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Twisted linear actions on projective spaces

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0. Introduction

In this paper, we shall study the twisted linear actions of non-compact Lie groups on complex projective spaces and quaternion projective spaces. A twisted linear action is defined by F. Uchida [4] who gave an example of $SL(n, \boldsymbol{R})$-actions on a $(2n-1)$-sphere. In contrast to compact Lie groups, those $SL(n, \boldsymbol{R})$-actions are uncountably many topologically distinct C^{ω}-actions (of. [4], [5]). From this point of view it seems interesting to study twisted linear actions of non-compact Lie groups on compact manifolds other than spheres. The remainder of this note is divided into three sections. In Section 1, we define twisted linear actions of Lie groups on $\boldsymbol{F}$-projective spaces, where $\boldsymbol{F}=\boldsymbol{C}$, $\boldsymbol{H}$ and show that twisted linear actions of compact Lie groups on these spaces are equivariantly diffeomorphic to linear actions (Theorem 1.4). In Section 2, we show that there are uncountably many topologically distinct C^{ω}-actions of $SL(n, \boldsymbol{F})$ on an $(nk-1)$-dimensional $\boldsymbol{F}$-projective space, where $n>k\geqq 2$ (Theorem 2.3). In Section 3, we show that there are uncountably many C^{1}-differentiably distinct but topologically equivalent C^{ω}-actions of $SL(n, \boldsymbol{F})$ on an m-dimensional $\boldsymbol{F}$-projective space, where $m\geqq n\geqq 2$ (Theorem 3.3 and Theorem 3.5). The author wishes to express his hearty gratitude to Professor Fuichi Uchida who offered this topic and helpful advice.

1. Twisted linear actions on projective spaces

Throughout this paper, let $\boldsymbol{F}$ be the field of complex numbers $\boldsymbol{C}$ or quaternions $\boldsymbol{H}$ and $M(n, m; \boldsymbol{F})$ the set of all $\boldsymbol{F}$-matrices of type (n, m). Moreover let $\boldsymbol{F}^{n}$ denote the right $\boldsymbol{F}$-vector space of all n-dimensional $\boldsymbol{F}$-column vectors and we set $M_{n}(\boldsymbol{F})=M(n, n; \boldsymbol{F})$. We denote the set of all square real matrices of degree n by $M_{n}(\boldsymbol{R})$. We define $\iota_{1}: M_{n}(\boldsymbol{C})\rightarrow M_{2n}(\boldsymbol{R})$ and $\iota_{2}: M_{n}(\boldsymbol{H})\rightarrow M_{2n}(\boldsymbol{C})$ by

$$\iota_{1}(A+iB)=\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \text{ and } \iota_{2}(C+iD)=\begin{pmatrix} C & -\bar{D} \\ D & \bar{C} \end{pmatrix},$$

where A, $B\in M_{n}(\boldsymbol{R})$ and C, $D\in M_{n}(\boldsymbol{C})$. Then we see that ι_{1} and ι_{2} are injective ring homomorphisms. We define $\iota : M_n(\boldsymbol{F}) \to M_{d_F n}(\boldsymbol{R})$ by $(d_F, \iota)=(2, \iota_1)$ or $(4, \iota_1\iota_2)$ for $\boldsymbol{F}=\boldsymbol{C}$ or $\boldsymbol{H}$, respectively. We set

$$SL(n, \boldsymbol{H})=\{A\in M_n(\boldsymbol{H})\ ;\ \det(\iota_2(A))=1\}.$$

Then we have $SL(n, \boldsymbol{H})=\iota_2^{-1}(SL(2n, \boldsymbol{C}))$. For $A\in M(n, m\ ; \boldsymbol{F})$, A^* denotes the adjoint matrix of A. We set $U(n, \boldsymbol{F})=\{A\in M_n(\boldsymbol{F})\ ;\ A^*A=I_n\}$. Then $U(n, \boldsymbol{C})$, $U(n, \boldsymbol{H})$ is equal to the unitary group $U(n)$, symplectic group $Sp(n)$, respectively. For $u, v\in \boldsymbol{F}^n$, we define their hermitian inner product by $\langle u, v\rangle = u^*v$ and the norm of u by $\|u\|=\sqrt{\langle u, u\rangle}$.

1. 1. We say that $X\in M_n(\boldsymbol{F})$ satisfies the condition (T) if $\frac{1}{2}(X+X^*)$ is a positive definite hermitian matrix. It is easy to see that X satisfies the condition (T) if and only (T') $\frac{d}{dt}\|\exp(tx)z\|>0$ for each $z\in \boldsymbol{F}_0^n=\boldsymbol{F}^n-\{0\}$, $t\in\boldsymbol{R}$. If X satisfies (T'), then

$$\lim_{t\to+\infty}\|\exp(tX)z\|=+\infty \text{ and } \lim_{t\to-\infty}\|\exp(tX)z\|=0$$

for each $z\in\boldsymbol{F}_0^n$ and hence there exists a unique real valued C^ω-function τ on $\boldsymbol{F}_0^n$ such that

$$\|\exp(\tau(z)X)z\|=1 \text{ for } z\in\boldsymbol{F}_0^n.$$

The following lemma is proved in [4, Lemma 2. 2].

LEMMA 1. 1. *Let $X\in M_n(\boldsymbol{F})$, where $\boldsymbol{F}=\boldsymbol{R}$, $\boldsymbol{C}$ or $\boldsymbol{H}$ and assume that all the eigenvalues of X have positive real parts. Then there exists $P\in GL(n, \boldsymbol{F})$ such that $P^{-1}XP$ satisfies the condition (T).*

Throughout this paper, $\boldsymbol{F}_0$ denotes the multiplicative group of non-zero elements of $\boldsymbol{F}$. For $X\in M_n(\boldsymbol{F})$ whose all eigenvalues have positive real parts, we define a real analytic right $\boldsymbol{F}_0$-action α_F^X on $\boldsymbol{F}_0^n$ as follows:

$$\alpha_F^X : \boldsymbol{F}_0^n\times\boldsymbol{F}_0 \longrightarrow \boldsymbol{F}_0^n,\ \alpha_F^X(z, \zeta)=\exp((\log|\zeta|)X)z(\zeta/|\zeta|).$$

By Lemma 1. 1, there exists $P\in GL(n, \boldsymbol{F})$ such that $X_0=P^{-1}XP$ satisfies the condition (T). For this matrix X_0, we define C^ω-diffeomorphism Φ_{X_0}, $\Psi_{X_0} : \boldsymbol{F}_0^n \longrightarrow \boldsymbol{F}_0^n$ by

$$\Phi_{X_0}(z)=\exp((\log\|z\|)X_0)z/\|z\|,$$
$$\Psi_{X_0}(w)=\exp(\tau(w)X_0)we^{-\tau(w)}.$$

