Hokkaido Mathematical Journal Vol. 18 (1989) p. 497-511

Generalized vector measures and path integrals for hyperbolic systems

Dedicated to Professor S. Koshi on his 60th birthday

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(Received March 23, 1988, Revised January 26, 1989)

§1. Introduction.

This paper concerns a path integral formula for the solution of the Cauchy problem for a hyperbolic system. Let us begin by considering an $N\times N$ hyperbolic system of the first order

$$(1.1)\qquad \frac{\partial}{\partial t}\Psi(t, x)=[\textstyle\sum_{l=1}^{d} P_{l}\frac{\partial}{\partial x_{l}}+iQ+V(x)]\Psi(t, x)\quad 0<t<T,\ x\in \boldsymbol{R}^{d},$$

where $0<T\leqq\infty$, and $V(x)$ is a complex-valued bounded Borel measurable function and the P_{l}, $1\leqq l\leqq d$, and Q are constant hermitian $N\times N$-matrices. For the case that the P_{l}'s are simultaneously diagonalizable, T. Ichinose made an elegant approach to the problem to obtain a path integral formula by constructing countably additive measures [3]. The Dirac equation in two space-time dimensions is applied to this case. As for the Dirac equation in four space-time dimensions, the P_{l}'s are not simultaneously diagonalizable. In this paper, we do not assume that the P_{l}'s are simultaneously diagonalizable. In this general case, note that the Cauchy problem for (1.1) is not L^{∞} well-posed but only L^{2} well-posed.

Concerning the Feynman-Kac formula for the Schrödinger group, I. Kluvanek has shown a complete space of integrable functions by using a seminorm[4]. In this paper, for hyperbolic systems we shall define the space $\mathfrak{G}$ of integrable functions with respect to μ_{t} which is an extension of tensor product spaces, where μ_{t} is an $\mathfrak{L}(L^{2}(\boldsymbol{R}^{d}; \boldsymbol{C}^{N}))$-valued generalized vector measure on the space $\tilde{X}_{t}$ of Lipschitz continuous paths $X:[0, t]\rightarrow \boldsymbol{R}^{d}$. However, μ_{t} is not countably additive. We shall show the construction of the integral of $\boldsymbol{C}^{N}$-valued functions on $\tilde{X}_{t}$ with respect to μ_{t}, where the integral of $G(X)g(X(0))$ [$G\in\mathfrak{G}$ and $g\in L^{2}(\boldsymbol{R}^{d}; \boldsymbol{C}^{N})$] is a limit of those of $\boldsymbol{C}^{N}$-valued simple functions. By this integral, we shall establish the path integral formula

$$\Psi(t, \cdot)=\int d\mu_{t}(X)\ \exp\{\int_{0}^{t}V(X(s))\ ds\}\ g(X(0)),$$

for the solution $\Psi(t, x)$ of the Cauchy problem for the hyperbolic system (1.1) with initial datum $\Psi(0, \cdot)=g$, which includes the Dirac equation in four space-time dimensions. In § 2, we shall explain some well-known results about hyperbolic systems for later use. § 3 is devoted to the study of the tensor product space $\boldsymbol{B}_{fin}(X_t : \otimes_\pi)$ and a bounded linear operator T_t of $\boldsymbol{B}_{fin}(X_t ; \otimes_\pi)$ into $\mathfrak{L}(L^2(\boldsymbol{R}^d ; \boldsymbol{C}^N))$, which is constructed by the fundamental solution of the Cauchy problem for (1.1) with $V=0$. We also study the set of functions expressed as $\Phi(X)=\exp\{\int_0^t V(X(s))\, ds\}$, where V is a complex-valued bounded Borel measurable function on $\boldsymbol{R}^d$. In § 4, we obtain main theorems (Theorems 2 and 3).

§ 2. The hyperbolic system of the first order.

Let $0<T\leqq\infty$ and consider the Cauchy problem for the hyperbolic system of the first order

$$(2.1)\qquad \begin{cases} [\partial_t-\sum_{l=1}^d P_l\partial_l]\Psi(t, x)=iQ\Psi(t, x) \quad 0<t<T,\ x\in\boldsymbol{R}^d \\ \Psi(0, x)=g(x), \end{cases}$$

where t and $x=(x_1, \cdots, x_d)$ are regarded as time and space variables respectively and the symbols $\partial_t=\partial/\partial t$ and $\partial_l=\partial/\partial x_l$ $(1\leqq l\leqq d)$ are used, $\Psi(t, x)$ is a $\boldsymbol{C}^N$-valued function and the P_l $(1\leqq l\leqq d)$ and Q are constant hermitian $N\times N$-matrices.

$\frac{1}{i}\sum_{l=1}^d P_l\partial_l+Q$ is, considered as an operator in $L^2(\boldsymbol{R}^d ; \boldsymbol{C}^N)$, essentially selfadjoint on $C_0^\infty(\boldsymbol{R}^d ; \boldsymbol{C}^N)$. Let H_0 be its selfadjoint extension and $\{U_t^0\}_{t\in\boldsymbol{R}}$ be the C_0-group of unitary operators on $L^2(\boldsymbol{R}^d ; \boldsymbol{C}^N)$ with the infinitesimal generator iH_0. Then

$$U_t^0\, g=\Psi(t, \cdot) \quad \text{for } g\in L^2(\boldsymbol{R}^d ; \boldsymbol{C}^N),$$

where $\Psi(t, \cdot)$ is the solution of (2.1) with initial datum $\Psi(0, \cdot)=g$.

For the solution Ψ of (2.1) with initial datum $g\in C_0^\infty(\boldsymbol{R}^d ; \boldsymbol{C}^N)$, we have the following equation

$$\Psi(t, x)=(U_t^0\, g)(x)=\int_{\boldsymbol{R}^d} K(t ; x, y)g(y)\, dy \quad 0<t<T,\ x\in\boldsymbol{R}^d$$

by using the fundamental solution $K(t ; x, y)$ of the Cauchy problem (2.1). It is also known that there is a finite propagation speed $v\geqq 0$ such that $K(t ; x, y)$ vanishes outside the backward conoid $\Gamma^{(t,x)}$, where

$$\Gamma^{(t,x)}=\{(s, y)\in\boldsymbol{R}\times\boldsymbol{R}^d ; 0\leqq s\leqq t,\ v\cdot(t-s)\geqq|x-y|\}$$

and $|x-y|$ is the Euclidean norm of $x-y$ in $\boldsymbol{R}^d$.

For $t\in[0, T)$ fixed, let $X_t=\prod_{[0,t]}\boldsymbol{R}^d$ be the product of the uncountably many $\boldsymbol{R}^d$.

§ 3. Tensor product spaces.

Let $\boldsymbol{B}(\boldsymbol{R}^d)$ be the space of complex-valued bounded Borel measurable functions on $\boldsymbol{R}^d$. For a finite partition Δ_n: $0=t_0<t_1<\cdots<t_n=t$ of the interval $[0, t]$, let $\boldsymbol{B}(X_t;\otimes_\pi, \Delta_n)$ denote the space of the complex-valued functions Ψ on X_t for which there exist functions $f_{j,k}\in\boldsymbol{B}(\boldsymbol{R}^d)$ $(j=0, 1, \cdots, n$ and $k=1, \cdots, m)$ such that

$$(3.1)\qquad \Psi(X)=(\textstyle\sum_{k=1}^m f_{0,k}\otimes\cdots\otimes f_{n,k})(X) = \sum_{k=1}^m\prod_{j=0}^n f_{j,k}(X(t_j))$$

equipped with π-norm.

