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# Gap theorems for Riemannian manifolds of constant curvature outside a compact set

Dedicated to Professor Noboru Tanaka on his 60th birthday

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## § 1. Introduction and Main Results.

In Riemannian geometry, the spaces of constant curvature are considered to be the typical models in various problems. For example, the famous sphere theorem deals with compact Riemannian manifolds whose sectional curvatures are similar to that of the standard sphere. For noncompact Riemannian manifolds, we have the following theorem. Let $K_M$ denote the sectional curvature of a Riemannian manifold $M$ and $k$ be a nonpositive constant.

GAP THEOREM 1.1 (Greene-Wu[5], Kasue-Sugahara[3]). *Let $M$ be a complete noncompact Riemannian manifold of dimension $n \geqq 3$. Suppose that $M$ satisfies the following three conditions.*
( i ) *$M$ has a pole $o$, i.e., the exponential mapping* $\exp_0$ *from the tangent space $T_0M$ to $M$ is a diffeomorphism.*
( ii ) *$K_M \leqq k$ or $K_M \geqq k$ everywhere.*
(iii) $\liminf_{r\to\infty} \max_{d(o,p)=r} k(r)|K_M - k| = 0$, *where $d(o, p)$ denotes the distance between two points $o$ and $p$ and $k(r) = r^2$ for $k=0$ and $k(r) = \exp(2r\sqrt{-k})$ for $k<0$.*
*Then $M$ is isometric to the $n$-dimensional simply connected space of constant curvature $k$.*

In this theorem, the third condition can be understood that $M$ is in a neighbourhood of the model in the set of Riemannian manifolds. This theorem asserts that there are gaps in the positive and negative sides of the model spaces in the set of Riemannian manifolds. But the first condition implies that $M$ is diffeomorphic to $\boldsymbol{R}^n$ and the assertion is restricted within metrics on $\boldsymbol{R}^n$.

We note that Gromov claimed that for $M$ with $K_M \geqq k = 0$ the first condition can be relaxed to
(i′) *$M$ is simply connected at infinity,*
where $M$ is said to be simply connected at infinity if for any compact

subset $V$ of $M$ there is a compact subset $V'$ of $M$ such that $V$ is contained in $V'$ and $M\setminus V'$ is connected and simply connected.

We generalize the claim of Gromov to the following

PROBLEM. *Classify Riemannian manifolds M which satisfy the following two conditions.*

(ii) $K_M \geqq k$ *or* $K_M \leqq k$ *everywhere.*

(iii) $K_M$ *converges to* $k$ *at infinity.*

In case of $K_M \leqq k$, we have the following lemma.

LEMMA 1.1(Kasue-Sugahara[3]. *If the universal covering of M is homeomorphic to* $\boldsymbol{R}^n (n \geqq 3)$ *and if M is simply connected at infinity, then M is simply connected.*

It follows from the Hadamard-Cartan theorem and this lemma that (i′) implies (i) if $K_M \leqq 0$. Therefore (i) can be replaced by (i′) in this case.

THEOREM 1.2(Kasue-Sugahara[3]). *Let M be a complete noncompact Riemannian manifold of dimension* $n \geqq 3$. *Suppose that M satisfies the following three conditions.*

(i′) *M is simply connected at infinity.*

(ii) $K_M \leqq k$ *everywhere.*

(iii) $\liminf_{r\to\infty} \max_{d(o,p)=r} k(r)|K_M - k| = 0$, *where* $k(r) = r^2$ *for* $k=0$ *and* $k(r) = \exp(2r\sqrt{-k})$ *for* $k<0$.

*Then M is isometric to the n-dimensional simply connected space of constant curvature k.*

Here in this paper we present the following

CONJECTURE 1. *Let M be a complete noncompact Riemannian manifold of dimension* $n \geqq 3$. *Suppose that M satisfies the following three conditions.*

(i′) *M is simply connected at infinity.*

(ii) $K_M \geqq k$ *everywhere.*

(iii) $\liminf_{r\to\infty} \max_{d(o,p)=r} k(r)|K_M - k| = 0$.

*Then M is isometric to the n-dimensional simply connected space of constant curvature k.*

The next theorem supports this conjecture.