Then we have $\Phi_{X_0}^{-1}=\Psi_{X_0}$. Moreover, for the above matrix X, we define a C^ω-diffeomorphism F_X of $\boldsymbol{F}_0^n$ by $F_X=L_P\circ\Phi_{X_0}$, where $L_P(z)=Pz$ for each z

$\in \boldsymbol{F}_0^n$. F_X depends probably on a choice of P. Then we have $F_X^{-1} = \Psi_{X_0} \circ L_P^{-1}$. Furthermore we have the commutative diagram

(1.1)
$$\begin{array}{ccc} \boldsymbol{F}_0^n \times \boldsymbol{F}_0 & \xrightarrow{\alpha_F^{I_n}} & \boldsymbol{F}_0^n \\ \Big\downarrow F_X \times 1 & & \Big\downarrow F_X \\ \boldsymbol{F}_0^n \times \boldsymbol{F}_0 & \xrightarrow{\alpha_F^X} & \boldsymbol{F}_0^n . \end{array}$$

We denote the orbit space of the action α_F^X by $P_{n-1}^X(\boldsymbol{F})$. Then $P_{n-1}^{I_n}(\boldsymbol{F})$ is the usual $(n-1)$-dimensional $\boldsymbol{F}$-projective space $P_{n-1}(\boldsymbol{F})$. We denote the α_F^X-orbit through $z \in \boldsymbol{F}_0^n$ by $[z]_X$ and the canonical projection of $\boldsymbol{F}_0^n$ onto $P_{n-1}^X(\boldsymbol{F})$ by π_X. *Then a* C^ω-manifold structure of $P_{n-1}^X(\boldsymbol{F})$ is canonically induced by the real analytic right $\boldsymbol{F}_0$-action α_F^X on $\boldsymbol{F}_0^n$ such that the projection $\pi_X : \boldsymbol{F}_0^n \longrightarrow P_{n-1}^X(\boldsymbol{F})$ is a C^ω-submersion. Furthermore we have the commutative diagram

(1.2)
$$\begin{array}{ccc} \boldsymbol{F}_0^n & \xrightarrow{F_X} & \boldsymbol{F}_0^n \\ \Big\downarrow \pi & & \Big\downarrow \pi_X \\ P_{n-1}(\boldsymbol{F}) & \xrightarrow{\widetilde{F}_X} & P_{n-1}^X(\boldsymbol{F}) \end{array}$$

where $\pi = \pi_{I_n}$, $\widetilde{F}_X$ is a C^ω-diffeomorphism defined by $\widetilde{F}_X([z]) = [F_X(z)]_X$ and $[z] = [z]_{I_n}$ for $z \in \boldsymbol{F}_0^n$.

1. 2. Let G be a Lie group, $\rho : G \longrightarrow GL(n, \boldsymbol{F})$ a matricial representation and X a square $\boldsymbol{F}$-matrix of degree n whose all eigenvalues have positive real parts. We call (ρ, X) an $\boldsymbol{F}TC$-pair of degree n, if $\rho(g)X = X\rho(g)$ for each $g \in G$. For an $\boldsymbol{F}TC$-pair (ρ, X) of degree n, we can define a C^ω-mapping

$$\xi_F : G \times P_{n-1}^X(\boldsymbol{F}) \longrightarrow P_{n-1}^X(\boldsymbol{F}) \text{ by } \xi_F(g, [z]_X) = [\rho(g)z]_X$$

and we see that ξ_F is a real analytic G-action on $P_{n-1}^X(\boldsymbol{F})$. We call $\xi_F = \xi_F^{(\rho, X)}$ a twisted linear action of G on $P_{n-1}^X(\boldsymbol{F})$ determined by the $\boldsymbol{F}TC$-pair (ρ, X) and we say that ξ_F is associated to the matricial representation ρ. Moreover we have a real analytic G-action $\xi_{\boldsymbol{F}}^{0}: G\times P_{n-1}(\boldsymbol{F}) \longrightarrow P_{n-1}(\boldsymbol{F})$ defined by

$$\xi_{\boldsymbol{F}}^{0}(g, [z])=[F_{X}^{-1}(\rho(g)F_{X}(z))].$$

Then the following diagram is commutative:

$$\begin{array}{ccc} G\times P_{n-1}(\boldsymbol{F}) & \xrightarrow{\xi_{\boldsymbol{F}}^{0}} & P_{n-1}(\boldsymbol{F}) \\ \downarrow 1\times\widetilde{F}_{X} & & \downarrow \widetilde{F}_{X} \\ G\times P_{n-1}^{X}(\boldsymbol{F}) & \xrightarrow{\xi_{\boldsymbol{F}}} & P_{n-1}^{X}(\boldsymbol{F}). \end{array} \tag{1.3}$$

We call also $\xi_{\boldsymbol{F}}^{0}$ a twisted linear action of G on $P_{n-1}(\boldsymbol{F})$ determined by the $\boldsymbol{F}TC$-pair (ρ, X) and we say that $\xi_{\boldsymbol{F}}^{0}$ is associated to the matricial representation ρ.

1.3. For a given Lie group G, we introduce certain equivalence relations on $\boldsymbol{F}TC$-pairs. Let (ρ, X) and (σ, Y) be of degree n, where ρ, $\sigma: G\longrightarrow GL(n, \boldsymbol{F})$ are matricial representations and X, Y are $\boldsymbol{F}$-matrices of degree n whose all eigenvalues have positive real parts. We say that (ρ, X) is algebraically equivalent to (σ, Y);
in case $\boldsymbol{F}=\boldsymbol{C}$, if there exist $A\in GL(n, \boldsymbol{C})$, a positive real number c and a real number d satisfying

$$Y=cAXA^{-1}+\sqrt{-1}dI_{n} \text{ and } \sigma(g)=A\rho(g)A^{-1} \tag{1.4}$$

for each $g\in G$ and in case $\boldsymbol{F}=\boldsymbol{H}$, if there exist $A\in GL(n, \boldsymbol{H})$ and a positive real number c satisfying

$$Y=cAXA^{-1} \text{ and } \sigma(g)=A\rho(g)A^{-1} \tag{1.5}$$

for each $g\in G$,
We say that (ρ, X) is C^{r}-equivalent to (σ, Y), if there exists a C^{r}-diffeomorphism $f: P_{n-1}^{X}(\boldsymbol{F})\longrightarrow P_{n-1}^{Y}(\boldsymbol{F})$ $(r=0, 1, 2, \cdots, \infty, \omega)$ such that the following diagram is commutative:

$$
\begin{array}{ccc}
G\times P^X_{n-1}(\boldsymbol{F}) & \xrightarrow{\quad \xi_{\boldsymbol{F}}^{(\rho,X)} \quad} & P^X_{n-1}(\boldsymbol{F}) \\
\Big\downarrow{\scriptstyle 1\times f} & & \Big\downarrow{\scriptstyle f} \\
G\times P^Y_{n-1}(\boldsymbol{F}) & \xrightarrow{\quad \xi_{\boldsymbol{F}}^{(\sigma,Y)} \quad} & P^Y_{n-1}(\boldsymbol{F}).
\end{array}
\tag{1.6}
$$

We call f a G-equivariant C^r-diffeomorphism. We prove the following lemma.

