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On the number of conjugacy classes in a finite p-group

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Abstract

Let G be a finite p-group of order $p^{m}=p^{2n+e}$, with n a non-negative integer, p a prime number and $e=0$ or 1, and let $r(G)$ be the number of conjugacy classes of elements of G. Then the following equality, due to P. Hall, holds ([4], p. 549):

$$r(G)=(p^{2}-1)n+p^{e}+k(p^{2}-1)(p-1),$$

For some non-negative integer k. In this paper, we obtain new properties relative to $r(G)$ by the analysis of the number $r_{G}(gN)$ of conjugacy classes of elements of G that intersect the coset gN, where N is a normal subgroup of G and g any element of G. It contains a number of equations and congruences relating $r(G)$ to other invariants of G. In particular, our results improve the above equality of P. Hall, when G has maximal nilpotent class or $n\leq p+1$. Examples are given, which make our improvements evident.

Introduction

The standard notation of the theory of groups is used in this paper. In the following, G will denote a finite non-abelian p-group of order $p^{m}=p^{2n+e}$, with n a positive integer, p a prime number, and $e=0$ or 1, and $r(G)$ denotes the number of conjugacy classes of elements of G. If S is a non-empty subset of G, $r_{G}(S)$ denotes the number of conjugacy classes of elements of G that intersect S. The lower central series of G is the series $G>Y_{2}>\ldots>Y_{c}=1$ of normal subgroups Y_{i} of G in which $Y_{2}=G'=[G, G]$ is the derived subgroup of G and Y_{i} is the subgroup generated by the set $\{[x, y]=x^{-1}y^{-1}xy \mid x\in G,\ y\in Y_{i-1}\}$ for each $i=3, \ldots, c$; the number $c-1$ is called the nilpotent class of G. G is said to have maximum degree of commutativity $d=d(G)$ if $[Y_{i}, Y_{j}]\leq Y_{i+j+d}$ for all $i, j=1,2,3, \ldots$ and d is the maximum such integer; obviously $d\geq 0$. It is well-known (cf. [4])

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that G has nilpotent class at most $m-1$. In case $c=m-1$, and $m \geqq 4$, we consider the subgroup $Y_1 \supseteq Y_4$ of G defined by: Y_1/Y_4 is the centralizer of Y_2/Y_4 in G/Y_4. Then $|G: Y_1|=p$ and $|Y_1: Y_2|=p$. If Y_1 is an abelian group, then we have $r(G)=p^2-1+p^{m-2}$. Therefore we can suppose that Y_1 is non-abelian.

Specifically we prove the following results:

A) There exist non-negative integer numbers k_1 and k_2 such that

i) $p \cdot r(G)=(p^2-1)(|Z(G)|+n+e+p-2)+p^{1-e}+k_1 \cdot (p^2-1)(p-1)$

ii) $p \cdot r(G)=(p^2-1)(|G/G'|/p+n+e+p-2)$
$+p^{1-e}+k_2 \cdot (p^2-1)(p-1)$.

In particular if $n \leq p+1$, $A)$ yields

B) There exist non-negative integer numbers k_3 and k_4 such that

i) $r(G)=(p^2-1)(|Z(G)|/p+n-1)+p^e+k_3 \cdot (p^2-1)(p-1)$.

ii) $r(G)=(p^2-1)(|G/G'|/p^2+n-1)+p^e+k_4 \cdot (p^2-1)(p-1)$.

Since $|Z(G)| \geq p$ and $|G/G'| \geq p^2$, it is evident that B) improves P. Hall's equality. On the other hand, if p divides $n-2$, then A) also yields

C) There exist non-negative integer numbers k_5 and k_6 such that

i) $r(G)=(p^2-1)(|Z(G)|/p+1+(n-2)/p)$
$+p^e+k_5 \cdot (p^2-1)(p-1)$.

ii) $r(G)=(p^2-1)(|G/G'|/p^2+1+(n-2)/p)$
$+p^e+k_6 \cdot (p^2-1)(p-1)$.

In case $p=2$, A) ii) yields the best bound possible in the case $|G| \leq 2^7$ for fixed values of $|G/G'|$ greater than 4.