For $\Psi=\sum_{k=1}^m f_{0,k}\otimes\cdots\otimes f_{n,k}$, its π-norm is defined as follows: $\|\Psi\|_\pi=\inf \sum_{k=1}^m\prod_{j=0}^n\|f_{j,k}\|_\infty$, where the infimum is taken over all representations of Ψ. If Δ_m is a refinement of Δ_n, every $\Psi\in\boldsymbol{B}(X_t;\otimes_\pi, \Delta_n)$ belongs to $\boldsymbol{B}(X_t;\otimes_\pi, \Delta_m)$ and the π-norm of Ψ considered as an element of $\boldsymbol{B}(X_t;\otimes_\pi, \Delta_n)$ is the same as that of $\boldsymbol{B}(X_t;\otimes_\pi, \Delta_m)$.

Let $\boldsymbol{B}_{fin}(X_t;\otimes_\pi)$ denote the space of functions Ψ on X_t for which there exists a finite partition Δ_n of $[0, t]$ such that $\Psi\in\boldsymbol{B}(X_t;\otimes_\pi, \Delta_n)$, equipped with π-norm. Let $T_t(\Delta_n)$ be a linear operator of $\boldsymbol{B}(X_t;\otimes_\pi, \Delta_n)$ into the space $\mathfrak{L}(L^2(\boldsymbol{R}^d;\boldsymbol{C}^N))$ of bounded linear operators on $L^2(\boldsymbol{R}^d;\boldsymbol{C}^N)$ defined by

$$(3.2)\qquad [T_t(\Delta_n)(f_0\otimes\cdots\cdots\otimes f_n)]g \equiv f_n U^0_{\Delta t_n} f_{n-1} U^0_{\Delta t_{n-1}}\cdots U^0_{\Delta t_2} f_1 U^0_{\Delta t_1}(f_0 g) = f_n\prod_{j=n-1}^1(U^0_{\Delta t_{j+1}} f_j)U^0_{\Delta t_1}(f_0 g)$$

for $f_0\otimes\cdots\cdots\otimes f_n\in\boldsymbol{B}(X_t;\otimes_\pi, \Delta_n)$ and $g\in L^2(\boldsymbol{R}^d;\boldsymbol{C}^N)$, where $\Delta t_j=t_j-t_{j-1}$ $(j=1, \cdots, n)$.

PROPOSITION 1. *For a finite partition Δ_n: $0=t_0<t_1<\cdots<t_n=t$ of $[0, t]$, $T_t(\Delta_n)$ is a bounded linear operator of $\boldsymbol{B}(X_t;\otimes_\pi, \Delta_n)$ into $\mathfrak{L}(L^2(\boldsymbol{R}^d;\boldsymbol{C}^N))$ such that*

$$\|T_t(\Delta_n)\Psi\|\leqq\|\Psi\|_\pi$$

holds for $\Psi\in\boldsymbol{B}(X_t;\otimes_\pi, \Delta_n)$.

PROOF. For $\Psi\in\boldsymbol{B}(X_t;\otimes_\pi, \Delta_n)$, there is a representation $\Psi=\sum_{k=1}^m f_{0,k}\otimes\cdots\otimes f_{n,k}$. Since U^0_s is a unitary operator, we get $\|[T_t(\Delta_n)(\sum_{k=1}^m f_{0,k}\otimes\cdots\otimes f_{n,k})]g\|_2\leqq\sum_{k=1}^m\|f_{n,k}\|_\infty\cdots\|f_{0,k}\|_\infty\|g\|_2$. The above relation holds for any representation $\sum_{k=1}^{l} f_{0,k}\otimes\cdots\otimes f_{n,k}$ of Ψ, and so it holds

$$\|(T_t(\Delta_n)\Psi)g\|_2 \leqq \|\Psi\|_\pi \|g\|_2,$$

which implies the desired result. □

LEMMA. *Let* $\sum_{k=1}^{r_1} f_{0,k}\otimes\cdots\otimes f_{n,k}$ *belong to* $\boldsymbol{B}(X_t; \otimes_\pi, \Delta_n)$ *and* $\sum_{l=1}^{r_2} g_{0,l}\otimes\cdots\otimes g_{m,l}$ *belong to* $\boldsymbol{B}(X_t; \otimes_\pi, \Delta_m)$.

If $(\sum_{k=1}^{r_1} f_{0,k}\otimes\cdots\otimes f_{n,k})(X) = (\sum_{l=1}^{r_2} g_{0,l}\otimes\cdots\otimes g_{m,l})(X)$ *holds for any* $X \in X_t$, *then we have* $T_t(\Delta_n)(\sum_{k=1}^{r_1} f_{0,k}\otimes\cdots\otimes f_{n,k}) = T_t(\Delta_m)(\sum_{l=1}^{r_2} g_{0,l}\otimes\cdots\otimes g_{m,l})$.

PROOF. Let Δ_r be a common refinement of Δ_n and Δ_m. Then both $\sum_{k=1}^{r_1} f_{0,k}\otimes\cdots\otimes f_{n,k}$ and $\sum_{l=1}^{r_2} g_{0,l}\otimes\cdots\otimes g_{m,l}$ can be considered as elements of $\boldsymbol{B}(X_t; \otimes_\pi, \Delta_r)$ by inserting the constant function 1. By the semigroup property of U_s^0 and the property of tensor product space, we can obtain the desired result. □

Now we define an operator T_t of $\boldsymbol{B}_{fin}(X_t; \otimes_\pi)$ into $\mathfrak{L}(L^2(\boldsymbol{R}^d; \boldsymbol{C}^N))$ by

$$T_t(\Psi) \equiv T_t(\Delta_n)(\Psi) \quad \text{for } \Psi \in \boldsymbol{B}(X_t; \otimes_\pi, \Delta_n).$$

Then it is well-defined by Lemma.

Let $\tilde{X}_t$ be the subset of those X in X_t for which $|X(s) - x(s')| \leqq v|s-s'|$ holds for any $0 \leqq s$, $s' \leqq t$, where v is the positive, finite propagation speed of the solution of (2.1) and $|X(s) - X(s')|$ is the Euclidean norm of $X(s) - X(s')$ in $\boldsymbol{R}^d$. Then we have

THEOREM 1. i) *T_t is a bounded linear operator of $\boldsymbol{B}_{fin}(X_t; \otimes_\pi)$ into $\mathfrak{L}(L^2(\boldsymbol{R}^d; \boldsymbol{C}^N))$ such that*

$$\|T_t\Psi\| \leqq \|\Psi\|_\pi$$

holds for $\Psi \in \boldsymbol{B}_{fin}(X_t; \otimes_\pi)$.

ii) Suppose that Φ is an element of $\boldsymbol{B}_{fin}(X_t; \otimes_\pi)$ *such that* $\Phi_{|\tilde{X}_t} = 0$. *Then* $T_t(\Phi) = 0$.