LEMMA 1.2. *If (ρ, X) is algebraically equivalent to (σ, Y), then (ρ, X) is C^{ω}-equivalent to (σ, Y).*

PROOF. We prove the lemma only in case $\boldsymbol{F}=\boldsymbol{C}$, since in case $\boldsymbol{F}=\boldsymbol{H}$, it is shown in the similar manner. Suppose that there exist $A\in GL(n, \boldsymbol{C})$, a positive real number c and a real number d satisfying (1.4). We define an automorphism of the Lie grouup $\boldsymbol{C}_0$ by

$$f(\zeta)=\zeta\exp((\log|\zeta|)(1-c-\sqrt{-1}d)/c).$$

Then the following diagram is commutative:

$$
\begin{array}{ccc}
\boldsymbol{C}^n_0\times \boldsymbol{C}_0 & \xrightarrow{\quad \alpha^X_{\boldsymbol{C}} \quad} & \boldsymbol{C}^n_0 \\
\Big\downarrow{\scriptstyle L_A\times f} & & \Big\downarrow{\scriptstyle L_A} \\
\boldsymbol{C}^n_0\times \boldsymbol{C}_0 & \xrightarrow{\quad \alpha^Y_{\boldsymbol{C}} \quad} & \boldsymbol{C}^n_0.
\end{array}
\tag{1.7}
$$

where $L_A(z)=Az$ for $z\in \boldsymbol{C}^n_0$. Now we define a C^{ω}-diffeomorphism $\tilde{L}_A: P^X_{n-1}(\boldsymbol{C})\longrightarrow P^Y_{n-1}(\boldsymbol{C})$ by $\tilde{L}_A([z]_X)=[Az]_Y$. It is easily shown that $\tilde{L}_A$ is G-equivariant. q. e. d.

Let $\rho: G\longrightarrow GL(n, \boldsymbol{F})$ be a matricial representation of a Lie group G. We say that ρ is in standard form, if there exist irreducible representations $\rho_j: G\longrightarrow GL(n_j, \boldsymbol{F})$ $(j=1, 2, \cdots, r)$ (furthermore $\rho_j(G)\subset GL(n_j, \boldsymbol{F}_j)$ in case $\boldsymbol{F}=\boldsymbol{H}$) such that

$$\rho=(\rho_1\otimes I_{k_1})\oplus\cdots\oplus(\rho_r\otimes I_{k_1}), \tag{1.8}$$

$$\mathrm{End}_G(\rho)=\begin{cases}(I_{n_1}\otimes M_{k_1}(\boldsymbol{C}))\oplus\cdots\oplus(I_{n_r}\otimes M_{k_r}(\boldsymbol{C})) & (\boldsymbol{F}=\boldsymbol{C})\\ (I_{n_1}\otimes M_{k_1}(K_{\boldsymbol{F}_1}))\oplus\cdots\oplus(I_{n_r}\otimes M_{k_r}(K_{\boldsymbol{F}_r})) & (\boldsymbol{F}=\boldsymbol{H}),\end{cases} \tag{1.9}$$

where $n=n_1k_1+\cdots+n_rk_r$, $\boldsymbol{F}_j=\boldsymbol{R}$, $\boldsymbol{C}$ or $\boldsymbol{H}$, $\mathrm{End}_G(\rho)=\{X\in M_n(\boldsymbol{F})\,;\, X\rho(g)=\rho(g)X \text{ for } g\in G\}$ and $K_{\boldsymbol{R}}=\boldsymbol{H}$, $K_{\boldsymbol{C}}=\boldsymbol{C}$, $K_{\boldsymbol{H}}=\boldsymbol{R}$. It is well known that any matricial representation of a compact Lie group is equivalent to one in standard form (cf. [1, ch. 3], [2, ch. II] or [3, ch. VI]). The following lemma is easily shown by (1.8), (1.9) and Lemma 1.1.

LEMMA 1.3. *Let ρ be a $\boldsymbol{F}$-matricial representation in standard form of a Lie group G. Let $X\in\mathrm{End}_G(\rho)$ and assume that all the eigenvalues of X have positive real parts. Then there exists $P\in\mathrm{Ant}_G(\rho)\cap GL(n, \boldsymbol{F})$ such that PXP^{-1} satisfies the condition (T).*

Now we prove the following theorem.

THEOREM 1.4. *Let G be a compact Lie group and $\rho: G\longrightarrow GL(n, \boldsymbol{F})$ a matricial representation. Then any $\boldsymbol{F}TC$-pair (ρ, X) is C^{ω}-equivalent to (ρ, I_n). In other words, any twisted linear acion of G on $P_{n-1}(\boldsymbol{F})$ associated to ρ is equivariantly C^{ω}-diffeomorphic to the linear action of G on $P_{n-1}(\boldsymbol{F})$ associated to ρ.*

PROOF. Since G is compact, there are $P_1\in GL(n, \boldsymbol{F})$ and a representation $\sigma: G\longrightarrow U(n, \boldsymbol{F})$ in standard form satisfying $\sigma(g)P_1=P_1\rho(g)$ for each $g\in G$. Then $P_1XP_1^{-1}\in\mathrm{End}_G(\rho)$ and all eigenvalues of $P_1XP_1^{-1}$ have positive real parts, so there exists $P_2\in\mathrm{Aut}_G(\sigma)$ such that $P_2P_1XP_1^{-1}P_2^{-1}$ satisfies the condition (T) by Lemma 1.3. Set $P=P_2P_1$ and $Y=PXP^{-1}$. Then (σ, Y) is an $\boldsymbol{F}TC$-pair. Moreover we have the commutative diagram

$$\begin{array}{ccccccc}
G\times P_{n-1}(\boldsymbol{F}) & \xrightarrow{1\times\tilde{L}_P} & G\times P_{n-1}(\boldsymbol{F}) & \xrightarrow{1\times\tilde{\Phi}_Y} & G\times P^Y_{n-1}(\boldsymbol{F}) & \xleftarrow{1\times\tilde{L}_P} & G\times P^X_{n-1}(\boldsymbol{F}) \\
\downarrow{\scriptstyle \xi_{\boldsymbol{F}}^{(\rho, I_n)}} & & \downarrow{\scriptstyle \xi_{\boldsymbol{F}}^{(\sigma, I_n)}} & & \downarrow{\scriptstyle \xi_{\boldsymbol{F}}^{(\sigma, Y)}} & & \downarrow{\scriptstyle \xi_{\boldsymbol{F}}^{(\rho, X)}} \\
P_{n-1}(\boldsymbol{F}) & \xrightarrow{\tilde{L}_P} & P_{n-1}(\boldsymbol{F}) & \xrightarrow{\tilde{\Phi}_Y} & P^Y_{n-1}(\boldsymbol{F}) & \xleftarrow{\tilde{L}_P} & P^X_{n-1}(\boldsymbol{F}),
\end{array}$$

where $\tilde{\Phi}_Y([z])=[\exp((\log\|z\|)Y)z/\|z\|]_Y$ (cf. (1.3)). It is easily seen that $\tilde{L}_P^{-1}\circ\tilde{\Phi}_Y\circ\tilde{L}_P$ is a C^{ω}-diffeomorphism. q. e. d.

