymbol{R}^m\times \boldsymbol{R}^m\longrightarrow \boldsymbol{R}_+$ such that

$$\varphi(t-s, u)\leq\varphi(t, K_1u)+F(t, s) \text{ for } s, t\in \boldsymbol{R}^m,\ u\in \boldsymbol{R}_+,$$

where F is measurable and periodic with period 1 with respect to the first variable, $h(s)=\int_Q F(t, s)dt\longrightarrow 0$ as $s=(s_1, \ldots, s_m)$, $s_i\longrightarrow 0_+$

for $i=1, 2, \ldots, m$ or $s_i \longrightarrow 1_-$ for $i=1, 2, \ldots, m$, and h is bounded in Q.

If φ does not depend on t, then the conditions (a) and (b) are satisfied always, with $K_1=1$, $F(t, s)=0$.

1. 3. Let

$$\omega_\varphi(f, \delta)=\sup_{|v|\leq\delta}\int_Q \varphi(t, |f(t+v)-f(t)|)\,dt,$$

where f is extended periodically with period 1 in each variable to $\boldsymbol{R}^m$. There is shown in [5], th. 7. 13, in case $m=1$ that if φ is τ-bounded and integrable and $f \in L^\varphi(Q)$, then $\omega_\varphi(cf, \delta) \longrightarrow 0$ *as* $\delta \longrightarrow 0_+$ *for some* $c>0$; the extension of this result to general m makes no difficulties.

1. 4. The following condition will be of use in the sequel:

(c) there are a set $A \subset Q$ of measure zero, a positive constant c and a matrix of nonnegative integrable functions $\varepsilon_{n,k}: Q \longrightarrow [0, +\infty)$, $k=(k_1, \ldots, k_m) \in \boldsymbol{N}^m$, $n=(n_1, \ldots, n_m) \in \boldsymbol{N}^m$, $k_i \leq n_i$ for $i=1, 2, \ldots, m$, such that

$$\varphi(t, u) \leq \varphi\left(\frac{t+k}{n}, cu\right)+\varepsilon_{n,k}(t)$$

for every $t \in Q \setminus A$, $u \geq 0$ and all k and n.

If φ does not depend on t, then (c) is satisfied with $c=1$, $\varepsilon_{n,k}(t) \equiv 0$.

1. 5. Let us remark that if φ satisfies condition 1. 2(b), then it satisfies also 1. 4(c) with $c=K_1$ and $\varepsilon_{n,k}(t)=F\left(\frac{k}{n}+\frac{1}{n}t, \frac{k}{n}-\left(1-\frac{1}{n}\right)t\right)$.

2. 1. NOTATION. We shall write in the following $x=(x_1, \ldots, x_m) \in \boldsymbol{R}^m$, $k=(k_1, \ldots, k_m) \in \boldsymbol{N}^m$, $n=(n_1, \ldots, n_m) \in \boldsymbol{N}^m$, $|n|=n_1 n_2 \ldots n_m$, $\sum_{k=0}^{n-1}$ will be the sum over all multiindices k such that $0 \leq k_i \leq n_i - 1$ for $i=1, 2, \ldots, m$. We shall write also

$$Q_k=\left[\frac{k_1}{n_1}, \frac{k_1+1}{n_1}\right] \times \ldots \times \left[\frac{k_m}{n_m}, \frac{k_m+1}{n_m}\right],$$

Given a function $f: Q \longrightarrow \boldsymbol{R}$, $f \in L^\varphi(Q)$, we denote the translated equidistant Riemann sums of f by

$$R_n(f, y)=\frac{1}{|n|}\sum_{k=0}^{n-1} f\left(\frac{y+k}{n}\right), \quad y \in Q,$$

where

$$\frac{y+k}{n}=\left(\frac{y_1+k_1}{n_1},\ldots,\frac{y_m+k_m}{n_m}\right).$$

Moreover, we write

$$I_\varphi(f)=\int_Q \varphi(t,|f(t)|)\,dt,$$

$$\Omega_n(f)=I_\varphi(R_n(f,\cdot)-\int_Q f(x)\,dx).$$

The problem consists in estimating the modular error $\Omega_n(f)$ of approximation of the integral $\int_Q f(x)\,dx$ by its Riemann sums $R_n(f,\cdot)$.