PROOF. i) Proposition 1 and Lemma show this fact.

ii) It is obtained by (3.2) and the fact that $K(t; x, y)$ vanishes outside the backward conoid $\Gamma^{(t,x)}$. □

PROPOSITION 2. *Let* $G = g_0\otimes\cdots\otimes g_m$ *be an element of* $\boldsymbol{B}(X_t; \otimes_\pi, \Delta_m)$ *with* Δ_m; $0 = t_0 < t_1 < \cdots < t_m = t$ *and put* $N_G = \{F = f_0\otimes\cdots\otimes f_m \in \boldsymbol{B}(X_t; \otimes_\pi, \Delta_m)$; $|f_j| \leqq g_j$ *for* $j = 0, \cdots, m\}$.

Suppose $\{F_n = f_{0,n}\otimes\cdots\otimes f_{m,n}\}$ *is a sequence of elements of* N_G *such that*

$$f_{j,0}(x) = \lim_{n\to\infty} f_{j,n}(x)$$

exists for every $x\in \boldsymbol{R}^d$ *and every* $j=0, \cdots, m$.

If we put $F_0=f_{0,0}\otimes\cdots\otimes f_{m,0}$, *then we have*

$$s\text{-}\lim_{n\to\infty}(T_t(F_n))h=(T_t(F_0))h$$

for any $h\in L^2(\boldsymbol{R}^d; \boldsymbol{C}^N)$.

PROOF. Since F_0 belongs to $\boldsymbol{B}(X_t; \otimes_\pi, \Delta_m)$, $T_t(F_0)$ is defined. Put $h_{0,n}=(f_{0,n}-f_{0,0})h$ and $h_{j,n}=(f_{j,n}-f_{j,0})\prod_{l=j-1}^{0}(U^0_{\Delta t_{l+1}}f_{l,0})h$ for $j=1, \cdots, m$ and $n\in \boldsymbol{N}$. Put $\Phi_0=2g_0h$ and $\Phi_j=2g_j\prod_{l=j+1}^{0}(U^0_{\Delta t_{l-1}}f_{l,0})h$ for $j=1, \cdots, m$. Then $|\Phi_j|^2(j=0, \cdots, m)$ is an integrable function on $\boldsymbol{R}^d$ with $|h_{j,n}|\leqq|\Phi_j|$ for $n\in\boldsymbol{N}$ and $\lim_{n\to\infty}h_{j,n}(x)=0$ almost everywhere. So by the Lebesgue dominated convergence theorem, $\lim_{n\to\infty}\|h_{j,n}\|_2=0$ $(j=0, \cdots, m)$. Then we get

$$\begin{aligned}
&\|(T_t(F_n))h-(T_t(F_0))h\|_2\\
&=\|(T_t(f_{0,n}\otimes\cdots\otimes f_{m,n}))h-(T_t(f_{0,0}\otimes\cdots\otimes f_{m,0}))h\|_2\\
&\leqq\textstyle\sum_{j=0}^{m}\|[T_t(f_{0,n}\otimes\cdots\otimes f_{j-1,n}(f_{j,n}-f_{j,0})f_{j+1,0}\otimes\cdots\otimes f_{m,0})]h\|_2\\
&\leqq\textstyle\sum_{j=0}^{m}\prod_{l=j+1}^{m}\|f_{l,n}\|_\infty\|h_{j,n}\|_2\\
&\leqq\textstyle\sum_{j=0}^{m}\prod_{l=j+1}^{m}\|g_l\|_\infty\|h_{j,n}\|_2
\end{aligned}$$

which converges to zero as $n\to\infty$. □

Let $\boldsymbol{B}(\tilde{X}_t; \otimes_\pi, \Delta_n)$[resp. $\boldsymbol{B}_{fin}(\tilde{X}_t; \otimes_\pi)$] be the space of functions F on $\tilde{X}_t$ such that there exists $\tilde{F}\in\boldsymbol{B}(X_t; \otimes_\pi, \Delta_n)$[resp. $\boldsymbol{B}_{fin}(X_t; \otimes_\pi)$] satisfying $F(X)=\tilde{F}(X)$ for $X\in\tilde{X}_t$. For $F\in\boldsymbol{B}_{fin}(\tilde{X}_t; \otimes_\pi)$, define T_t *by*

(3.3) $T_tF\equiv T_t\tilde{F}$, where $\tilde{F}\in\boldsymbol{B}_{fin}(X_t; \otimes_\pi)$ is an extension of F. The above definition is well-defined by Theorem 1 ii).

REMARK 1. T_t can also be considered as an operator of $\boldsymbol{B}_{fin}(\tilde{X}_t; \otimes_\pi)$ into $\mathfrak{L}(L^2(\boldsymbol{R}^d; \boldsymbol{C}^N))$.

Hereafter we shall consider $\tilde{X}_t$ instead of X_t. Let S be the set of those functions Φ on $\tilde{X}_t$ for which there exists $V\in\boldsymbol{B}(\boldsymbol{R}^d)$ satisfying

$$\Phi(X)=\exp\{\int_0^t V(X(s))\ ds\} \text{ for any } X\in\tilde{X}_t,$$

which is well-defined for $X\in\tilde{X}_t$ and $V\in\boldsymbol{B}(\boldsymbol{R}^d)$, since $V(X(\cdot))$ is a measurable function on $[0, t]$.

For $n\in\boldsymbol{N}$, let $\tilde{\Delta}_n$; $0=t_0<t_1<\cdots<t_n=t$ be the partition of $[0, t]$ such that $t_j=\frac{j}{n}t$ for $j=1, 2, \cdots, n$. For $\Phi\in S$, i.e. $\Phi(X)=\exp\{\int_0^t V(X(s))\ ds\}$, define the function $\Phi_{(n)}$ on $\tilde{X}_t$ by

(3.4) $\Phi_{(n)}(X) \equiv \exp\{\sum_{j=1}^{n} V(X(t_j))\frac{t}{n}\}$ for $X \in \tilde{X}_t$.

Then $\Phi_{(n)} \in \boldsymbol{B}(\tilde{X}_t; \otimes_\pi, \tilde{\Delta}_n)$. As for $T_t(\Phi_{(n)})$, we have

PROPOSITION 3. *For $\Phi \in S$, there exists $s\text{-}\lim_{n\to\infty}(T_t(\Phi_{(n)}))g$ in $L^2(\boldsymbol{R}^d; \boldsymbol{C}^N)$ for any $g \in L^2(\boldsymbol{R}^d; \boldsymbol{C}^N)$.*

PROOF. $\Phi \in S$ can be expressed as $\Phi(X) = c.\exp\{\int_0^t V(X(s))\, ds\}$ with $c \in \boldsymbol{R}$ and Re $V(x) \leqq 0$ for any $x \in \boldsymbol{R}^d$. Put $H = H_0 + \frac{1}{i}V$. Then Trotter's product formula shows that $T_t(\Phi_{(n)})g = c\cdot(e^{\frac{t}{n}V}\ U^0_{t/n})^n g$ converges to $c\cdot e^{iHt} g$ as $n \to \infty$, since $\{U^0_s\}_{s\in\boldsymbol{R}}$ and $\{e^{V(\cdot)s}\}_{s\in\boldsymbol{R}}$ are contraction semigroups on $L^2(\boldsymbol{R}^d; \mathbf{C}^N)$. □