For each real number x, $[x]$ denotes the integral part of x. Then $Y_{[(c-d+1)/2]}$ is an abelian group and for each natural number $j \leq (c-d+1)/2$ such that Y_j is abelian, there exists a non-negative integer k such that

$$|G|r(G)=(k+1)\sum_{i=3}^{j}|Y_i||Y_{c-(i-1)-d}|(|Y_{i-1}/Y_i|-1)$$
$$+p^{2m_j}+p^{m_2}(p^{2(m-m_2)}-1)+k \cdot p^{\min(m_2,m_j)} \cdot (p^2-1)(p-1),$$

in which $c-1$ is the nilpotent class of G, d the degree of commutativity of G and p^{m_i} the order of the ith term Y_i of the lower central series of G.

In particular, if G has maximum nilpotent class $(m-1)$ and $j \leq n$ is a natural number such that Y_j is abelian, then the following equalities hold:

D) i) $p \cdot r(G)=(p^2-1)(j+p-1)+p^{m-2j+1}+k \cdot (p^2-1)(p-1)$ for some $k \geq 0$.

ii) If $d \geq 1$ or $j \leq p$, then we have

$$r(G)=(p^2-1)j+p^{m-2j}+k'\cdot(p^2-1)(p-1)$$

for some $k' \geq 0$

(notice that $d \geq 1$, whenever m is odd or $m \geq p+2$ (cf. |1|)).
In general, if $d \geq 1$, there exists $k'' \geq 0$ such that

iii) $r(G)=(p^2-1)(p^{d-1}(j-2)+2)+p^{m-2j}+k''\cdot(p^2-1)(p-1)$.

(putting $j=n$ in D) ii) we get P. Hall's equality). In addition, by using results of N. Blackbum, C. R. Leedham-Green and Susan MoKay, and R. Shepherd (cf. |1|, |5|, |8|), we get:

iv) 1) If $p=3$ and $m \geq 5$, then we have $r(G)=16+3^{m-4}+k_1\cdot 16$ for some $k_1 \geq 0$.

2) If $p=5$ and $m \geq 6$, then D) iii) is satisfied substituting d for $[(m-5)/2]$ and putting $j=[(m-[(m-5)/2])/2]$.

3) If $p=7$ and $m \geq 9$, then D) iii) is satisfied substituting d for $[(m-8)/2]$ and putting $j=[(m-[(m-8)/2])/2]$.

4) If $p \geq 11$ and $m > 3p-7$, then D) iii) is satisfied substituting d for $[(m-3p+7)/2]$ and putting $j=[(m-[(m-3p+7)/2])/2]$.

Theorems and Proofs

THEOREM 1. *Suppose that G is a non-abelian p-group of order $p^m=p^{2n+e}$ with n a positive integer, p a prime number and $e=0$ or 1. Then there exist non-negative integer numbers k_1 and k_2 such that*

i) $p\cdot r(G)=(p^2-1)(|Z(G)|+n+e+p-2)+p^{1-e}+k_1\cdot(p^2-1)(p-1)$.

ii) $p\cdot r(G)=(p^2-1)(|G/G'|/p+n+e+p-2)$
$+p^{1-e}+k_2\cdot(p^2-1)(p-1)$,

where $r(G)$ denotes the number of conjugacy classes of elements of G.

PROOF. We claim that there exists $M \trianglelefteq G$ satisfying the following conditions: $G/M \simeq C_p$, $Z(G) \leq M$, and $|Z(M)| \geq p^2$. In fact, we consider $N \trianglelefteq G$ such that $|N|=p^2$. Then $\mathrm{Aut}(N) \simeq C_{p(p-1)}$ or $GL(2, p)$ and consequently $G/C_G(N) \leq C_p$. If N is contained in $Z(G)$ and M is a maximal subgroup of G such that $Z(G) \leq M$ then the above conditions are satisfied. Otherwise, we have $G/C_G(N) \simeq C_p$ and we take $M=C_G(N)$. Thus, in the following we can assume the existence of M. Set $G/M=\langle \bar{g} \rangle \simeq C_p$. Then arguing as in Note E of |2| we have $p\cdot r(G)=(p^2-1)s_g+r(M)$, where s_g is the number of conjugacy M-clases of M fixed by the conjugation-automorphism induced by g. Evidently, we have $s_g=|Z(G)|+k_1'.\ (p-1)$ for some $k_1' \geq 0$, since M contains $Z(G)$, and also by using a result of J.