2. 2. LEMMA. *There holds the inequality*

$$\Omega_n(f)\le\sum_{k=0}^{n-1}\int_{Q_k}\varphi(nt-k,\ |n|\int_{Q_k}(f(t)-f(x))\,dx)\,dt.$$

PROOF. Indeed, we have

$$R_n(f,y)-\int_Q f(x)\,dx=\frac{1}{|n|}\sum_{k=0}^{n-1}\int_{Q_k}|n|(f\left(\frac{y+k}{n}\right)-f(x))\,dx,$$

Hence, by convexity of the functional I_φ,

$$\Omega_n(f)\le\frac{1}{|n|}\sum_{k=0}^{n-1}\int_Q\varphi(y,\ \int_{Q_k}|n|(f\left(\frac{y+k}{n}\right)-f(x))\,dx)\,dy,$$

Substituting $t=(y+k)/n$, we obtain the desired result.

2. 3. In order to formulate the next lemma we need some further auxiliary notation. Let $\bar{i}=\{i_1, i_2, \ldots, i_l\}$ be a fixed subset of the set $\bar{m}=\{1, 2,\ldots, m\}$, where $1\le i_1<i_2<\ldots<i_l\le m$, and let its complement $\bar{m}\setminus\bar{i}=\{i_{l+1}, i_{l+2},\ldots, i_m\}$, where $1\le i_{l+1}<\ldots<i_m\le m$, Moreover, we shall write for $k=(k_1,\ldots,k_m)\in\boldsymbol{N}^m$, $(t_1,\ldots,t_m)\in Q$, $u=(u_1,\ldots,u_m)\in\boldsymbol{R}^m$,

$$U_{\bar{i}}(u)=\left[\frac{k_{i_1}}{n_{i_1}}-u_{i_1},\ \frac{k_{i_1}+1}{n_{i_1}}\right]\times\ldots\times\left[\frac{k_{i_l}}{n_{i_l}}-u_{i_l},\ \frac{k_{i_l}+1}{n_{i_l}}\right]\ \text{for}\ \bar{i}\ne\phi,$$

$$V_{\bar{i}}(u)=\left[\frac{k_{i_{l+1}}}{n_{i_{l+1}}},\ \frac{k_{i_{l+1}}+1}{n_{i_{l+1}}}-u_{i_{l+1}}\right]\times\ldots\times\left[\frac{k_{i_m}}{n_{i_m}},\ \frac{k_{i_m}+1}{n_{i_m}}-u_{i_m}\right]\ \text{for}\ \bar{i}\ne\bar{m},$$

$$U_{\bar{i}}=\left[-\frac{1}{n_{i_1}},\ 0\right]\times\ldots\times\left[-\frac{1}{n_{i_l}},\ 0\right]\ \text{for}\ \bar{i}\ne\phi,$$

$$V_{\bar{i}}=\left[0,\ \frac{1}{n_{i_{l+1}}}\right]\times\ldots\times\left[0,\ \frac{1}{n_{i_m}}\right]\ \text{for}\ \bar{i}\ne\bar{m},$$

$$P_{i,k}(t)=\left[\frac{k_i}{n_i}-t_i,\ 0\right],\ \text{for}$$

$$i=1, 2,\ldots, m\ \text{if}\ \frac{k_i}{n_i}-t_i\le0,\ \text{otherwise}\ P_{i,k}(t)=\phi,$$

$$R_{i,k}(t)=\left[0,\ \frac{k_i+1}{n_i}-t_i\right],\ \text{for}$$

$$i=1,\ 2,\dots,\ m\ \text{if}\ \frac{k_i+1}{n_i}-t_i\geq 0,\ \text{otherwise}\ R_{i,k}(t)=\phi,$$