THEOREM 1.3(Greene-Wu[5]). *Let M be a complete noncompact*

*Riemannian manifold of dimension $n \geqq 3$. Suppose that $M$ satisfies the following three conditions.*
(i′) *$M$ is simply connected at infinity.*
(ii) *$K_M \geqq 0$ everywhere.*
(iii′) *$M$ is flat outside a compact set.*
*Then $M$ is isometric to the $n$-dimensional euclidean space $\boldsymbol{R}^n$.*

In this paper we consider the following conjecture as a step to prove Conjecture 1.

CONJECTURE 2. *Let $M$ be a complete noncompact Riemannian manifold of dimension $n \geqq 3$. Suppose that $M$ satisfies the following three conditions.*
(i′) *$M$ is simply connected at infinity.*
(ii) *$K_M \geqq -1$ everywhere.*
(iii′) *$K_M = -1$ outside a compact set.*
*Then $M$ is isometric to the $n$-dimensional hyperbolic space of constant curvature $-1$.*

DEFINITION. A Riemannian manifold $M$ is said to be *of constant curvature $k$ at infinity* if there is a compact proper subset $V$ of $M$ such that $K_M = k$ outside $V$. Two Riemannian manifolds $M$ and $N$ are said to be *isometric at infinity* if there are compact proper subsets $V_M \subset M$ and $V_N \subset N$ and an isometry between $M \setminus V_M$ and $N \setminus V_N$.

In Theorem 1.3 and Conjecture 2, the condition (i′) restricts the dimension of $M$ to be greater than or equal to 3. But the next lemma enables us to rewrite them in the form which contains the case of dimension 2 as follows.

LEMMA 1.2 (Greene-Wu [5]). *Let $M$ be a noncompact complete Riemannian manifold of dimension $n \geqq 3$. Suppose that $M$ is simply connected at infinity and that $M$ is of constant curvature $k$ at infinity. Then $M$ is isometric at infinity to the simply connected space of constant curvature $k$.*

THEOREM 1.4. *Let $M$ be a complete Riemannian manifold of dimension $n \geqq 2$. Suppose that $K_M \geqq 0$ and $M$ is isometric to the $n$-dimensional Euclidean space at infinity. Then $M$ is isometric to the $n$-dimensional Euclidean space.*

We note that this theorem is a weak version of the Geroch conjecture [4] in dimension 3.

CONJECTURE 3. *Let $M$ be a complete noncompact Riemannian manifold of dimension $n \geqq 2$. Suppose that $K_M \geqq -1$ and $M$ is isometric at infinity to the n-dimensional real hyperbolic space of constant curvature $-1$. Then $M$ is isometric to the n-dimensional hyperbolic space of constant curvature $-1$, i.e., the real hyperbolic space cannot be changed its structure inside any compact subset with $K_M \geqq -1$.*

We present here the elliptic version for this conjecture.

CONJECTURE 4. *Let $M$ be an open hemisphere of the n-dimensional standard sphere $S^n(1)$ of constant curvature 1. Then its structure cannot be changed in any compact subset with $K_M \geqq 1$.*

In this paper, we give two proofs of Theorem 1.4. One is reduced to the splitting theorem of Cheeger-Gromoll [2] for Riemannian manifolds of nonnegative curvature and the other can be applied to the proof of conjectures 3 and 4 with the assumption that the compact sets are small. We shall prove

THEOREM 1.5. *Let $H^2(-1)$ be the 2-dimensional hyperbolic space of constant curvature $-1$. Then $H^2(-1)$ cannot be changed its (topological and Riemannian) structure inside any compact set with the sectional curvature $K \geqq -1$.*

THEOREM 1.6. *Let $H^n(-1)$ be the hyperbolic space of constant curvature $-1$. Let $V_H(r)$ be a metric ball of radius $r < r_{-1} = \log \frac{20\sqrt{2}-1}{17}$. Then $H^n(-1)$ cannot be changed its (topological and Riemannian) structure in $V_H(r)$ with the sectional curvature $K \geqq -1$.*

THEOREM 1.7. *Let $S^n(1)$ be the sphere of constant curvature 1. Let $V_S(r)$ be a metric ball of radius $r < r_{+1} = \pi/4$. Then $S^n(1)$ cannot be changed its (topological and Riemannian) structure in $V_S(r)$ with the sectional curvature $K \geqq 1$.*

As for the Euclidean space, we can shrink compact sets by homothety. Therefore the size of a set has no sense. On the other hand, a homothety of the hyperbolic space or the sphere changes the curvature of the space.