Poland (cf. |6| Th. (4.2)) we have

$$r(M)=(n+e-1)(p^2-1)+p^{1-e}+(p^2-1)(p-1)$$
$$+k_2'(p^2-1)(p-1) \text{ for some } k_2'\geq 0,$$

because $|M|=p^{2(n+e-1)+1-e}$ and M is not of maximal class (for example, $Z(M)\neq C_p$). Now we conclude

$$p\cdot r(G)=(p^2-1)(|Z(G)|+n+e+p-2)+p^{1-e}$$
$$+k_1\cdot(p^2-1)(p-1) \text{ for some } k_1\geq 0.$$

On the other hand, we have $s_g=r_G(gM)\geq r_{G/G'}(gM/G')=|M/G'|=|G/G'|/p$, hence $s_g=|G/G'|/p+k_3'(p-1)$ for some $k_3'\geq 0$, and arguing as above we get the second equality.

COROLLARY 2. *Suppose that $n\leq p+1$. Then there exist non-negative integers k_3 and k_4 such that*
i) $r(G)=(p^2-1)(|Z(G)|/p+n-1)+p^e+k_3\cdot(p^2-1)(p-1)$.
ii) $r(G)=(p^2-1)(|G/G'|/p^2+n-1)+p^e+k_4\cdot(p^2-1)(p-1)$.

PROOF. From Theorem 1 we get $k_i\equiv n-2$ (mod. p) and the conditions $k_i\geq 0$ and $n-2<p$ imply $k_i=n-2+k_{i+2}\cdot p$ for some $k_{i+2}\geq 0$. Now substituting these values into the equalities of Theorem 1 we get

$$\begin{aligned} r(G)&=(p^2-1)(|Z(G)|/p+n-1)\\ &\quad+((p^2-1)e+p^{1-e})/p+k_3(p^2-1)(p-1)\\ &=(p^2-1)(|G/G'|/p^2+n-1)\\ &\quad+((p^2-1)e+p^{1-e})/p+k_4(p^2-1)(p-1). \end{aligned}$$

Finally we notice that $((p^2-1)e+p^{1-e})/p=p^e$ and therefore we obtain the desired equalities.

Evidently, the equalities given in Corollary 2 improve the following congruence of P. Hall

$$r(G)=(p^2-1)n+p^e+k\cdot(p^2-1)(p-1) \text{ for}$$
$$\text{some } k\geq 0 \text{ (cf. |4|V. 15.2)},$$

whenever $m\leq 2(p+1)+e$.

For example, let us suppose that $p=2$. A theorem of O. Taussky (cf. |4| III. 11.9. a)) asserts that the only non-abelian 2-groups for which $|G:G'|=4$ are the dihedral, semidihedral and generalized quaternion groups. In each of these groups, the number of conjugacy classes is $r(G)=3+2^{m-2}$. Thus we can assume that $|G/G'|\geq 8$ and Corollary 2 yields $r(G)=3(n+1)+2^e+k.3$, for some $k\geq 0$, improving the information given in the above equality of P. Hall. Furthermore, in case $|G|\leq 2^6$ and by using Hall-

Senior's notation (cf. |3|) we obtain best possible bounds. Indeed,

For $|G|=32$ and $|G'|=2$, Corollary 2 yields $r(G)=17+3.k$ and the lower bound $r(G)=17$ is attained for the stem groups of the family Γ_5.

For $|G|=32$ and $|G'|=4$, Corollary 2 yields $r(G)=11+3.k$ and the lower bound $r(G)=11$ is attained for the stem groups of the family Γ_6, Γ_7.

For $|G|=64$ and $|G'|=4$, Corollary 2 yields $r(G)=19+3.k$ and the lower bound $r(G)=19$ is attained for the stem groups of the family Γ_{13}.

For $|G|=64$ and $|G'|=8$, Corollary 2 yields $r(G)=13+3.k$ and the lower bound $r(G)=13$ is attained for the stem groups of the family Γ_{22} and Γ_{23}.