Moreover, χ_A will denote the characteristic function of the set A. Finally we write

$$B^{\bar{i}}_{n,k}=|n|\int_{U_{\bar{i}}}\int_{V_{\bar{i}}}\Big(\int_{U_{\bar{i}}(u)}\int_{V_{\bar{i}}(u)}\varphi(nt-k,\ 2^m|f(t)-f(t+u)|)\,dt_{\bar{i}}dt_{\bar{i}'}\Big)$$

$du_{\bar{i}}du_{\bar{i}'}$,

where $dt_{\bar{i}}=dt_{i_1}\dots dt_{i_l}$, $dt_{\bar{i}'}=dt_{i_{l+1}}\dots dt_{i_m}$, $du_{\bar{i}}=du_{i_1}\dots du_{i_l}$, $du_{\bar{i}'}=du_{i_{l+1}}\dots du_{i_m}$. In case of $\bar{i}=\phi$, we omit here integration over $U_{\bar{i}}$ and $U_{\bar{i}}(u)$, and in case of $\bar{i}=\bar{m}$, we omit integration over $V_{\bar{i}}$ and $V_{\bar{i}}(u)$. Then there holds

2. 4. LEMMA.

$$\Omega_n(f)\leq\frac{1}{2^m}\sum_{k=0}^{n-1}\sum_{\bar{i}\subset\bar{m}}B^{\bar{i}}_{n,k}.$$

PROOF. Let $t\in Q_k$, Substituting $x=t+u$ in the integral $\int_{Q_k}(f(t)-f(x))\,dx$ we observe, that the interval $\left[\frac{k_i}{n_i},\ \frac{k_i+1}{n_i}\right]$ in the variable x_i is transformed in the interval $\left[\frac{k_i}{n_i}-t_i,\ \frac{k_i+1}{n_i}-t_i\right]=P_{i,k}(t)\cup R_{i,k}(t)$ in the variable u_i. Thus, writing $P^{\bar{i}}_k(t)=P_{i_1,k}(t)\times\dots\times P_{i_l,k}(t)$ and $P^{\bar{i}}_k(t)=R_{i_{l+1},k}(t)\times\dots\times R_{i_m,k}(t)$, we obtain

$$\int_{Q_k}(f(t)-f(x))\,dx=\Big(\int_{P_{1,k}(t)}+\int_{R_{1,k}(t)}\Big)\dots\Big(\int_{P_{m,k}(t)}+\int_{R_{m,k}(t)}\Big)$$
$$(f(t)-f(t+u))\,du=\sum_{\bar{i}\subset\bar{m}}\int_{P^{\bar{i}}_k(t)}\Big(\int_{R^{\bar{i}}_k(t)}(f(t)-f(t+u))\,du_{\bar{i}'}\Big)du_{\bar{i}},$$

with the same convention as is the definition of $B^{\bar{i}}_{n,k}$.
Hence, by Lemma 2. 2 and convexity of φ,

$$\Omega_n(f)\leq\sum_{k=0}^{n-1}\int_{Q_k}\frac{1}{2^m}\sum_{\bar{i}\subset\bar{m}}\varphi(nt-k,$$
$$\int_{P^{\bar{i}}_k(t)}\Big(\int_{R^{\bar{i}}_k(t)}2^m|n||f(t)-f(t+u)|\,dx_{\bar{i}'}\Big)du_{\bar{i}}\Big)\,dt=\frac{1}{2^m}\sum_{k=0}^{n-1}\sum_{\bar{i}\subset\bar{m}}A^{\bar{i}}_{n,k},$$

where

$$A^{\bar{i}}_{n,k}=\int_{Q_k}\varphi\Big(nt-k,\int_{P^{\bar{i}}_k(t)}\Big(\int_{R^{\bar{i}}_k(t)}2^m|n||f(t)-f(t+u)|\,du_{\bar{i}'}\Big)du_{\bar{i}}\Big)\,dt$$

$$=\int_{Q_k}\varphi(nt-k,\int_{U_{\bar{i}}}(\int_{V_{\bar{i}}}2^m|n||f(t)-f(t+u)|\chi_{R_{i_{l+1},k}(t)}(u_{i_{l+1}})\ldots$$
$$\chi_{R_{i_m,k}(t)}(u_{i_m})du_{\bar{i}'})\chi_{P_{i_1,k}(t)}(u_{i_1})\ldots\chi_{P_{i_l,k}(t)}(u_{i_l})du_{\bar{i}})dt.$$