By Proposition 3, we can extend T_t to an operator of S into $\mathfrak{L}(L^2(\boldsymbol{R}^d; \boldsymbol{C}^N))$ by

(3.5) $(T_t\Phi)g \equiv s\text{-}\lim_{n\to\infty}[T_t(\Phi_{(n)})]g$ for $\Phi \in S$

and for $g \in L^2(\boldsymbol{R}^d; \boldsymbol{C}^N)$. As for elements of S, we have

PROPOSITION 4. i) *For $V_1, V_2 \in \boldsymbol{B}(\boldsymbol{R}^d)$, put $\Phi(X) = \exp\{\int_0^t V_1(X(s))ds\}$ and $\Psi(X) = \exp\{\int_0^t V_2(X(s))ds\}$. If $V_1(x) = V_2(x)$ holds almost everywhere, then we have $(T_t\Phi)g = (T_t\Psi)g$ for any $g \in L^2(\boldsymbol{R}^d; \boldsymbol{C}^N)$.*

ii) *Let $\Psi(X) = \exp\{\int_0^t V(X(s))ds\}$ be an element of S with* $\sup\{\text{Re } V(x) : x \in \boldsymbol{R}^d\} \leqq 0$. *Put*

$N_\Psi = \{\Phi(X) = \exp\{\int_0^t U(X(s))ds\} \in S; |U| \leqq |V|$, Re $U(x) \leqq 0$ *for* $x \in \boldsymbol{R}^d\}$.

Suppose $\{\Phi_n(X) = \exp\{\int_0^t V_n(X(s))ds)\}$ is a sequence of elements of N_Ψ such that

$$V_0(x) = \lim_{n\to\infty} V_n(x)$$

exists for every $x \in \boldsymbol{R}^d$.

Then by putting $\Phi_0(X) = \exp\{\int_0^t V_0(X(s))ds\}$ we have

$$\Phi_0 \in S \text{ and } s\text{-}\lim_{n\to\infty}(T_t(\Phi_n))g = (T_t(\Phi_0))g$$

for any $g\in L^2(\boldsymbol{R}^d;\boldsymbol{C}^N)$.

PROOF. i) By the equation $T_t(\Phi_{(n)})g=(e^{\frac{t}{n}V_1}U^0_{t/n})^n g$, we have $T_t(\Phi_{(n)})g=T_t(\Psi_{(n)})g$ for any $n\in\boldsymbol{N}$, since $\exp\frac{1}{n}V_1(x)=\exp\frac{1}{n}V_2(x)$ holds almost everywhere. So we have the desired result.

ii) Put $H_n=H_0+\frac{1}{i}V_n$ for $n\in\boldsymbol{N}$ and $H_{00}=H_0+\frac{1}{i}V_0$. Then by the proof of Proposition 3. $(T_t\Phi_0)g=e^{iH_{00}t}g$ and $(T_t\Phi_n)g=e^{iH_nt}g$ hold for $n\in\boldsymbol{N}$. Since we have $\|e^{iH_nt}\|\leqq 1$ and $(\lambda-H_n)^{-1}h\to(\lambda-H_{00})^{-1}h$ as $n\to\infty$ for every $h\in L^2(\boldsymbol{R}^d;\boldsymbol{C}^N)$ and λ with Re $\lambda>0$, $e^{iH_nt}g\to e^{iH_{00}t}g$ holds for $t\geqq 0$ [5 Theorem 4.2], which proves the proposition. □

§ 4. Generalized vector measures and path integrals.

Let $\boldsymbol{B}_{fin}(\tilde{X}_t;\hat{\otimes}_\pi)$ be the completion of $\boldsymbol{B}_{fin}(\tilde{X}_t;\otimes_\pi)$. Then by Theorem 1, T_t can be extended to a continuous linear operator from the Banach space $\boldsymbol{B}_{fin}(\tilde{X}_t;\hat{\otimes}_\pi)$ of complex functions on $\tilde{X}_t$ into $\mathfrak{L}(L^2(\boldsymbol{R}^d;\boldsymbol{C}^N))$. We shall associate with T_t an $\mathfrak{L}(L^2(\boldsymbol{R}^d;\boldsymbol{C}^N))$-valued finitely additive vector measure μ_t on $\tilde{X}_t$ and determine integrable functions with respect to μ_t.

We shall consider a field generated by subsets of $\tilde{X}_t$. Let $\mathfrak{B}$ be the set of Borel subsets of $\boldsymbol{R}^d$. For a partition $\Delta_n;0=t_0<t_1<\cdots<t_n=t$ of $[0,t]$ and $B_j\in\mathfrak{B}(j=0,1,\cdots,n)$, put $J(B_0,B_1,\cdots,B_n;\Delta_n)\equiv\{X\in\tilde{X}_t;X(t_j)\in B_j\ (j=0,1,\cdots,n)\}$. Let $\mathfrak{J}$ be the set $\{J(B_0,B_1,\cdots,B_n;\Delta_n);\Delta_n$ is a partition of $[0,t]$, $B_j\in\mathfrak{B}\}$ and $\mathfrak{F}$ be the field generated by $\mathfrak{J}$. Let $\mathfrak{S}$ be the space of $\mathfrak{F}$-measurable finitely-valued numerical functions on $\tilde{X}_t$. Then $\mathfrak{S}$ is a subspace of $\boldsymbol{B}_{fin}(\tilde{X}_t;\otimes_\pi)$.

We shall define the space of integrable functions with respect to μ_t, which includes S. For $\Phi\in S$, we have defined the function $\Phi_{(n)}$. To define the corresponding function for a function on $\tilde{\tilde{X}}_t$, we shall introduce a subset $\tilde{\tilde{X}}_t$ of X_t defined as follows.

For $n\in\boldsymbol{N}$. let $\tilde{\Delta}_n:0=t_0<t_1<\cdots<t_n=t$ be the partition of $[0,t]$ such that $t_j=\frac{j}{n}t$ for $j=1,2,\cdots,n$. For $X\in X_t$, define $X^{\tilde{\Delta}_n}\in X_t$ by

$$X_{\tilde{\Delta}_n}(s)\equiv X(t_j)\ \text{for}\ t_{j-1}<s\leqq t_j\ (j=1,\cdots,n)$$

and $X_{\tilde{\Delta}_n}(0)\equiv X(0)$.

Let $\tilde{\tilde{X}}_t$ be the subset of those X in X_t for which either $X\in\tilde{X}_t$ or there exist $\tilde{X}\in\tilde{X}_t$ and $n\in\boldsymbol{N}$ such that $\tilde{X}_{\tilde{\Delta}_n}=X$. For a function F on $\tilde{\tilde{X}}_t$, define the function $F_{(\tilde{n})}$ on $\tilde{X}_t$ by

(4.1) $F_{(\tilde{n})}(X)\equiv F(X_{\tilde{\Delta}_n})$ for $X\in\tilde{X}_t$.