Thus our results are best posible, in case $n\leq p+1$. Suppose now that $|G|=2^7$ (and $|G/G'|\geq 8$), then Theorem 1 yields $2.r(G)=3(|G/G'|/2+3+1)+1+k.3$, with $k\geq 0$; necessarily k is an odd number, that is, $k=1+2k'$ with $k'\geq 0$ and consequently $r(G)=3(|G/G'|/4+2)+2+3.k'$. For $|G/G'|=8$ we have $r(G)=14+3.k'$ and the lower bound $r(G)=14$ is attained for the stem groups of the family Γ_{106} (cf. |7|). In general, if $|G|=2^{4t+3}$ for some $t\geq 0$, then Theorem 1 yields

$$r(G)=3.(|G/G'|/4+t+1)+2+3.k \text{ for some } k\geq 0.$$

COROLLARY 3. *Suppose that p divides n-2. Then there exist non-negative integers k_5 and k_6 such that*

i) $r(G)=(p^2-1)(|Z(G)|/p+1+(n-2)/p)+p^e+k_5\cdot(p^2-1)(p-1)$.

ii) $r(G)=(p^2-1)(|G/G'|/p^2+1+(n-2)/p)+p^e+k_6\cdot(p^2-1)(p-1)$.

PROOF. This result follows immediately from Theorem 1, arguing as in Corollary 2.

In the following, let d be the degree of commutativity of G and let $c-1$ be the nilpotent class of G. If $(c-d)/2$ is an integer, we have

$$[Y_{(c-d)/2}, Y_{(c-d)/2}]\leq Y_{2(c-d)/2+d}=Y_c=1,$$

On the other hand, if $(c-d+1)/2$ is an integer, then we have

$$[Y_{(c-d+1)/2}, Y_{(c-d+1)/2}]\leq Y_{c-d+1+d}=Y_{c+1}=1,$$

Thus, Y_j is an abelian group, in case $j=[(c-d+1)/2]$ (and evidently, Y_v is abelian for each $v\geq(c-d+1)/2$).

In the following we assume that j is any natural number satisfying $j\leq(c-d+1)/2$ and Y_j is abelian. For each $i\leq j$ we have $i\leq(c-d+1)/2$, hence $c-(i-1)-d\geq i$ and consequently $Y_{c-(i-1)-d}\leq Y_i$. Moreover

$$[Y_{c-(i-1)-d}, Y_{i-1}]\leq Y_{c-(i-1)+i-1+d-d}=Y_c=1$$

whence

$$Y_{c-(i-1)-d} \leq Z(Y_{i-1}) \cap Y_i \text{ for each } i \leq j \tag{1}$$

Next, let us consider a series

$$1 = N_m < N_{m-1} < \ldots < N_1 < N_0 = G$$

of normal subgroups N_i of G such that $N_{i-1}/N_i = \langle \bar{x}_i \rangle \simeq C_p$ and $Y_t = N_{m-m_t}$ for each $t = 2, \ldots, c$. Then we have

$$r(G) = s_1(p^2-1)/p + s_2(p^2-1)/p^2 + \ldots + s_m(p^2-1)/p^m + 1/|G|$$

where $s_i = r_{N_{i-1}}(x_i N_i)$ is the number of conjugacy N_i-classes of N_i fixed by the automorphism $f_i : N_i \longrightarrow N_i$ defined by $f_i(z) = z^{x_i}$ for all $z \in N_i$ (cf. Note E of |2|) We have $N_i \leq Y_j$ if and only if $|N_i| = p^{m-i} \leq |Y_j| = p^{m_j}$, i.e., $m - i \leq m_j$, and N_i is abelian in this case. Furthermore, $s_i = r_{N_{i-1}}(x_i N_i) = |N_i|$ if N_{i-1} is abelian, that is, in case $i \geq m - m_j + 1$. Therefore we have $s_i = p^{m-i}$ for each $i = m - m_j + 1, \ldots, m$ and consequently

$$\sum_{i=m-m_j+1}^{m} s_i/p^i = \sum_{i=m-m_j+1}^{m} p^{m-i}/p^i = (p^{2m_j}-1)/(p^m(p^2-1)) \tag{2}$$

Thus we have the following decomposition of the number $|G| r(G)$:

$$|G| r(G) = \sum_{i=1}^{m} s_i p^{m-i}(p^2-1) + 1 = \sum_{i=1}^{m-m_j} s_i p^{m-i}(p^2-1) + p^{2m_j} \tag{3}$$