2. First typical examples

Here we shall study twisted linear actions of $G=SL(n, \boldsymbol{F})$ on the $(nk-1)$-dimensional $\boldsymbol{F}$-projective space associated to a representation $\rho=\rho_n\otimes I_k$, that is, $\rho(A)=A\otimes I_k$ for each $A\in G$.

2. 1. Suppose that $A\in M_n(\boldsymbol{C})$, $B\in M_k(\boldsymbol{C})$ or $A\in M_n(\boldsymbol{H})$, $B\in M_k(\boldsymbol{R})$ or $A\in M_n(\boldsymbol{R})$, $B\in M_k(\boldsymbol{H})$. Denote by $A\otimes B$ the Kronecker product written in the form

$$A\otimes B=\begin{pmatrix} b_{11}A & \cdots & b_{1k}A \\ \vdots & & \vdots \\ b_{k1}A & \cdots & b_{kk}A \end{pmatrix}\in M_{nk}(\boldsymbol{F}).$$

Let $u_1, \ldots, u_k$ be column vectors in $\boldsymbol{F}^n$. Then the correspondence

$$(u_1, \cdots, u_k)\longmapsto\begin{bmatrix} u_1 \\ \vdots \\ u_k \end{bmatrix}$$

defines a linear $\boldsymbol{F}$-isomorphism $\lambda: M(n, k; \boldsymbol{F})\longrightarrow\boldsymbol{F}^{nk}$, where we regard $M(n, k; \boldsymbol{F})$ as a right $\boldsymbol{F}$-vector space. As usual, for $X, Y\in M(n, k; \boldsymbol{F})$, we define their hermitian inner product by

$$\langle X, Y\rangle=\mathrm{trace}(X^*Y)$$

and the norm of X by $\|X\|=\sqrt{\langle X, X\rangle}$. Then λ is an isometry. Furthermore the equality

$$(A\otimes B)\lambda(X)=\lambda(AX\,{}^tB)$$

holds, where $A\in M_n(\boldsymbol{C})$(resp. $M_n(\boldsymbol{H})$), $B\in M_k(\boldsymbol{C})$(resp. $M_k(\boldsymbol{R})$) and $X\in M(n, k; \boldsymbol{C})$(resp. $M(n, k; \boldsymbol{H})$). In this section, we shall identify $\boldsymbol{F}^{nk}$ with $M(n, k; \boldsymbol{F})$ via the isometry λ.

2. 2. We obtain the following lemma directly.

LEMMA 2. 1. *Let $\hat{M}$ be a square $\boldsymbol{F}$-matrix of degree nk. Then*

$$\hat{M}(A\otimes I_k)=(A\otimes I_k)\hat{M}$$

for each $A\in SL(n, \boldsymbol{F})$ if and only if $\hat{M}=I_n\otimes M$, where in case $\boldsymbol{F}=\boldsymbol{C}$ (resp. $\boldsymbol{H}$), M is a certain square $\boldsymbol{C}$(resp. $\boldsymbol{R}$)-matrix of degree k. Furthermore all the eigenvalues of $I_n\otimes M$ have positive real parts if and only if all the eigenvalues of M have positive real parts.

Consequently, $(\rho_n\otimes I_k, I_n\otimes M)$ is a CTC(resp. HTC)-pair for any square $\boldsymbol{C}$(resp. $\boldsymbol{R}$)-matrix M of degree k whose all eigenvalues have positive real parts and any FTC-pair $(\rho_n\otimes I_k, \hat{M})$ is written in such form. Furthermore CTC(resp. HTC)-pairs $(\rho_n\otimes I_k, I_n\otimes M)$ and $(\rho_n\otimes I_k, I_n\otimes N)$

are algebraically equivalent if and only if there exist $X\in GL(k, \boldsymbol{C})$ (resp. $GL(k, \boldsymbol{R})$), a positive real number c and a real number d (resp. a positive real number c) satisfying $N=cXMX^{-1}+\sqrt{-1}dI_k$ (resp. $N=cXMX^{-1}$).

2. 3. In case $\boldsymbol{F}=\boldsymbol{C}$ (resp. $\boldsymbol{H}$), let M be a square $\boldsymbol{C}$ (resp. $\boldsymbol{R}$)-matrix of degree k whose all eigenvalues have positive real parts. Denote by $\zeta_{\boldsymbol{F}}^{M}$ the twisted linear $SL(n, \boldsymbol{F})$-action on the $(nk-1)$-dimensional $\boldsymbol{F}$-projective space determined by the $\boldsymbol{F}TC$-pair $(\rho_n\otimes I_k, I_n\otimes M)$. We identify $\boldsymbol{F}_0^{nk}$ with $M(n, k; \boldsymbol{F})_0=M(n, k; \boldsymbol{F})-\{0\}$ via the isometry λ. Moreover we identify the orbit space of a right $\boldsymbol{F}_0$-action $\alpha_{\boldsymbol{F}}^{M}$ on $M(n, k; \boldsymbol{F})_0$ defined by

$$\alpha_{\boldsymbol{F}}^{M}(Z, \zeta)=Z\frac{\zeta}{|\zeta|}\exp((\log|\zeta|)^{t}M)$$

with $P_{nk-1}^{\hat{M}}(\boldsymbol{F})$ via λ where $\hat{M}=I_n\otimes M$. We denote the $\alpha_{\boldsymbol{F}}^{M}$-orbit through $Z\in M(n, k; \boldsymbol{F})_0$ by $[Z]_{\hat{M}}\in P_{nk-1}^{\hat{M}}(\boldsymbol{F})$. Then the C^{ω}-diffeomorphism $\tilde{F}_{\hat{M}}$ of $P_{nk-1}(\boldsymbol{F})=P_{nk-1}^{\hat{I}_k}(\boldsymbol{F})$ onto $P_{nk-1}^{\hat{M}}(\boldsymbol{F})$ is given by

$$\tilde{F}_{\hat{M}}([Z])=[F_{\hat{M}}(Z)]_{\hat{M}}=[\Phi_{\hat{M}_0}(Z)^{t}P]_{\hat{M}},$$

where in case $\boldsymbol{F}=\boldsymbol{C}$ (resp. $\boldsymbol{H}$), for some $P\in GL(k, \boldsymbol{C})$ (resp. $GL(k, \boldsymbol{R})$) $M_0=P^{-1}MP$ satisfies the condition (T) and