## § 2. The First Proof of Theorem 1.4.

LEMMA 2.1. *Suppose $M$ is isometric to the n-dimensional Euclidean space $\boldsymbol{R}^n$ at infinity. Then $M$ has a line, where a geodesic $\gamma : \boldsymbol{R} \to M$ is said to be a line if $\gamma$ restricted to any subinterval is distance-minimizing.*

PROOF. Let $V_M$ and $V_E$ be compact sets of $M$ and $\boldsymbol{R}^n$ such that there is an isometry $\varphi: \boldsymbol{R}^n \setminus V_E \to M \setminus V_M$. Let $\alpha: \boldsymbol{R} \to \boldsymbol{R}^n$ be a straight line. Let $\alpha_t$ be a minimizing geodesic from $\varphi(\alpha(-t))$ to $\varphi(\alpha(t))$ in $M$. If $\alpha_t$ does not pass $V_M$, then the geodesic $\varphi^{-1} \circ \alpha_t$ coincides $\alpha|_{[-t,t]}$. Therefore $\alpha_t$ passes a compact set $\{\varphi(\alpha(0))\} \cup V_M$ for any $t$. Then $\{\alpha_t\}_{t \in \boldsymbol{R}}$ contains a convergent subsequence. It is clear that the limit is a line since it is the limit of minimizing curves.

PROOF OF THEOREM 1.4. From the splitting theorem for Riemannian manifolds of nonnegative curvature [2] and the lemma above, we get a Riemannian decomposition $M = \boldsymbol{R}^m \times M'$ with $m > 0$. Since $M$ is flat at infinity, $M'$ must be flat and so is $M$.

It follows from Lemma 1.1 that $M$ is simply connected if $n = \dim M \geqq 3$, which implies that $M$ is isometric to $\boldsymbol{R}^n$.

Suppose that $M$ is of two dimension. Then $\dim M' = 0$ or 1. Since $M$ is complete, $M'$ is isometric to $S^1$, $\boldsymbol{R}^1$ or a point. Since $M$ is isometric to $\boldsymbol{R}^2$ at infinity, $M'$ is isometric to $\boldsymbol{R}^1$ or a point. Therefore $M$ is isometric to $\boldsymbol{R}^2$.

## § 3. The Second Proof of Theorem 1.4.

Let $V(r)$ be a ball of radius $r$ centered at $o$ in the n-dimensional Euclidean space $\boldsymbol{R}^n$.

LEMMA 3.1. *Let $p$ and $q$ be points in $\boldsymbol{R}^n$. Let $\theta$ denote the angle $\angle opq$. If*

$$|po| - r > |po| \cos\theta + r,$$

*then the segment $pq$ is shorter than the $\boldsymbol{R}^n \setminus V(r)$-part of any curve from $p$ to $q$.*

PROOF. Let $\alpha$ denot the angle $\angle oqp$. Let $c$ be a curve from $p$ to $q$. If $c$ does not pass $V(r)$, it is clear that $length(c) \geqq |pq|$. Suppose that $c$ passes $V(r)$. Let $x$ denote the first point at which $c$ crosses the boundary of $V(r)$ and $y$ the last point. Then we have

$$\begin{aligned} length(c) &\geqq |px| + |qy| \geqq (|po| - r) + (|qo| - r) \\ &> |po| \cos\theta + |qo| \geqq |po| \cos\theta + |qo| \cos\alpha \\ &= |pq|. \end{aligned}$$

Let $M$ be a complete Riemannian manifold of nonnegative curvature. Let $V_M$ be a compact set of $M$ and $\varphi$ an isometry from $\boldsymbol{R}^n \setminus V(r)$ to $M \setminus V_M$. Let $S$ be a hypersphere in $\boldsymbol{R}^n$ of radius $l \geqq 8r$ centered at $o$. Let

$p$ be a point of $S$ and $q$ be the antipodal point of $p$.