Consider the abelian group $G/G' = G/Y_2$ of order p^{m-m_2}. For each i such that $1 \leq i \leq m - m_2$ it is $N_{m-m_2} = Y_2 \leq N_i < N_{i-1}$ and

$$s_i = r_{N_{i-1}}(x_i N_i) \geq r_{N_{i-1}/G'}(\tilde{x}_i N_i/G') = |N_i/G'| = p^{m-i}/p^{m_2} = p^{m-m_2-i},$$

hence $s_i = p^{m-m_2-i} + k_i \cdot (p-1)$ for some $k_i \geq 0$ and consequently

$$\sum_{i=1}^{m-m_2} s_i p^{m-i}(p^2-1) = p^{m_2}(p^{2(m-m_2)}-1) + k' p^{m_2}(p^2-1)(p-1) \text{ for some } k' \geq 0 \tag{4}$$

We now analyse the numbers s_i for $i = m - m_2 + 1, \ldots, m - m_j$, these corresponding to groups $N_i < N_{i-1}$ situated into the following chain

$$N_{m-m_j} = Y_j < Y_{j-1} < \ldots < Y_3 < Y_2 = N_{m-m_2}.$$

We define $I_i = \{u \mid Y_i \leq N_u < N_{u-1} \leq Y_{i-1}\}$ for each $i = 2, \ldots, j$. From (1) we get $s_u = |Y_{c-(i-1)-d}| + k_{iu}(p-1)$ for some $k_{iu} \geq 0$ and for all $u \in I_i$, consequently

$$\sum_{u \in I_i} s_u p^{m-u}(p^2-1) = |Y_{c-(i-1)-d}| \sum_{u=m-m_{i-1}+1}^{m-m_i} p^{m-u}(p^2-1)$$

$$+k_i p^{m_i}(p^2-1)(p-1)$$

for some $k_i \geq 0$ (since $|Y_{i-1}/Y_i| = p^{m_{i-1}-m_i}$ implies $|I_i| = m_{i-1}-m_i$).

Finally, inasmuch as

$$\{m-m_2+1, \dots, m-m_j\} = \bigcup_{i=3}^{j} I_i$$

and

$$\sum_{u=m-m_{i-1}+1}^{m-m_i} p^{m-u} = p^{m_i}((p^{m_{i-1}-m_i}-1)/(p-1))$$
$$= |Y_i|((|Y_{i-1}/Y_i|-1)/(p-1))$$

the following theorem holds:

THEOREM 4. *Let j be a natural number such that $j \leq (c-d+1)/2$ and Y_j is an abelian group. Then there exists a non-negative integer number k such that*

$$|G|r(G) = \sum_{i=3}^{j} |Y_i||Y_{c-(i-1)-d}|(p^2-1)((|Y_{i-1}/Y_i|-1)/(p-1))$$
$$+p^{2m_j}+p^{m_2}(p^{2(m-m_2)}-1)+k.\ p^{\min.\{m_2,m_j\}}(p^2-1)(p-1),$$

in which, $c-1$ is the nilpotent class of G, d is the degree of commutativity of G and p^{m_u} is the order of u-th term Y_u of the lower central series of G.

Next, we analyse the case $c=m$, i.e., G has maximal class $m-1$. In this case, we have $G/Y_2 \simeq C_p \times C_p$ and $Y_{i-1}/Y_i \simeq C_p$ for each $i=1, \dots, c$. Therefore $m_i = m-i$ and we have

$$\sum_{i=3}^{j} |Y_i||Y_{m-(i-1)-d}|((p-1)/(p-1)) = \sum_{i=3}^{j} p^{m-i}p^{i-1+d} = p^{m-1+d}(j-2),$$

and Theorem 4 yields $|G|r(G) = (p^2-1)p^{m-1+d}(j-2)+p^{2(m-j)}+p^{m-2}(p^4-1)+k.\ p^{m-j}(p^2-1)(p-1)$ for some $k \geq 0$ and we have

$$p^2 r(G) = (p^2-1)p^{1+d}(j-2)+p^{m-2j+2}$$
$$+p^4-1+k'(p^2-1)(p-1) \text{ for some } k' \geq 0 \qquad (5)$$