$$\Phi_{\hat{M}_0}(Z)=\frac{Z}{\|Z\|}\exp((\log\|Z\|)^{t}M).$$

Moreover we can describe $\zeta_{\boldsymbol{F}}^{M}: SL(n, \boldsymbol{F})\times P_{nk-1}^{\hat{M}}(\boldsymbol{F})\longrightarrow P_{nk-1}^{\hat{M}}(\boldsymbol{F})$ by

$$\zeta_{\boldsymbol{F}}^{M}(A, [Z]_{\hat{M}})=[AZ]_{\hat{M}}.$$

Let $I(M)$ and $O(M)$ denote the isotropy group at

$$\left[\begin{pmatrix} I_k \\ 0 \end{pmatrix}\right]_{\hat{M}}$$

and the orbit through that point, respectively, with respect to the twisted linear action $\zeta_{\boldsymbol{F}}^{M}$. We define a homomorphism $g_{\boldsymbol{F}}^{M}: \boldsymbol{F}_0\longrightarrow GL(k, \boldsymbol{F})$ by

$$g_{\boldsymbol{F}}^{M}(\zeta)=\frac{\zeta}{|\zeta|}\exp((\log|\zeta|)^{t}M).$$

We obtain the following lemma.

LEMMA 2. 2. *Suppose that* $n>k\geqq 2$. *Then*

(i) *the isotropy group* $I(M)$ *is written in the form*

$$I(M)=\left\{\begin{pmatrix} g_F^M(\zeta) & * \\ 0 & * \end{pmatrix}\in SL(n, \boldsymbol{F})\,;\ \zeta\in \boldsymbol{F}_0\right\};$$

(ii) *the orbit* $O(M)$ *is equal to*

$$\{[Z]_{\hat{M}}\in P^{\hat{M}}_{nk-1}(\boldsymbol{F})|Z\in M(n, k\,;\boldsymbol{F})_0\,;\ rank(Z)=k\}\,;$$

(iii) *the orbit* $O(M)$ *is an open dense subset of* $P^{\hat{M}}_{nk-1}(\boldsymbol{F})$.

2.4. The purpose of this section is to prove the following theorem.

THEOREM 2.3. *Let* $n>k\geqq 2$. *Then any two of* $\boldsymbol{F}$*TC-pairs in the form* $(\rho_n\otimes I_k, I_n\otimes M)$ *are algebraically equivalent if and only if they are* C^0*-equivalent.*

First we prepare two lemmas for the proof.

LEMMA 2.4. *Set* $K_{\boldsymbol{C}}=\boldsymbol{C}$ *and* $K_{\boldsymbol{H}}=\boldsymbol{R}$. *For* $M\in M_k(K_{\boldsymbol{F}})$ *whose all eigenvalues have positive real parts, the homomorphism* $g_F^M:\boldsymbol{F}_0\longrightarrow GL(k, \boldsymbol{F})$ *defined in Subsection* **2.3** *is an into-homeomorphism.*

PROOF. There exists $P\in GL(k, K_{\boldsymbol{F}})$ such that ${}^tP^{-1}{}^tM{}^tP={}^tM^{\circ}$ satisfies the condition (T). Then $F_{{}^tM}:\boldsymbol{F}_0^k\longrightarrow \boldsymbol{F}_0^k$ defined by

$$F_{{}^tM}(z)={}^tP\exp((\log\|z\|){}^tM_0)z/\|z\|$$

is a C^ω-diffeomorphism (see 1.1). Hence a product mapping of k $F^t{}_M$'s

$$F_{{}^tM}:=F_{{}^tM}\times\cdots\times F_{{}^tM}:\boldsymbol{F}_0^k\times\cdots\times\boldsymbol{F}_0^k=(\boldsymbol{F}_0^k)^k\longrightarrow(\boldsymbol{F}_0^k)^k$$

is a C^ω-diffeomorphism. We regard $(\boldsymbol{F}_0^k)^k$ as an open subset of $M_k(\boldsymbol{F})$. A mapping $j_k:\boldsymbol{F}_0\longrightarrow GL(k, \boldsymbol{F})$ defined by $j_k(\zeta)=\zeta I_k$ is an into-homeomorphism. Therefore $F^k_{{}^tM}\circ j_k:\boldsymbol{F}_0\longrightarrow(\boldsymbol{F}_0^k)^k$ is an into-homeomorphism. It is easily seen that

$$g_F^M(\zeta)=(F^k_{{}^tM}(j_k(\zeta))){}^tP^{-1}$$

for each $\zeta\in\boldsymbol{F}_0$. Hence g_F^M is an into-homeomorphism. q. e. d.

LEMMA 2.5. *Let* f_F *be an automorphism of the Lie group* F_0. *The Lie algebra automorphism* $\tilde{f}_F$ *of* $\boldsymbol{F}$ *uniquely induced by* f_F *is given by the following form*:

$$\tilde{f}_C(x+\sqrt{-1}y)=(a+\sqrt{-1}b)x+\sqrt{-1}\delta y\qquad(x+\sqrt{-1}y\in\boldsymbol{C}),$$

for some real numbers a, b and δ ; $a\neq 0$ *and* $\delta=\pm 1$,

$$\tilde{f}_H(x+z)=ax+qz\bar{q}\quad (x\in \boldsymbol{R},\ \bar{z}=-z\in \boldsymbol{H})$$

for some $a\in \boldsymbol{R}$; $a\neq 0$ *and* $q\in \boldsymbol{H}$; $|q|=1$.

PROOF. We define a Lie group homomorphism $\lambda: \boldsymbol{F}_0 \longrightarrow \boldsymbol{R}$ by $\lambda(\zeta)=\log|\zeta|$, where we regard $\boldsymbol{R}$ as an additive group. Then we have the following short exact sequence:

$$1\longrightarrow U(1, \boldsymbol{F})\longrightarrow \boldsymbol{F}_0 \xrightarrow{\lambda} \boldsymbol{R}\longrightarrow 0.$$

Each automorphism f_F of $\boldsymbol{F}_0$ induces uniquely automorphisms f'_F, f''_F of $U(1, \boldsymbol{F})$, $\boldsymbol{R}$, respectively and the diagram

$$\begin{array}{ccccc}
U(1, \boldsymbol{F}) & \longrightarrow & \boldsymbol{F}_0 & \xrightarrow{\lambda} & \boldsymbol{R} \\
\downarrow{\scriptstyle f'_F} & & \downarrow{\scriptstyle f_F} & & \downarrow{\scriptstyle f''_F} \\
U(1, \boldsymbol{F}) & \longrightarrow & \boldsymbol{F}_0 & \xrightarrow{\lambda} & \boldsymbol{R}
\end{array}$$

is commutative. It is well known that each automorphism f'_F of $U(1, \boldsymbol{F})$ is gievn by $f_C(\zeta)=\zeta$ or $\bar{\zeta}$ $(\zeta\in U(1, \boldsymbol{C})=U(1))$, $f_H(\zeta)=q\zeta\bar{q}$ $(\zeta\in U(1, \boldsymbol{H})=Sp(1))$ for some $q\in \boldsymbol{H}$; $|q|=1$. By these facts, The lemma is easily shown. q. e. d.