LEMMA 3.2. *The image of the segment $pq$ by $\varphi$ outside $V(r)$ is a part of a minimizing geodesic between $\varphi(p)$ and $\varphi(q)$. And the distance between $\varphi(p)$ and $\varphi(q)$ is equal to the length $|pq|$.*

PROOF. Let $x$ be a point in $S$ with $xo \perp po$. Then from Lemma 3.1, we see that the segments $px$ and $qx$ does not pass $V(r)$ and that the images $\varphi(px)$ and $\varphi(qx)$ by $\varphi$ are minimizing geodesics. Let $\gamma$ be a minimizing geodesic from $\varphi(p)$ to $\varphi(q)$. We consider a geodesic triangle $\Delta_M\varphi(p)\varphi(q)\varphi(x)$ with sides $\gamma$, $\varphi(qx)$ and $\varphi(xp)$. We compare this triangle with $\Delta pqx$. Since they have the same angle at $x$ and $\varphi(x)$, we see $|pq| \geqq l\text{ength}(\gamma)$ from Toponogov's comparison theorem. Let $\Delta pq'x$ be a triangle in $\boldsymbol{R}^n$ with $|pq'| = \mathit{length}(\gamma) \leqq |pq|$ and $\angle xpq' = \angle\varphi(x)\varphi(p)\varphi(q)$. Then from Toponogov's comparison theorem we get

$$(*) \qquad |qx| = \mathit{length}\varphi(qx) \leqq |q'x|$$

If the image of the segment $pq$ outside $V(r)$ by $\varphi$ is not a part of $\gamma$, then we may choose $x$ so that the angle of $\Delta_M\varphi(p)\varphi(q)\varphi(x)$ at $\varphi(p)$ is less than $\angle xpq = \pi/4$. It is clear that if $\angle xpq' < \angle xpq = \pi/4$ or $|pq'| < |pq|$, then $|q'x| < |qx|$, which contradicts $(*)$. Hence $\mathit{length}(\gamma) = |pq|$ and the image of $pq$ by $\varphi$ outside $V(r)$ is a part of $\gamma$.

LEMMA 3.3. *Let $\gamma: [0, 2l] \to M$ be a unit speed geodesic which starts from $\varphi(S)$ in the direction of an inside normal to $\varphi(S)$ and $X$ a Jacobi field along $\gamma$ with $X(0)$, $X'(0) \in T_{\gamma(0)}\varphi(S)$. Then $X$ vanishes at $\gamma(l)$ and $\gamma(l)$ is the first focal point of $\varphi(S)$ along $\gamma$. Moreover the sectional curvature of 2-planes which is tangent to $\gamma$ is 0.*

PROOF. Let $p = \gamma(0)$ and $q = \gamma(2l)$. From Lemma 3.2, we see that $q \in \varphi(S)$ and $\gamma|_{[0,2l]}$ is minimizing. Let $X$ be a Jacobi field along $\gamma$ with $X(0)$, $X'(0) \in T_p(\varphi(S))$. Then $X$ is the infinitesimal variation of a variation of geodesics of length $2l$ which start from $\varphi(S)$ in the normal direction. From Lemma 3.2, we see that these geodesics arrive at $\varphi(S)$ normally at the other ends. Therefore $X(2l)$, $X'(2l) \in T_q(\varphi(S))$. We consider the geodesic $\tilde{\gamma}(t) = \gamma(2l\text{-}t)$ and the Jacobi field $\tilde{X}(t) = X(2l\text{-}t)$ along $\tilde{\gamma}$. From a generalization of the Rauch-Berger comparison theorem by Warner [6], $X$ and $\tilde{X}$ must vanish before or at $l$. Suppose $X$ vanishes before $l$. Then $X$ vanishes two distinct points in the interval $(0, 2l)$. Since $\gamma|_{[0,2l]}$ is minimal, it is impossible (cf. [1] Corollary 1.24). From the equality condition in the comparison theorem, we see that the sectional curvature is 0.

LEMMA 3.4. *Let*

$$F=\{\exp l\nu\ ;\ \nu \text{ are inside unit normals to } \varphi(S)\}.$$

*Then $F$ consists of a single point.*

PROOF. Let $\nu(s)$ be a 1-parameter family of inside unit normal vectors to $\varphi(S)$. Then $X_s(t)=\frac{\partial}{\partial s}\exp t\nu(s)$ is a Jacobi field along $t\mapsto \exp t\nu(s)$ with $X_s(0)$, $X_s'(0)\in T(\varphi(S))$. Then from Lemma 3.3, $X_s(l)=\frac{\mathrm{d}}{\mathrm{ds}}\exp l\nu(s)=0$. Hence $\{\exp l\nu(s)\}$ is a single point and the lemma follows.