Since $V(X(\cdot))$ is a measurable function on $[0, t]$ for $V\in\boldsymbol{B}(\boldsymbol{R}^d)$ and $X\in\tilde{\tilde{X}}_t$, a function $\Phi\in S$ $[\Phi(X)=\exp\{\int_0^t V(X(s))\ ds\}$ with $V\in\boldsymbol{B}(\boldsymbol{R}^d)]$ can be considered as a function $\tilde{\tilde{\Phi}}$ on $\tilde{\tilde{X}}_t$ satisfying

$$\tilde{\tilde{\Phi}}(X)=\exp\{\int_0^t V(X(s))\ ds\} \text{ for } X\in\tilde{\tilde{X}}_t.$$

Let $\tilde{S}$ be the space of such functions $\tilde{\tilde{\Phi}}$ on $\tilde{\tilde{X}}_t$. For $\Phi\in S$, we have $\Phi_{(n)}=\tilde{\tilde{\Phi}}_{(\tilde{n})}$, where $\Phi_{(n)}$ and $\tilde{\tilde{\Phi}}_{(\tilde{n})}$ are defined by (3.4) and (4.1). Let $\boldsymbol{B}_{fin}(\tilde{\tilde{X}}_t\,;\hat{\otimes}_\pi)$ be the space of functions on $\tilde{\tilde{X}}_t$ which are restrictions of elements of $\boldsymbol{B}_{fin}(X_t\,;\hat{\otimes}_\pi)$ to $\tilde{\tilde{X}}_t$.

Let $\tilde{\mathfrak{G}}$ be the set of those functions Ψ on $\tilde{\tilde{X}}_t$ such that

$$\Psi_{(\tilde{n})}\in\boldsymbol{B}_{fin}(\tilde{X}_t\,;\otimes_\pi) \text{ for any } n\in\boldsymbol{N}, \text{ and}$$
$$s\text{-}\lim_{n\to\infty} T_t(\Psi_{(\tilde{n})})g \text{ exists for any } g\in L^2(\boldsymbol{R}^d\,;\boldsymbol{C}^N).$$

REMARK 2. T_t can also be considered as an operator of $\boldsymbol{B}_{fin}(\tilde{\tilde{X}}_t\,;\hat{\otimes}_\pi)$ into $\mathfrak{L}(L^2(\boldsymbol{R}^d\,;\boldsymbol{C}^N))$ such that $T_tF=T_t(F|_{\tilde{X}_t})$ for $F\in\boldsymbol{B}_{fin}(\tilde{\tilde{X}}_t\,;\hat{\otimes}_\pi)$, by (3.3) and Remark 1. T_t can also be considered as an operator of $\tilde{S}$ into $\mathfrak{L}(L^2(\boldsymbol{R}^d\,;\boldsymbol{C}^N))$.

DEFINITION 1. For $J=J(B_0, B_1, \cdots, B_n\,;\Delta_n)\in\mathfrak{J}$, we shall define an operator $\mu_t(J)\in\mathfrak{L}(L^2(\boldsymbol{R}^d\,;\boldsymbol{C}^N))$ by

(4.2) $(\mu_t(J))g\equiv(T_t(\chi_J))g$ for $g\in L^2(\boldsymbol{R}^d\,;\boldsymbol{C}^N)$,

where $\chi_J\equiv\chi_{B_0}\otimes\chi_{B_1}\otimes\cdots\otimes\chi_{B_n}$ is the characteristic function of the set J.

Then μ_t is an $\mathfrak{L}(L^2(\boldsymbol{R}^d\,;\boldsymbol{C}^N))$-valued finitely additive vector measure μ_t on $\mathfrak{J}$. We shall construct the integral of a $\boldsymbol{C}^N$-valued function on $\tilde{X}_t$ with respect to μ_t. Put $\mathfrak{J}_0=\{J(B_0, B_1, \cdots, B_n\,;\Delta_n)\in\mathfrak{J}\,;\ B_0$ is relatively compact$\}$. We shall say that Θ is a $\boldsymbol{C}^N$-valued $\mathfrak{J}_0$-simple function on $\tilde{X}_t$ if there exist $k\in\boldsymbol{N}$, $\bar{a}_j\in\boldsymbol{C}^N$ and $J_j\in\mathfrak{J}_0$ satisfying

$$\Theta=\sum_{j=1}^k \bar{a}_j\,\chi_{J_j}.$$

Consider $\bar{a}\chi_m\in L^2(\boldsymbol{R}^d\,;\boldsymbol{C}^N)$ such that

$$\bar{a}\chi_m(x)=\begin{cases}\bar{a} & \text{for } \|x\|\leqq m\\ \bar{0} & \text{for } \|x\|>m\end{cases},$$

where $\bar{a}\in\boldsymbol{C}^N$ and $\bar{0}$ is th zero element of $\boldsymbol{C}^N$.

PROPOSITION 5. *For a $\boldsymbol{C}^N$-valued $\mathfrak{J}_0$-simple function $\Theta=\bar{a}\,\mathrm{X}_J$ on $\tilde{X}_t$ with $J=J(B_0, B_1, \cdots, B_n; \Delta_n)\in\mathfrak{J}_0$ and $\bar{a}\in\boldsymbol{C}^N$, we have*

$$s\text{-}\lim_{m\to\infty}\mu_t(J)(\bar{a}\mathrm{X}_m)=\mu_t(J)(\bar{a}\mathrm{X}_{B_0}) \ \text{ in } L^2(\boldsymbol{R}^d; \boldsymbol{C}^N).$$

PROOF. Since B_0 is relatively compact, $\bar{a}\mathrm{X}_{B_0}\in L^2(\boldsymbol{R}^d; \boldsymbol{C}^N)$. By the relation $\mu_t(J)(\bar{a}\mathrm{X}_m)=\mu_t(J)(\bar{a}\mathrm{X}_m\mathrm{X}_{B_0})$, we have

$$s\text{-}\lim_{m\to\infty}\mu_t(J)(\bar{a}\mathrm{X}_m)=\mu_t(J)(\bar{a}\mathrm{X}_{B_0}). \qquad \square$$

DEFINITION 2. We shall define the integral of a $\boldsymbol{C}^N$-valued $\mathfrak{J}_0$-simple function $\Theta=\sum_{j=1}^{k}\bar{a}_jX_{J_j}$ on $\tilde{X}_t$ with respect to μ_t by

$$\int_{\tilde{X}_t}d\mu_t(X)\Theta(X)\equiv s\text{-}\lim_{m\to\infty}\sum_{j=1}^{k}\mu_t(J_j)(\bar{a}_j\mathrm{X}_m). \tag{4.3}$$

By Proposition 5, (4.3) is well-defined. It is equal to $\sum_{j=1}^{k}\mu_t(J_j)\,\bar{a}_jX_{B^0_j}$, where $J_j=J(B_0^j, B_1^j, \cdots, B_n^j; \Delta_n)$,, and belongs to $L^2(\boldsymbol{R}^d; \boldsymbol{C}^N)$. We have

PROPOSITION 6. *Suppose $G\in\mathfrak{S}$ and $g\in L^2(\boldsymbol{R}^d; \boldsymbol{C}^N)$ is a simple function, i.e. $G=\sum_{j=1}^{k}b_j\mathrm{X}_{J_j}$ with $b_j\in\boldsymbol{C}$ and $g=\sum_{i=1}^{l}\bar{a}_i\mathrm{X}_{C_i}$ with $\bar{a}_i\in\boldsymbol{C}^N$ and relatively compact $C_i\in\mathfrak{B}$. Then we have*

$$\int_{\tilde{X}_t}d\mu_t(X)G(X)g(X(0))=\sum_{j=1}^{k}b_j\cdot\mu_t(J_j)g=(T_tG)g. \tag{4.4}$$