PROOF OF THE THEOREM 2. 3. Since the necessary condition follows immediately from Lemma 1.2, we have only to show the sufficient condition. Let M, $N\in M_k(K_F)$, where $K_C=\boldsymbol{C}$ and $K_H=\boldsymbol{R}$. Suppose that all the eigenvalues of M, N have positive real parts. Let $n>k\geqq 2$. Assume that there exists an $SL(n, \boldsymbol{F})$-equivariant homeomorphism f of $P^{\hat{M}}_{nk-1}(\boldsymbol{F})$ with a twisted linear action ζ^M_F onto $P^{\hat{N}}_{nk-1}(\boldsymbol{F})$ with a twisted linear action ζ^N_F. Then we obtain $f(O(M))=O(N)$ and hence $I(M)$ and $I(N)$ are conjugate in $SL(n, \boldsymbol{F})$. Thus there exists $T\in SL(n, \boldsymbol{F})$ such that $I(S)=TI(M)T^{-1}$. Then it easily shown that

$$T=\begin{pmatrix} X & C \\ 0 & Y \end{pmatrix},$$

for some $X\in GL(k, \boldsymbol{F})$, $Y\in GL(n-k, \boldsymbol{F})$ and $C\in M(k, n-k\,; \boldsymbol{F})$. Now we can assign each $\zeta\in \boldsymbol{F}_0$ to $\zeta'\in \boldsymbol{F}_0$ such that $Xg^M_F(\zeta)X^{-1}=g^N_F(\zeta')$. Then we obtain an automorphism f_F of an abstract group F_0 such that $f_F(\zeta)=\zeta'$ for each $\zeta\in F_0$. By Lemma 2.4, f_F is a homeomorphism of F_0 onto F_0. Hence f_F is an automorphism of the Lie group F_0. By Lemma 2.5, we see that for some a, $b\in \boldsymbol{R}$; $a\neq 0$ and $\delta=\pm 1$,

$$f_C(e^{x+\sqrt{-1}y})=e^{\tilde{f}_C(x+\sqrt{-1}y)}=e^{(a+\sqrt{-1}b)x+\sqrt{-1}\delta y},$$

where $x+\sqrt{-1}y\in \boldsymbol{C}$ and that for some $a\in\boldsymbol{R}$; $a\neq 0$ and $q\in \boldsymbol{H}$; $|q|=1$,

$$f_H(e^{x+z})=e^{\tilde{f}_H(x+z)}=(e^a)^x qe^z\bar{q},$$

where $x\in \boldsymbol{R}$ and $\bar{z}=-z\in \boldsymbol{H}$. First we shall investigate the case $\boldsymbol{F}=\boldsymbol{C}$. By the above observations, we have

$$\exp(x(X{}^tMX^{-1})+\sqrt{-1}yI_k)=\exp(ax{}^tN+\sqrt{-1}(bx+\delta y)I_k) \tag{2.1}$$

for each $x+\sqrt{-1}y\in\boldsymbol{C}$. Therefore we obtain

$$X{}^tMX^{-1}=a{}^tN+\sqrt{-1}bI_k \text{ and } \delta=1.$$

Since all the eigenvalues of M, N have positive real parts, a is a positive real number. Hence it is shown that in this case, $\boldsymbol{C}TC$-pairs $(\rho_n\otimes I_k, I_n\otimes M)$ and $(\rho_n\otimes I_k, I_n\otimes N)$ are algebraically equivalent (see **2. 2**). Next we shall investigate the case $\boldsymbol{F}=\boldsymbol{H}$. By the above observations, we have

$$Xe^z\exp(x{}^tM)X^{-1}=qe^z\bar{q}\exp(ax{}^tN) \tag{2.2}$$

for each $x+z\in\boldsymbol{H}(x\in\boldsymbol{R}, \bar{z}=-z\in\boldsymbol{H})$. In particular, we have $X\zeta X^{-1}=q\zeta\bar{q}I_k$ for each $\zeta\in\boldsymbol{H}$; $|\zeta|=1$. This fact implies $X_0=\bar{q}X\in GL(k, \boldsymbol{R})$. Hence by (2. 2), we have

$$\exp(x(X_0^tMX_0^{-1}))=\exp(x(a{}^tN))$$

for each $x\in\boldsymbol{R}$. Therefore for some positive real number a, $X_0^tMX_0^{-1}=a{}^tN$ holds good. Hence it follows from this fact that $\boldsymbol{H}TC$-pairs $(\rho_n\otimes I_k, I_n\otimes M)$ and $(\rho_n\otimes I_k, I_n\otimes N)$ are algebraically equivalent. q. e. d.

3. Second typical examples

Here we shall study twisted linear actions of $G=SL(n, \boldsymbol{F})$ on the $(n+k-1)$-dimensional $\boldsymbol{F}$-projective space associated to a representation $\rho=\rho_n\oplus I_k$, that is, $\rho(A)=A\oplus I_k$ for each $A\in G$.

3. 1. Let $A\in M_n(\boldsymbol{F})$ and $B\in M_k(\boldsymbol{F})$. We denote by $A\oplus B$ the matrix

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in M_{n+k}(\boldsymbol{F}).$$

We obtain the following lemma.

LEMMA 3.1. *Let $n \geqq 2$ and $k \geqq 1$. Let $X \in M_{n+k}(\boldsymbol{F})$. Then*

$$X(A \oplus I_k) = (A \oplus I_k)X$$

for $A \in SL(n, \boldsymbol{F})$ if and only if $X = cI_n \oplus M$ for some $M \in M_k(\boldsymbol{F})$ and $c \in K_{\boldsymbol{F}}$, where $K_{\boldsymbol{C}} = \boldsymbol{C}$ and $K_{\boldsymbol{H}} = \boldsymbol{R}$. Furthermore all the eigenvalues of X have positive real parts if and only if c and all the eigenvalues of M have positive real parts.