However, we observe easily that for $t\in Q_k$ we have for nonempty $P_{i_j,k}(t)$ resp. $R_{i_j,k}(t)$

$$\chi_{P_{i_j,k}(t)}(u_{i_j})=\chi_{\left[\frac{k_{i_j}}{n_{i_j}}-u_{i_j},\ \frac{k_{i_j}+1}{n_{i_j}}\right]}(t_{i_j}),$$
$$\chi_{R_{i_j,k}(t)}(u_{i_j})=\chi_{\left[\frac{k_{i_j}}{n_{i_j}},\ \frac{k_{i_j}+1}{n_{i_j}}-u_{i_j}\right]}(t_{i_j}).$$

Hence, applying again convexity of φ and then changing the variables, we obtain

$$A_{n,k}^{\bar{i}}\leq|n|\int_{U_{\bar{i}}}\int_{V_{\bar{i}}}(\int_{U_{\bar{i}}(u)}\int_{V_{\bar{i}}(u)}\varphi(nt-k,\ 2^m|f(t)-f(t+u)|)$$
$$dt_{\bar{i}'}dt_{\bar{i}})du_{\bar{i}'}du_{\bar{i}}=B_{n,k}^{\bar{i}}.$$

3.1. THEOREM. *Let φ satisfy the condition* 1.4(c) *and let*

(1) $\varepsilon_{n,k}^{\bar{i}}(s)=\varepsilon_{n,k}(v_1,\ldots,v_m),$

where $v_i=s_i$ if $i\in\bar{i}$, $v_i=1-s_i$ if $i\notin\bar{i}$. Then there holds the inequality

$$\Omega_n(f)\leq\omega_\varphi\left(2^m\ cf,\ \frac{1}{|n|}\right)+\frac{|n|}{2^m}\sum_{k=0}^{n-1}\sum_{\bar{i}\subset\bar{m}}\int_{I_n}s_1\ldots s_m,\ \varepsilon_{n,k}^{\bar{i}}(ns)ds,$$

where $I_n=\left[0,\ \frac{1}{n_1}\right]\times\ldots\times\left[0,\ \frac{1}{n_m}\right]$ *for arbitrary $n\in N^m$.*

PROOF. Let us remark that 1.4(c) implies

$$\varphi(nt-k,\ u)\leq\varphi(t,\ cu)+\varepsilon_{n,k}(nt-k)$$

for $t\in Q_k$, $u\geq 0$. Hence

$$B_{n,k}^{\bar{i}}\leq|n|\int_{U_{\bar{i}}}\int_{V_{\bar{i}}}(\int_{Q_k}\varphi(t,\ 2^mc|f(t)-f(t+u)|)dt)du_{\bar{i}'}du_{\bar{i}}$$
$$+|n|\int_{U_{\bar{i}}}\int_{V_{\bar{i}}}(\int_{U_{\bar{i}}(u)}\int_{V_{\bar{i}}(u)}\varepsilon_{n,k}(nt-k)dt_{\bar{i}'}dt_{\bar{i}})du_{\bar{i}'}du_{\bar{i}}=\tilde{J}_{n,k}^{\bar{i}}+J_{n,k}^{\bar{i}}.$$

Hence, by Lemma 2.4,

(2) $$\Omega_n(f)\leq\frac{1}{2^m}\sum_{k=0}^{n-1}\sum_{\bar{i}\subset\bar{m}}\tilde{J}_{n,k}^{\bar{i}}+\frac{1}{2^m}\sum_{k=0}^{n-1}\sum_{\bar{i}\subset\bar{m}}J_{n,k}^{\bar{i}}.$$