PROOF. $\Theta(X)=G(X)g(X(0))$ is a $\boldsymbol{C}^N$-valued $\mathfrak{J}_0$-simple function on $\tilde{X}_t$, since we have $\mathrm{X}_J(X)\,\bar{a}\mathrm{X}_C(X(0))=\bar{a}\mathrm{X}_{J\circ C}(X)$, where $J\circ C=J(B_0, B_1, \cdots, B_n; \Delta_n)\circ C\equiv J(B_0\cap C, B_1, \cdots, B_n; \Delta_n)$. By the relation $\mu_t(J\circ C)(\bar{a}\mathrm{X}_C)=\mu_t(J)(\bar{a}\mathrm{X}_C)$, (4.2) and (4.3), we have

$$\int_{\tilde{X}_t}d\mu_t(X)G(X)g(X(0))=\sum_{j=1}^{k}\sum_{i=1}^{l}b_j\cdot\mu_t(J_j)(\bar{a}_i\mathrm{X}_{C_i})$$
$$=\sum_{j=1}^{k}b_j\cdot\mu_t(J_j)g=(T_tG)g. \qquad \square$$

As for convergence of the integral of $\{G_n\Phi_m\}$ with respect to μ_t for $G_n\in\mathfrak{S}$ and a simple function $\Phi_m\in L^2(\boldsymbol{R}^d; \boldsymbol{C}^N)$, we have the following

PROPOSITION 7. *Let $\{G_n\}$ be a sequence in $\mathfrak{S}$ such that $\lim_{n,m\to\infty}\|G_n-G_m\|_\pi=0$ and $\{\Phi_n\}$ be a sequence of simple functions in $L^2(\boldsymbol{R}^d; \boldsymbol{C}^N)$ such that $|\Phi_n|\leqq|g|$ with $g\in L^2(\boldsymbol{R}^d; \boldsymbol{C}^N)$ and $\lim_{n\to\infty}\Phi_n(x)$ exists almost everywhere.*

Then there exists a subsequence $\{\Phi_{j(n)}\}$ of $\{\Phi_n\}$ such that

$s\text{-}\lim_{n\to\infty}\int_{\tilde{\tilde{X}}_t} d\mu_t(X)G_n(X)\Phi_{j(n)}(X(0))$ *exists.*

Moreover, for any subsequence $\{\Phi_{k(n)}\}$ *of* $\{\Phi_n\}$, $\int_{\tilde{\tilde{X}}_t} d\mu_t(X)G_n(X)\Phi_{k(n)}(X(0))$ *converges to the same element as* $n\to\infty$ *if its limit exists.*

PROOF. Put $h(x)=\lim_{n\to\infty}\Phi_n(x)$ a.e. Then for any $n\in\boldsymbol{N}$, we have $s\text{-}\lim_{j\to\infty}(T_tG_n)\Phi_j=(T_tG_n)h$ by the Lebesgue dominated convergence theorem. Let $\{\Phi_{j(n)}\}$ be a subsequence of $\{\Phi_n\}$ such that $\|(T_tG_n)\Phi_{j(n)}-(T_tG_n)h\|\leqq 1/n$ for any $n\in\boldsymbol{N}$. By Theorem 1 we have $\|T_t(G_n)\Phi_j-T_t(G_m)\Phi_j\|\leqq G_n-G_m\|_\pi\cdot\|\Phi_j\|\leqq\|G_n-G_m\|_\pi\cdot\|g\|$ for any $j\in\boldsymbol{N}$. So by the relation (4.4), there exists $s\text{-}\lim_{n\to\infty}\int_{\tilde{\tilde{X}}_t} d\mu_t(X)G_n(X)\Phi_{j(n)}(X(0))$.

By the relation $\lim_{n,m\to\infty}\|G_n-G_m\|_\pi=0$, there exists $F\in\boldsymbol{B}_{fin}(\tilde{\tilde{X}}_t\,;\hat{\otimes}_\pi)$ such that $\lim_{n\to\infty}\|G_n-F\|_\pi=0$. Then for any subsequence $\{\Phi_{k(n)}\}$ of $\{\Phi_n\}$, $s\text{-}\lim_{n\to\infty}\int_{\tilde{\tilde{X}}_t} d\mu_t(X)G_n(X)\Phi_{k(n)}(X(0))$ *is equal to* $(T_tF)g$ if its limit exists. □

As a consequence, we have

COROLLARY. *For* $F\in\boldsymbol{B}_{fin}(\tilde{\tilde{X}}_t\,;\hat{\otimes}_\pi)$ *and* $g\in L^2(\boldsymbol{R}^d\,;\boldsymbol{C}^N)$, *there exist a sequence* $\{G_n\}$ *in* $\mathfrak{S}$ *and a sequence* $\{\Phi_n\}$ *of simple functions in* $L^2(\boldsymbol{R}^d\,;\boldsymbol{C}^N)$ *satisfying*

i) $\lim_{n\to\infty}\|G_n-F|_{\tilde{X}_t}\|_\pi=0$

ii) $g(x)=\lim\Phi_n(x)$ *a.e.*

iii) $s\text{-}\lim_{n\to\infty}\int_{\tilde{\tilde{X}}_t} d\mu_t(X)G_n(X)\Phi_n(X(0))$ *exists.*

Moreover, $s\text{-}\lim_{n\to\infty}\int_{\tilde{\tilde{X}}_t} d\mu_t(X)G_n(X)\Phi_n(X(0))$ *is the same for any sequences* $\{G_n\}$ *and* $\{\Phi_n\}$ *satisfying* i)~iii).

DEFINITION 3. i) For $F\in\boldsymbol{B}_{fin}(\tilde{\tilde{X}}_t\,;\hat{\otimes}_\pi)$ and $g\in L^2(\boldsymbol{R}^d\,;\boldsymbol{C}^N)$, there exist a sequence $\{G_n\}$ in $\mathfrak{S}$ and a sequence $\{\Phi_n\}$ of simple functions in $L^2(\boldsymbol{R}^d\,;\boldsymbol{C}^N)$ *satisfying the condition* i)~iii) *in Corollary to Proposition 7. So we shall define the integral of the function* $F(X)g(X(0))$ on $\tilde{\tilde{X}}_t$ with respect to μ_t by

$$\int_{\tilde{\tilde{X}}_t} d\mu_t(X)F(X)g(X(0))\equiv s\text{-}\lim_{n\to\infty}\int_{\tilde{\tilde{X}}_t} d\mu_t(X)G_n(X)\Phi_n(X(0)). \tag{4.5}$$

ii) For $\Psi\in\tilde{\mathfrak{S}}$, its integral is defined by

$$(4.6) \qquad \int_{\tilde{X}_t} d\mu(X)\Psi(X)g(X(0)) \equiv s\text{-}\lim_{n\to\infty} \int_{\tilde{X}_t} d\mu_t(X)\Psi_{(\tilde{n})}(X)g(X(0))$$

for any $g \in L^2(\boldsymbol{R}^d; \boldsymbol{C}^N)$.

The above definitions are well-defined by the defintion of $\tilde{\mathfrak{G}}$ and Corollary to Proposition 7.