3. 2. Let $M \in M_k(\boldsymbol{F})$ and suppose that all the eigenvalues of M have positive real parts. Denote by $\chi_{\boldsymbol{F}}^M$ the twisted linear $SL(n, \boldsymbol{F})$-action on the $(n+k-1)$-dimensional $\boldsymbol{F}$-projective space $P_{n+k-1}^{(M)}(\boldsymbol{F})$ determined by the $\boldsymbol{F}TC$-pair $(\rho_n \oplus I_k, (M))$, where $(M) = I_n \oplus M$. Then $\chi_{\boldsymbol{F}}^M$ is written in the form

$$\chi_{\boldsymbol{F}}^M(A, [u \oplus v]_{(M)}) = [(Au) \oplus v]_{(M)}.$$

3. 3. Let us define closed subgroups $L_{\boldsymbol{F}}(n)$ and $N_{\boldsymbol{F}}(n)$ of $SL(n, \boldsymbol{F})$ by the forms

$$L_{\boldsymbol{C}}(n) = \left\{ \begin{pmatrix} 1 & * \cdots\cdots * \\ 0 & \\ \vdots & A_0 \\ 0 & \end{pmatrix} ; A_0 \in SL(n-1, \boldsymbol{C}) \right\},$$

$$N_{\boldsymbol{C}}(n) = \left\{ \begin{pmatrix} \lambda & * \cdots\cdots * \\ 0 & \\ \vdots & A_0 \\ 0 & \end{pmatrix} ; \begin{array}{l} \lambda \in \boldsymbol{C}_0,\ A_0 \in GL(n-1, \boldsymbol{C}) \\ \&\ \lambda \det A_0 = 1 \end{array} \right\},$$

$$L_{\boldsymbol{H}}(n) = \left\{ \begin{pmatrix} 1 & * \cdots\cdots * \\ 0 & \\ \vdots & A_0 \\ 0 & \end{pmatrix} ; A_0 \in SL(n-1, \boldsymbol{H}) \right\},$$

$$N_{\boldsymbol{H}}(n) = \left\{ \begin{pmatrix} \lambda & * \cdots\cdots * \\ 0 & \\ \vdots & A_0 \\ 0 & \end{pmatrix} ; \begin{array}{l} \lambda \in \boldsymbol{H}_0,\ A_0 \in GL(n-1, \boldsymbol{H}) \\ \&\ |\lambda|^2 \det(\iota_2(A_0)) = 1 \end{array} \right\}.$$

Denote by $F(M)$ the fixed point set of $L_F(n)$ with respect to the twisted linear action χ_F^M. Then we have the following lemma.

LEMMA 3.2. *With respect to the twisted linear action* χ_F^M,

$$F(M)=\{[e_1a\oplus v]_{(M)}\in P_{n+k-1}^{(M)}(\boldsymbol{F})|a\in\boldsymbol{F},\ v\in\boldsymbol{F}^k;|a|^2+\|v\|^2\neq 0\},$$

where $e_1={}^t(1,0,\cdots,0)\in\boldsymbol{F}^n$. *The isotropy group at* $[0_n\oplus v]_{(M)}$ *coincides with* $SL(n,\boldsymbol{F})$, *the one at* $[e_1\oplus 0_k]_{(M)}$ *coincides with* $N_F(n)$ *and if* $a\|v\|\neq 0$, *then the one of at* $[e_1a\oplus v]_{(M)}$ *coincides with* $L_F(n)$, *where* 0_n (*resp.* 0_k) *is the zero vector of* $\boldsymbol{F}^n$ (*resp.* $\boldsymbol{F}^k$).

3.4. Notice that the normalizer $N(L_F(n))$ of $L_F(n)$ acts on $F(M)$ naturally via χ_F^M, $N(L_F(n))$ coincides with $N_F(n)$ and the factor group $N(L_F(n))/L_F(n)=N_F(n)/L_F(n)$ is naturally isomorphic to the Lie group $\boldsymbol{F}_0$. The $\boldsymbol{F}_0$-action $\hat{\chi}_F^M:\boldsymbol{F}_0\times F(M)\longrightarrow F(M)$ induced naturally by χ_F^M is as follows:

$$\hat{\chi}_F^M(\lambda,\ [e_1a\oplus v]_{(M)})=[e_1\lambda a\oplus v]_{(M)}.$$

Here we shall show the following theorem.

THEOREM 3.3. *Suppose that* $M\in M_k(\boldsymbol{F})$ *is an arbitrary matrix whose all eigenvalues have positive real parts. Then the* **FTC**-*pair* $(\rho_n\oplus I_k,(M))$ *is* C^0-*equivalent to* $(\rho_n\oplus I_k,(I_k))$. *In other words, there exists an* $SL(n,\boldsymbol{F})$-*equivariant homeomorphism of* $P_{n+k-1}^{(M)}(\boldsymbol{F})$ *with a twisted linear action* χ_F^M *onto* $P_{n+k-1}(\boldsymbol{F})$ *with a linear action associated to* $\rho_n\oplus I_k$.

PROOF. By the above remark, we can construct uniquely an $\boldsymbol{F}_0$-equivariant homeomorphism f_0 of $F(I_k)$ onto $F(M)$ satisfying the following conditions

$$f_0([e_1a\oplus v])=[e_1a\oplus F_M(v)]_{(M)}\ for\ v\in\boldsymbol{F}_0^k$$

and

$$f_0([e_1\oplus 0_k])=[e_1\oplus 0_k]_{(M)},$$

where F_M is defined in **1.1** and 0_k is the zero vector of $\boldsymbol{F}^k$. Next we consider the diagram

$$\begin{array}{ccc} SU(n, \boldsymbol{F}) \times F(I_k) & \xrightarrow{\quad \psi \quad} & P_{n+k-1}(\boldsymbol{F}) \\ \Big\downarrow 1\times f_0 & & \vdots\, f \\ SU(n, \boldsymbol{F}) \times F(M) & \xrightarrow{\quad \psi_M \quad} & P^{(M)}_{n+k-1}(\boldsymbol{F}), \end{array}$$

where $SU(n, \boldsymbol{C})=SU(n)$, $SU(n, \boldsymbol{H})=Sp(n)$ and

$$\psi(K, [e_1a\oplus v])=[Ke_1a\oplus v],$$
$$\psi_M(K, [e_1a\oplus v]_{(M)})=[Ke_1a\oplus v]_{(M)}.$$