But

$$\frac{1}{2^m}\sum_{k=0}^{n-1}\sum_{\bar{i}\subset\bar{m}}\tilde{J}_{n,k}^{\bar{i}}\leq\frac{|n|}{2^m}\sum_{k=0}^{n-1}\int_{Q_k}(\sum_{\bar{i}\subset\bar{m}}\int_{U_{\bar{i}}}\int_{V_{\bar{i}}}\varphi(t,\ 2^mc|f(t)-f(t+u)|)$$

$$du_{i'}du_{\bar{i}})\,dt=\frac{|n|}{2^m}\sum_{\bar{i}\subset\bar{m}}\int_{U_{\bar{i}}}\int_{V_{\bar{i}}}(\int_Q\varphi(t,\ 2^m c|f(t)-f(t+u)|)\,dt)\,du_{\bar{i}'}du_{\bar{i}}$$
$$\leq\omega_\varphi\left(2^m\ cf,\ \frac{1}{|n|}\right)$$

Moreover, by changing variables u_i and t_i and then substituting

$$t_i-\frac{k_i}{n_i}=s_i \text{ if } i\in\bar{i},\ \frac{k_i+1}{n_i}-t_i=s_i \text{ if } i\notin\bar{i},$$

we obtain easily

$$J^{\bar{i}}_{n,k}=|n|\int_{I_n}s_1\ldots s_m\varepsilon^{\bar{i}}_{n,k}(ns)\,ds_1\ldots ds_m.$$

Hence, by inequality (2), we obtain the desired result.

3. 2. COROLLARY. *Let φ satisfy the condition* 1. 4(c) *with $\varepsilon_{n,k}$ constant. Then there holds the inequality*

$$\Omega_n(f)\leq\omega_\varphi\left(2^m\ cf,\ \frac{1}{|n|}\right)+\frac{1}{2^m}\frac{1}{|n|}\sum_{k=0}^{n-1}\varepsilon_{n,k}.$$

This corollary follows from Theorem 3. 1, *immediately.*

3. 3. THEOREM. *Let φ satisfy the conditions* 1. 2(a), (b) *and* 1. 4(c) *with $\varepsilon_{n,k}$ such that*

$$\lim_{n\to\infty}|n|\sum_{k=0}^{n-1}\int_{I_n}s_1\ldots s_m\varepsilon^{\bar{i}}_{n,k}(ns)\,ds=0$$

for every $\bar{i}\subset\bar{m}$, where $n\to\infty$ means $n_i\to\infty$ for $i=1,\ldots,m$. Then

$$(3)\qquad R_n(f,\cdot)\longrightarrow\int_Q f(x)\,dx\qquad as\ n\to\infty$$

in the sense of modular convergence in $L^\varphi(Q)$ (for the definition see [5], 5. 1).

This theorem follows from Theorems 1. 3 *and* 3. 1. *Similarly as Corollary* 3. 2, *we get the following*

3. 4. COROLLARY. *Let φ satisfy* 1. 2(a), (b) *and* 1. 4(c) *with constant $\varepsilon_{n,k}$ such that*

$$\lim_{n\to\infty}\frac{1}{|n|}\sum_{k=0}^{n-1}\varepsilon_{n,k}=0$$

Then (3) *holds in the sense of modular convergence in $L^\varphi(Q)$.*

In case of φ independent of the variable t, the above results take the *following simplified form.*

3.5. COROLLARY. *Let $\varphi : \boldsymbol{R}_+ \longrightarrow \boldsymbol{R}_+$ be a convex φ-function. Then*

$$\Omega_n(f) \leq \omega_\varphi\left(2^m\, cf, \frac{1}{|n|}\right) \text{for } n=1, 2, \ldots$$

and so (3) *holds in the sense of modular convergence in the Orlicz space* $L^\varphi(Q)$.

In particular case of power function and dimension one, the above estimation is exactly that obtained in [1].

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