Let $\nu$ and $\nu'$ be distinct inside unit normal vectors to $\varphi(S)$. Since the geodesics $\exp t\nu$ and $\exp t\nu'$ $(0\leqq t\leqq l)$ are minimizing, they have no common points except $\exp l\nu=\exp l\nu'$. Therefore the map $\exp_F:\{v\in T_F(M)\ ;\ |v|\leqq l\}\to M$ is injective and the boundary of the image is $\varphi(S)$. Since the sectional curvature of 2-planes which are tangent to geodesics emanating from $F$ is 0, the image is isometric to a ball of radius $l$ in $\boldsymbol{R}^n$ and $M$ is isometric to the Euclidean space.

## §4. The Second Proof of Theorem 1.4: dim M=2.

Let $M$ be a complete 2-dimensional Riemannian manifold of non-negative curvature. Let $V_M$ and $V_E$ be compact sets of $M$ and $\boldsymbol{R}^2$ respectively. Suppose that there is an isometry $\varphi:\boldsymbol{R}^2\setminus V_E\to M\setminus V_M$. Let $S$ be a circle of radius $l$ in $\boldsymbol{R}^2$ which contains $V_E$ in its interior. We denote the inside of $S$ by $D_E$ and the inside of $\varphi(S)$ by $D_M$.

LEMMA 4.1. *$D_M$ is orientable.*

PROOF. If $D_M$ is not orientable, there is a double cover $\pi:D'\to D_M$, where $D'$ is orientable. Let $S'$ be the boundary of $D'$. Then we have the Gauss-Bonnet formula for $D'$.

$$\int_{D'}K'+\int_{S'}k'=2\pi\chi(D'),$$

where $K'$ denotes the Gauss curvature of $D'$ and $k'$ denotes the geodesic curvature of $S'$. Here we have

$$\int_{D'}K'\geqq 0=2\int_{D_E}K_E,$$
$$\int_{S'}k'=2\int_S k_S=4\pi,$$
$$\chi(D')\leqq 1=\chi(D_E).$$

Therefore we get

$$2\int_{D_E} K_E + 2\int_S k_S \leqq \int_{D'} K' + \int_{S'} k' = 2\pi\chi(D') \leqq 2\pi\chi(D_E).$$

These inequalities contradict the Gauss-Bonnet formula

$$\int_{D_E} K_E + \int_S k_S = 2\pi\chi(D_E)$$

for the domain $D_E$.

LEMMA 4.2. *$D_M$ is flat and its Euler number is* 1.

PROOF. As in the proof of Lemma 4.1, we have the Gauss-Bonnet formulas for domains $D_E$ and $D_M$

$$\int_{D_E} K_E + \int_S k_S = 2\pi\chi(D_E)$$
$$\int_{D_M} K_M + \int_{\varphi(S)} k_{\varphi(S)} = 2\pi\chi(D_M)$$

and relations

$$K_M \geqq 0 = K_E$$
$$\int_{\varphi(S)} k_{\varphi(S)} = \int_S k_S = 2\pi$$
$$\chi(D_M) \leqq 1 = \chi(D_E).$$

Therefore we get

$$\int_{D_E} K_E + \int_S k_S \leqq \int_{D_M} K_M + \int_{\varphi(S)} k_{\varphi(S)} = 2\pi\chi(D_M) \leqq 2\pi\chi(D_E),$$

which proves the lemma.

Therefore $M$ is flat and simply connected, i.e., $M$ is isometric to the 2-dimensional Euclidean space.

## §5. Proof of Theorem 1.5.

Let $M$ be a complete 2-dimensional Riemannian manifold with $K_M \geqq -1$. Let $V_M$ and $V_H$ be compact sets of $M$ and $H^2(-1)$ respectively. Suppose that there is an isometry $\varphi: H^2(-1)\setminus V_H \to M\setminus V_M$. Let $S$ be a circle of radius $l$ in $H^2(-1)$ which contains $V_H$ in its interior. We denote the inside of $S$ by $D_H$ and the inside of $\varphi(S)$ by $D_M$.

LEMMA 5.1. *$D_M$ is orientable.*

PROOF. If $D_M$ is not orientable, there is a double cover $\pi: D' \to D_M$, where $D'$ is orientable. Let $S'$ be the boundary of $D'$. Let $K'$ denote the sectional curvature of $D'$. Then it follows from a generalization of the Rauch-Berger comparison theorem by Waner[6] that the focal radius of $S'$ in $D'$ is less than or equal to $l$ and the area of $D'$ is less than or equal to $2area(D_H)$. Hence we have

$$\int_{D'} K' \geqq 2\int_{D_H} K_H = -2\cdot area(D_H).$$

Then the lemma can be proved as in the proof of Lemma 4.1.