From the above equality we deduce that p divides $-1+k'(p^2-1)(p-1)$, hence $k'=1+k''p$ for some $k'' \geq 0$ and $-1+k'(p^2-1)(p-1) = p^3-p^2-p+k''(p^2-1)(p-1)$. *By substituting this latter number in (5) we get*

$$p.\ r(G) = (p^2-1)p^d(j-2)+p^{m-2j+1}+p^3+p^2-p-1$$
$$+k''(p^2-1)(p-1)$$
$$=(p^2-1)(p^d(j-2)+p+1)+p^{m-2j+1}$$

$$+k''(p^2-1)(p-1) \tag{6}$$

Suppose that $d=0$. In this case, (6) implies that p divides $-(j-1)+k''$, so, if $j-1\not\equiv p$, necessarily we have $k''=j-1+k'''(p^2-1)(p-1)$ for some $k'''\geq 0$ and (6) yields

$$r(G)=(p^2-1)j+p^{m-2j}+k'''(p^2-1)(p-1).$$

Suppose that $d\geq 1$. In this case, $k''=1+k_1''p$ for some $k_1''\geq 0$ and (6) yields

$$r(G)=(p^2-1)(p^{d-1}(j-2)+2)+p^{m-2j}+k_1(p^2-1)(p-1)$$

for some $k_1\geq 0$.

Thus we have showed

COROLLARY 5. *Let G be a p-group of maximal class $m-1$ and let j be a natural number smaller than or equal to $(m-d)/2$ such that Y_j is an abelian group. Then there exists a non-negative integer k such that*

$$p.\ r(G)=(p^2-1)(p^d(j-2)+p+1)+p^{m-2j+1}+k.\ (p^2-1)(p-1).$$

In particular, if $d\geq 1$ or $j\leq p$ then there exists $k'\geq 0$ *such that*

$$r(G)=(p^2-1)j+p^{m-2j}+k'(p^2-1)(p-1). \tag{7}$$

Furthermore, in case $d\geq 1$ we have

$$r(G)=(p^2-1)(p^{d-1}(j-2)+2)+p^{m-2j}+k''(p^2-1)(p-1)$$

for some $k''\geq 0$.

It is well-known that $d\geq 1$ whenever m is an odd number or $m\geq p+2$ (cf. |1|), thus (7) improves P. Hall's result (obtained putting $j=n$ in (7)), indeed if j is smaller than n, then (7) can be written in the following way

$$r(G)=(p^2-1)(j+p^e(p^{n-j-1}+p^{n-j-2}+\dots+p+1))$$
$$+p^e+k'(p^2-1)(p-1).$$

In addition, we have

COROLLARY 6. *Let G be a finite p-group of maximal class $m-1$. Then the following equalities hold*:

1) *If $p=3$ and $m\geq 5$,* then $r(G)=16+3^{m-4}+k_1.16$ *for some $k_1\geq 0$.*

2) *If $p=5$ and $m\geq 6$, then we have*

$r(G)=(5^2-1)(5^{[(m-5)/2]-1}([(m-[(m-5)/2])/2]-2)+2)+5^{m-2[(m-5)/2]}+k_2(5^2-1)(5-1)$

for some $k_2\geq 0$.

3) *If $p=7$ and $m\geq 9$, then we have*

$r(G)=(7^2-1)(7^{[(m-8)/2]-1}([(m-[(m-8)/2])/2]-2)+2)+7^{m-2[(m-8)/2]}+k_3(7^2-1)(7-1)$

for some $k_3\geq 0$.

4) *If $p\geq 11$ and $m\geq 3p-6$ we have*

$r(G)=(p^2-1)(p^{[(m-3p+7)/2]-1}([(m-[(m-3p+7)/2]-2)+2)+p^{m-2[(m-3p+7)/2]}+k_4(p^2-1).\ (p-1)$

for some $k_4\geq 0$.

PROOF. This result follows directly from the following inequalities (cf. |1|, |5|, |8|) $d\geq m-4$ if $p=3$; $d\geq[(m-5)/2]$ if $p=5$; $d\geq[(m-8)/2]$ if $p=7$ and $d\geq[(m-3p+7)/2]$ for any prime number p.

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