DEFINITION 4. Let $\mathfrak{G}$ be the linear span of $\boldsymbol{B}_{fin}(\tilde{\tilde{X}}_t; \hat{\otimes}_\pi)$ and $\tilde{\mathfrak{G}}$. We shall call the members of $\mathfrak{G}$ to be *integrable functions with respect to* μ_t.

REMARK 3. For $\Phi \in S$ i.e. $\Phi(X) = \exp\{\int_0^t V(X(s))\ ds\}$, $\lim_{n\to\infty}\Phi_{(n)}(X) = \Phi(X)$ does not necessarily hold for $X \in \tilde{X}_t$ if $V(x)$ is not Riemann integrable, but $\tilde{\tilde{\Phi}}$ belongs to $\mathfrak{G}$.

Though μ_t is not countably additive, we have constructed the integral of $\boldsymbol{C}^N$-valued functions on $\tilde{X}_t$ with respect to μ_t and it has the property of some kind of a dominated convergence theorem as shown in the following proposition 8. So we shall call μ_t a *generalized vector measure on* $\tilde{X}_t$. By a generalized measure we mean a measure which is not necessarily countably additive but has some more property than a merely finitely additive measure. [1]

THEOREM 2. *There exist a $\mathfrak{L}(L^2(\boldsymbol{R}^d; \boldsymbol{C}^N))$-valued generalized vector measure μ_t on $\mathfrak{J}$ which represents T_t in the sense that*

i) $(\mu_t(J))g = (T_t(\chi_J))g$

for $J = J(B_0, B_1, \cdots, B_n; \Delta_n) \in \mathfrak{J}$ and $g \in L^2(\boldsymbol{R}^d; \boldsymbol{C}^N)$, where $\chi_J = \chi_{B_0} \otimes \chi_{B_1} \otimes \cdots \otimes \chi_{B_n}$ is the characteristic function of the set J.

ii) *For $F \in \mathfrak{G}$(= the space of integrable functions), and $g \in L^2(\boldsymbol{R}^d; \boldsymbol{C}^N)$, there is a sequence $\{\Theta_n\}$ of $\boldsymbol{C}^N$-valued $\mathfrak{J}_0$-simple functions on $\tilde{X}_t$ such that*

$$\int_{\tilde{X}_t} d\mu_t(X)F(X)g(X(0)) = s\text{-}\lim_{n\to\infty} \int_{\tilde{X}_t} d\mu_t(X)\Theta_n(X).$$

iii) *Every $\Psi \in \boldsymbol{B}_{fin}(\tilde{\tilde{X}}_t; \hat{\otimes}_\pi) \cup \tilde{S}$ is an integrable function with respect to μ_t and*

$$(T_t\Psi)g = \int_{\tilde{X}_t} d\mu_t(X)\Psi(X)g(X(0)) \ \textit{holds}$$

for any $g \in L^2(\boldsymbol{R}^d; \boldsymbol{C}^N)$.

iv) *For $J = J(B_0, B_1, \cdots, B_n; \Delta_n) \in \mathfrak{J}$, $m(B_0 \times \cdots \times B_n) = 0$ implies $\mu_t(J) = 0$, where m is the Lebesgue measure.*

PROOF. i) follows from the definition of μ_t.

ii) For $F\in \boldsymbol{B}_{fin}(\tilde{\tilde{\bar{X}}}_t;\hat{\otimes}_\pi)$, put $\tilde{F}=F|_{\tilde{X}_t}$. Let $g\in L^2(\boldsymbol{R}^d;\boldsymbol{C}^N)$. Then by (4.5), there exist a sequence $\{G_n\}$ in $\mathfrak{S}$ such that $\lim_{n\to\infty}\|G_n-\tilde{F}\|_\pi=0$ and a sequence $\{\Phi_n\}$ of simple functions in $L^2(\boldsymbol{R}^d;\boldsymbol{C}^N)$ satisfying

$$\int_{\tilde{X}_t} d\mu_t(X)F(X)g(X(0))\equiv s\text{-}\lim_{n\to\infty}\int_{\tilde{X}_t} d\mu_t(X)G_n(X)\Phi_n(X(0)).$$

Put $\Theta_n(X)=G_n(X)\Phi_n(X(0))$ for any $X\in\tilde{X}_t$. Then $\{\Theta_n\}$ is a desired sequence of $\boldsymbol{C}^N$-valued $\mathfrak{J}_0$-simple functions on $\tilde{X}_t$.

For $\Phi\in\tilde{\mathfrak{S}}$, $\Phi_{(\tilde{n})}$ belongs to $\boldsymbol{B}_{fin}(\tilde{X}_t;\otimes_\pi)$. So the above statement shows that there exists a $\boldsymbol{C}^N$-valued $\mathfrak{J}_0$-simple function Θ_n on $\tilde{X}_t$ such that

$$\left\|\int_{\tilde{X}_t} d\mu_t(X)\Phi_{(\tilde{n})}(X)g(X(0))-\int_{\tilde{X}_t} d\mu_t(X)\Theta_n(X)\right\|\leqq\frac{1}{n}.$$

By using th definition (4.6), we have

$$\int_{\tilde{X}_t} d\mu_t(X)\Theta(X)g(X(0))=s\text{-}\lim_{n\to\infty}\int_{\tilde{X}_t} d\mu_t(X)\Theta_n(X).$$

iii) follows from (3.3), (3.5), (4.2), (4.5), (4.6) and Remarks 1 and 2.

iv) For $J=J(B_0,B_1,\cdots,B_n;\Delta_n)\in\mathfrak{J}$, $m(B_0\times\cdots\times B_n)=0$ implies $(T_t(\Delta_n)(\chi_{B_0}\otimes\chi_{B_1}\otimes\cdots\otimes\chi_{B_n}))g=0$ for any $g\in L^2(\boldsymbol{R}^d;\boldsymbol{C}^N)$ by the definition of T_t. So $\mu_t(J)=0$. □

The generalized measure μ_t defined above is not countably additive, but Propositions 2 and 4 show that it has the property of some kind of a dominated convergence theorem as shown in the following.

PROPOSITION 8. *Let $G=g_0\otimes\cdots\otimes g_m$ be an element of $\boldsymbol{B}(\tilde{X}_t;\otimes_\pi,\Delta_m)$ with $\Delta_m;0=t_0<t_1<\cdots<t_m=t$ and put $N_G=\{F=f_0\otimes\cdots\otimes f_m\in\boldsymbol{B}(\tilde{X}_t;\otimes_\pi,\Delta_m);|f_j|\leqq g_j$ for $j=0,\cdots,m\}$.*

Suppose $\{F_n=f_{0,n}\otimes\cdots\otimes f_{m,n}\}$ is a sequence of elements of N_G such that

$$f_{j,0}(x)=\lim_{n\to\infty}f_{j,n}(x)$$

exists for every $x\in\boldsymbol{R}^d$ and every $j=0,\cdots,m$.

If we put $F_0=f_{0,0}\otimes\cdots\otimes f_{m,0}$, then we have

$$s\text{-}\lim_{n\to\infty}\int_{\tilde{X}_t} d\mu_t(X)F_n(X)h(X(0))=\int_{\tilde{X}_t} d\mu_t(X)F_0(X)h(X(0))$$

for any $h\in L^2(\boldsymbol{R}^d;\boldsymbol{C}^N)$.