By the construction of f_0 and the diagram (1. 1), we see that $\psi(K, [e_1a\oplus v])=\psi(K', [e_1a'\oplus v'])$ if and only if $\psi_M(K, f_0([e_1a\oplus v]))=\psi_M(K', f_0([e_1a'\oplus v']))$ and hence we obtain unique bijection f of $P_{n+k-1}(\boldsymbol{F})$ onto $P^{(M)}_{n+k-1}(\boldsymbol{F})$ satisfying

$$f\circ\psi=\psi_M\circ(1\times f_0).$$

Then f is a homeomorphism, because ψ and ψ_M are closed continuous mappings. Finally, we show that f is $SL(n, \boldsymbol{F})$-equivariant. Let $A\in SL(n, \boldsymbol{F})$, $K\in SU(n, \boldsymbol{F})$ and $[e_1a\oplus v]\in F(I_k)$. Then, there are $B\in SU(n, \boldsymbol{F})$ and $U\in N_{\boldsymbol{F}}(n)$ such that $AK=BU$ and hence

$$\begin{aligned} f(\chi_{\boldsymbol{F}}(A, \psi(K, z))) &= f(\chi_{\boldsymbol{F}}(AK, z))=f(\chi_{\boldsymbol{F}}(BU, z)) \\ &= f(\psi(B, \chi_{\boldsymbol{F}}(U, z))) \\ &= \psi_M(B, f_0(\chi_{\boldsymbol{F}}(U, z))) \\ &= \psi_M(B, \chi^M_{\boldsymbol{F}}(U, f_0(z)))=\chi^M_{\boldsymbol{F}}(BU, f_0(z)) \\ &= \chi^M_{\boldsymbol{F}}(AK, f_0(z))=\chi^M_{\boldsymbol{F}}(A, \psi_M(K, f_0(z))) \\ &= \chi^M_{\boldsymbol{F}}(A, f(\psi(K, z))), \end{aligned}$$

where $\chi_{\boldsymbol{F}}=\chi^{I_k}_{\boldsymbol{F}}$ and $z=[e_1a\oplus v]\in F(I_k)$. Cosequently, we see that f is an $SL(n, \boldsymbol{F})$-equivariant homeomorphism of $P_{n+k-1}(\boldsymbol{F})$ with a linear action associated to $\rho_n\oplus I_k$ onto $P^{(M)}_{n+k-1}(\boldsymbol{F})$ with a twisted linear action $\chi^M_{\boldsymbol{F}}$. q. e. d.

3. 5. Denote by $F(M)_0$ an open subset of $F(M)$ consisting of $[e_1a\oplus v]_{(M)}$ with $a\in\boldsymbol{F}_0$ and define $\omega^M_{\boldsymbol{F}}: F(M)_0\longrightarrow\boldsymbol{F}^k$ by

$$\omega^M_{\boldsymbol{F}}([e_1a\oplus v]_{(M)})=\exp((-\log|a|)M)v\bar{a}/|a|=\alpha^M_{\boldsymbol{F}}(v, a^{-1}),$$

where $\alpha^M_{\boldsymbol{F}}$ is defined in **1. 1.** $\chi^M_{\boldsymbol{F}}$ induces naturally a real analytic left $\boldsymbol{F}_0$-action $\beta^M_{\boldsymbol{F}}$ on $\boldsymbol{F}^k$ by

$$\beta_F^M(\zeta, v) = \exp((-\log|\zeta|)M)v\bar{\zeta}/|\zeta| = \alpha_F^M(v, \zeta^{-1}).$$

Then ω_F^M is an equivariant C^ω-diffeomorphism satisfying $\omega_F^M([e_1 \oplus 0_k]_{(M)}) = 0_k$, where 0_k is the zero vector of $\boldsymbol{F}^k$. For any linear action of a Lie group on a Euclidean space, the tangential representation of its isotropy group at the origin of the Euclidean space is equivalent to itself. Hence we have the following lemma.

LEMMA 3.4. *For the $\boldsymbol{F}_0$-action χ_F^M on $F(M)$, the tangential representation of the isotropy group $\boldsymbol{F}_0$ on $T(F(M))_p$ is equivalent to β_F^M, where $p = [e_1 \oplus 0_k]_{(M)}$.*

Finally we shall show the following result.

THEOREM 3.5. *Let $M, N \in M_k(\boldsymbol{F})$ and suppose that all eigenvalues of M, N have positive real parts. If there exists an $SL(n, \boldsymbol{F})$-equivariant C^1-diffeomorphism f of $P_{n+k-1}^{(M)}(\boldsymbol{F})$ with a twisted linear action χ_F^M onto $P_{n+k-1}^{(N)}(\boldsymbol{F})$ with a twisted linear action χ_F^N, then*

$$\iota(N) = P\iota(M)P^{-1}$$

for some $P \in GL(d_F k, \boldsymbol{R})$, where $\iota : M_k(\boldsymbol{F}) \longrightarrow M_{d_F k}(\boldsymbol{R})$ is the ring homomorphism defined in Section 1.

PROOF. By the existence of such an equivariant diffeomorphism f, we obtain an $\boldsymbol{F}_0$-equivariant C^1-diffeomorphism $f_0 : F(M) \longrightarrow F(N)$. Considering points whose isotropy groups coincide with $\boldsymbol{F}_0$, we can assume $f_0([e_1 \oplus 0_k]_{(M)}) = [e_1 \oplus 0_k]_{(N)}$. Then we obtain an $\boldsymbol{F}_0$-equivariant $\boldsymbol{R}$-linear isomorphism

$$(df_0)_p : T(F(M))_p \longrightarrow T(F(N))_q$$

of tangential representation spaces of the isotropy group $\boldsymbol{F}_0$, where $p = [e_1 \oplus 0_k]_{(M)}$ and $q = [e_1 \oplus 0_k]_{(N)}$. $(df_0)_p$ induces an $\boldsymbol{F}_0$-equivariant $\boldsymbol{R}$-linear automorphism ϕ of $\boldsymbol{F}^k$ such that the diagram

$$\begin{array}{ccc}
T(F(M))_p & \xrightarrow{(df_0)_p} & T(F(N))_q \\
\Big\downarrow{\scriptstyle (d\omega_F^M)_p} & & \Big\downarrow{\scriptstyle (d\omega_F^N)_q} \\
\boldsymbol{F}^k & \xrightarrow{\ \ \phi\ \ } & \boldsymbol{F}^k
\end{array}$$

is commutative. Since ϕ is $\boldsymbol{F}_0$-equivariant, we have

$$\phi\circ\beta_{\boldsymbol{F}}^{M}=\beta_{\boldsymbol{F}}^{N}\circ(1\times\phi).$$

Hence we have $\phi(\exp(-tM)\phi^{-1}(u))=\exp(-tN)u$ for each $t\in\boldsymbol{R}$, $u\in\boldsymbol{F}^{k}$. It follows from this fact that there exists $P\in GL(d_{F}k, \boldsymbol{R})$ such that $P\exp(-t\iota(M))P^{-1}=\exp(-t\iota(N))$ for each $t\in\boldsymbol{R}$. Therefore we have $\iota(N)=P\iota(M)P^{-1}$ for some $P\in GL(d_{F}k, \boldsymbol{R})$. q. e. d.

References

[1] J. F. ADAMS, Lectures on Lie Groups, Benjamin Inc., 1969.

[2] T. BRÖCKER and T. tom Dieck, Representations of Compact Lie Groups, Springer-Verlag, 1985.

[3] C. CHEVALLEY, Theory of Lie Groups I, Princeton Univ. Press, 1946.

[4] F. UCHIDA, On a method to construct analytic actions of non-compact Lie groups on a sphere, Tohoku Math. J. 39 (1987), 61-69.

[5] F. UCHIDA, Certain aspects of twisted linear actions, Osaka J. Math. 25 (1988), 343-352.

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