LEMMA 5.2. *$D_M$ is of constant curvature $-1$ and its Euler number is* 1.

PROOF. As in the proof of Lemma 5.1, We have an inequality

$$\int_{D_H} K_H \leqq \int_{D_M} K_M.$$

Then the lemma can be proved as in the proof of Lemma 4.2.

Therefore $M$ is of constant curvature $-1$ and simply connected, i.e., $M$ is isometric to the 2-dimensional Hyperbolic space.

## § 6. Proof of Theorem 1.6.

Let $H^n(-1)$ be the $n$-dimensional hyperbolic space of constant curvature $-1$. Let $o$ be a point of $H^n(-1)$. Let $x$ and $y$ be points of $H^n(-1)$ with $xo \perp yo$ and $|xo| = |yo|$, where $xo$ denotes the geodesic segment between $x$ and $o$ and $|xo|$ denotes its length. Let $S_x(|xy|/2)$ and $S_y(|xy|/2)$ denote the metric spheres of radius $|xy|/2$ centered at $x$ and $y$. Then $S_x(|xy|/2)$ and $S_y(|xy|/2)$ converge to horospheres which tangent each other as $|xo| = |yo| \to \infty$ and the distance between $o$ and the horospheres is $r_{-1}$. Therefore we have the following

LEMMA 6.1. *Let $V(r)$ be the metric ball of radius $r < r_{-1}$ centered at $o$. We take $x$ and $y$ far away from $o$ so that metric spheres $S_x(|xy|/2)$ and $S_y(|xy|/2)$ do not intersect $V(r)$. Then the geodesic segment $xy$ does not pass $V(r)$ and is shorter than the $H^n(-1)\setminus V(r)$-part of any curve from $x$ to $y$.*

Let $M$ be a complete Riemannian manifold with curvature $K_M \geqq -1$. Let $V_M$ be a compact set of $M$. Let $V(r)$ be the metric ball of radius $r < r_{-1}$ in $H^n(-1)$ centered at $o$. Suppose there is an isometry $\varphi$: $H^n(-1)\setminus V(r) \to M\setminus V_M$. Let $S$ be the metric sphere of radius $l$ in $H^n(-1)$ centered at $o$ whose inside contains $V(r)$. Let $p$ be a point of

$S$ and $q$ the antipodal point of $p$ in $S$. Let $x$ be a point of $S$ with $xo \perp po$. From Lemma 6.1, we may assume that the geodesic segments $px$ and $qx$ are shorter than $H^n(-1)\setminus V(r)$-part of any curve between their ends. Then we can prove lemmas 3.2, 3.3, 3.4 and consequently Theorem 1.6 as in section 3.

## § 7. Proof of Theorem 1.7.

Let $S^n(1)$ be the $n$-dimensional sphere of constant curvature 1. Let $o$ be a point of $S^n(1)$. Let $x$ and $y$ be points of $S^n(1)$ with $xo \perp yo$ and $|xo| = |yo| < \pi/2$, where $xo$ denotes the geodesic segment between $x$ and $o$ and $|xo|$ denotes its length. Then next lemma is clear.

LEMMA 7.1. *Let $V(r)$ be the metric ball of radius $r < r_{+1} = \pi/4$ centered at $o$. We take $x$ and $y$ far away from $o$ so that $r + \pi/4 < |xo| = |yo| < \pi/2$. Then the geodesic segment $xy$ does not pass $V(r)$ and is shorter than the $S^n(1)\setminus V(r)$-part of any curve from $x$ to $y$.*

Let $M$ be a complete Riemannian manifold with curvature $K_M \geqq 1$. Let $V_M$ be a compact set of $M$. Let $V(r)$ be the metric ball of radius $r < r_{+1}$ in $S^n(1)$ centered at $o$. Suppose there is an isometry $\varphi: S^n(1)\setminus V(r) \to M\setminus V_M$. Let $S$ be the metric sphere of radius $l < \pi/2$ in $S^n(1)$ centered at $o$ whose inside contains $V(r)$. Let $p$ be a point of $S$ and $q$ the antipodal point of $p$ in $S$. Let $x$ be a point of $S$ with $xo \perp po$. From Lemma 7.1, we may assume that the geodesic segments $px$ and $qx$ are shorter than the $S^n(1)\setminus V(r)$-part of any curve between their ends. Then we can prove lemmas 3.2, 3.3, 3.4 and consequently Theorem 1.7 as in section 3.

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