PROPOSITION 9. *Let $\Psi(X)=\exp\{\int_0^t V(X(s))ds\}$ be an element of S*

with sup {Re $V(x)$: $x \in \boldsymbol{R}^d$} $\leqq 0$. *Put* $N_\Psi = \{\Phi(X) = \exp\{\int_0^t U(X(s))ds\} \in S;\ |U| \leqq |V|$, Re $U(x) \leqq 0$ *for* $x \in \boldsymbol{R}^d\}$.

Suppose $\{\Phi_n(X) = \exp\{\int_0^t V_n(X(s))ds\}\}$ *is a sequence of elements of* N_Ψ *such that*

$$V_0(x) = \lim_{n\to\infty} V_n(x)$$

exists for every $x \in \boldsymbol{R}^d$.

Then by putting $\Phi_0(x) = \exp\{\int_0^t V_0(X(s))ds\}$
we have

$$s\text{-}\lim_{n\to\infty} \int_{\tilde{X}_t} d\mu_t(X)\Phi_n(X)g(X(0) = \int_{\tilde{X}_t} d\mu_t(X)\Phi_0(X)g(X(0))$$

for any $g \in L^2(\boldsymbol{R}^d; \boldsymbol{C}^N)$.

REMARK 4. Proposition 8 shows that μ_t has the property of some kind of the dominated convergence theorem, but it is not countably additive on the σ-field generated by the $\mathfrak{J}_{\Delta_m} = \{J \in \mathfrak{J}; J = \{J(B_0, B_1, \cdots, B_m; \Delta_m); B_j \in \mathfrak{B}\}$, $m = 1, 2, \cdots$. Let $K_j \subset \boldsymbol{R}^d$ $(j = 1, 2, \cdots, m)$ be compact and put $K = K_m \times \cdots \times K_0$. If μ_t is restricted to $C_c^\infty(K) \equiv \{f \in C^\infty(\boldsymbol{R}^{d(m+1)})$; supp $f \subset K\}$, it has a kind of countable additivity as shown in the following. Since $C_c^\infty(K_j)$ $(j = 0, 1, \cdots, m)$ is a nuclear space [6, p. 530], the π- and ε- tensor product topologies coincide: $C_c^\infty(K_m) \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi C_c^\infty(K_0) = C_c^\infty(K_m) \hat{\otimes}_\varepsilon \cdots \hat{\otimes}_\varepsilon C_c^\infty(K_0) = C_c^\infty(K)$. By this fact, for f, $g \in L^2(\boldsymbol{R}^d; \boldsymbol{C}^N)$ fixed, there exist regular measures $\{\nu_t^{m,\alpha};\ |\alpha| = N(m)\}$ on K [2, p. 344] such that

$$\langle f, T_t(\Delta_m)(F)g \rangle = \int_K \sum_{|\alpha|=N(m)} \partial^\alpha F(x)\ d\nu_t^{m,\alpha}(x)$$

holds for $F \in C_c^\infty(K)$. In this case, the countable additive measure $\nu_t^{m,\alpha}$ does not act on F but on the partial derivative $\partial^\alpha F$ with $|\alpha| = N(m)$. If the set $\{N(m);\ m \in \boldsymbol{N}\}$ is bounded, the countable additive measure $\nu_t^{m,\alpha}$ may be extended to a finitely additive measure on $\mathfrak{J}$, but the author is not sure about its boundedness.

Now we consider the hyperbolic system of the first order

$$(4.7) \qquad \begin{cases} \dfrac{\partial}{\partial t}\Psi(t, x) = [\sum_{l=1}^d P_l \dfrac{\partial}{\partial x_l} + iQ + V(x)]\Psi(t, x) \\ \qquad\qquad\qquad\qquad\qquad 0 < t < T,\ x \in \boldsymbol{R}^d \\ \Psi(0, x) = g(x), \end{cases}$$

where V is a complex-valued bounded Borel measurable function on $\boldsymbol{R}^d$.

By theorem 2, T_t may be regarded as a $\mathfrak{L}(L^2(\boldsymbol{R}^d; \boldsymbol{C}^N))$-valued generalized vector measure μ_t on $\tilde{X}_t$ and so we have the following theorem.

THEOREM 3. *There exists a $\mathfrak{L}(L^2(\boldsymbol{R}^d; \boldsymbol{C}^N))$-valued generalized vector measure μ_t on $\tilde{X}_t$ such that the solution $\Psi(t, \cdot)$ of the Cauchy problem for the hyperbolic system* (4.7) *with initial datum $\Psi(0, \cdot)=g\in L^2(\boldsymbol{R}^d; \boldsymbol{C}^N)$ is expressed as follows;*

$$\Psi(t, \cdot)=\int_{\tilde{X}_t} d\mu_t(X) \exp\{\int_0^t V(X(s))\, ds\}\, g(X(0)).$$

PROOF. H_0 is a selfadjoint operator in $L^2(\boldsymbol{R}^d; \boldsymbol{C}^N)$ and V is a bounded Borel measurable function on $\boldsymbol{R}^d$. So by using Trotter's product formula, we have

$$\Psi(t, \cdot)=s\text{-}\lim_{n\to\infty}(e^{\frac{t}{n}V}U^0_{t/n})^n g.$$ Put

$\Phi(X)=\exp\{\int_0^t V(X(s))\, ds\}$. Then Φ belongs to S and we have

$$(T_t(\Phi))g=s\text{-}\lim_{n\to\infty}(T_t(\Phi_{(n)}))g=s\text{-}\lim_{n\to\infty}(e^{\frac{t}{n}V}U^0_{t/n})^n g.$$

So by using Theorem 2, we obtain the desired result. □

REMARK 5. The special case of (4.7) is the Dirac equation in four space-time dimensions, which describes the motion of a spin 1/2 particle with non-zero rest mass under the influence of an electrostatic potential V;

$$(4.8) \qquad \begin{cases} \partial_t\Phi(t, x)=[\sum_{k=1}^3 \alpha_k\partial_k+i\alpha_4+iV(x)]\Phi \\ \Phi(0, x)=g(x) \end{cases}$$

where α_1, α_2, α_3 and α_4 are hermitian 4×4-matrices satisfying the anticommutation relations; $\alpha_j\alpha_k+\alpha_k\alpha_j=2\delta_{jk}I\ (j, k=1, 2, 3, 4)$ and $V\in\boldsymbol{B}(\boldsymbol{R}^3)$ is a real-valued function. Then Theorem 3 implies that there exists a $\mathfrak{L}(L^2(\boldsymbol{R}^3; \boldsymbol{C}^4))$-valued generalized vector measure μ_t on $\tilde{X}_t$ such that the solution $\Phi(t, \cdot)$ of the Cauchy problem for the Dirac equaiton (4.8) with initial datum $\Phi(0, \cdot)=g\in L^2(\boldsymbol{R}^3; \boldsymbol{C}^4)$ is expressed as follows;

$$\Phi(t, \cdot)=\int_{\tilde{X}_t} d\mu(x) \exp\{i\int_0^t V(X(s))\, ds\}\, g(X(0)).$$

The author would like to express her gratitude to Prof. Y. Kōmura for introducing her to the concept of Feynman's path integral and for numerous enlightening discussions on the subject of this paper and to Prof. S. Miyajima for many valuable suggestions. She is also grateful to the referee for useful many comments which led to an improved version of this paper.

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