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The generalized Burnside ring of a finite group*

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1 Introduction

Let G be a finite group and let $\mathfrak{X}$ be a family of subgroups of G closed under G-conjugation. We consider the free abelian group $\Omega(G, \mathfrak{X})$ generated by the isomorphism classes of transitive G-sets of the form $[G/S]$, where$S\in\mathfrak{X}$. This group $\Omega(G, \mathfrak{X})$ is a subgroup of the ordinary Burnside ring $\Omega(G)$ of the category of finite G-sets and G-maps. The purpose of this paper is to study when $\Omega(G, \mathfrak{X})$ has a ring structure called a generalized Burnside ring. It is easily checked that if $\mathfrak{X}$ is closed under intersection, then $\Omega(G, \mathfrak{X})$ is a subring of the Burnside ring $\Omega(G)$. But the structure of the generalized Burnside ring can be introduced under some weaker conditions.

As in the case of usual Burnside rings, there is a homomorphism

$$\varphi_S:\Omega(G, \mathfrak{X})\rightarrow \boldsymbol{Z}:[X]\mapsto |X^S|,$$

where X^S denotes the set of S-fixed points in a $(G, \mathfrak{X})$-set X, that is, a G-set in which the stabilizer of each point belongs to $\mathfrak{X}$. Thus taking the direct product on the set $C(\mathfrak{X})$ of the conjugacy classes (S) of subgroups S in $\mathfrak{X}$, we have an additive homomorphism

$$\varphi:=(\varphi_S)_{(S)}:\Omega(G, \mathfrak{X})\rightarrow\tilde{\Omega}(G, \mathfrak{X}):=\prod_{(S)\in C(\mathfrak{X})}\boldsymbol{Z}.$$

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In the case of the usual Burnside ring $\Omega(G)$, this homomorphism is a ring homomorphism and is called a Burnside homomorphism or mark homomorphism. However $\Omega(G, \mathfrak{X})$ has not yet possessed a ring structure, and so φ has not been a ring homomorphism. We want a ring structure on $\Omega(G, \mathfrak{X})$ for which φ is an injective ring homomorphism into the ring $\tilde{\Omega}(G, \mathfrak{X})$. When $\Omega(G, \mathfrak{X})$ has such a ring structure, we call this ring a generalized Burnside ring.

We use the notation $\bar{H} \leq G$ for any subgroup H defined by

$$\bar{H} := \cap\{S \in \mathfrak{X} \mid H \subseteq S\}.$$

Then the following theorem holds. For the proof, see Theorem 3.11.

THEOREM A. *Assume that the family satisfies the following condition*:

$(\mathrm{C})_\infty \qquad S \in \mathfrak{X},\ g \in WS \Longrightarrow \overline{\langle g \rangle S} \in \mathfrak{X}$

Then $\Omega(G, \mathfrak{X})$ is a generalized Burnside ring.

Such rings are not subrings of the usual Burnside ring $\Omega(G)$ in general, but many properties which the usual Burnside rings satisfy hold also for generalized Burnside rings.

First of all, the fundamental theorem holds. (The proof will be given in Section 3 Theorem 3.10.)

THEOREM B. *Under the condition $(\mathrm{C})_\infty$, there is an exact sequence of abelian groups as follows*:

$$0 \longrightarrow \Omega(G, \mathfrak{X}) \xrightarrow{\varphi} \tilde{\Omega}(G, \mathfrak{X}) \xrightarrow{\psi} \mathrm{Obs}(G, \mathfrak{X}) \longrightarrow 0.$$

Here similarly to usual Burnside rings, we define

$$\mathrm{Obs}(G, \mathfrak{X}) := \prod_{(S) \in C(\mathfrak{X})} (\boldsymbol{Z}/|WS|\boldsymbol{Z}),$$

where $WS := N_G(S)/S$, and furthermore, ψ is defined by

$$\begin{aligned} \psi : \tilde{\Omega}(G, \mathfrak{X}) &\longrightarrow \mathrm{Obs}(G, \mathfrak{X}) \\ : (\chi(S))_{(S)} &\mapsto \Big(\sum_{gS \in WS} \chi(\overline{\langle g \rangle S}) \bmod |WS|\Big)_{(S)}. \end{aligned}$$

This map ψ is called a *Cauchy-Frobenius homomorphism* because the Cauchy-Frobenius lemma proves the fact that $\psi\varphi = 0$.

This theorem is essential in the study of generalized Burnside rings. In Section 4, as an application of the fundamental theorem, we will show that primitive idempotents in $\Omega(G, \mathfrak{X})_{(p)}$, the generalized Burnside ring localized at a prime p, are bijectively corresponding to equivalence classes of an equivalence relation $\sim_p$.

About prime ideals, some similar facts as in the case of usual Burnside rings hold. See Section 5.

In Section 6, we study functorial properties of generalized Burnside rings. That is, we can construct restriction maps, induction maps, inflation maps, fixed-point maps, etc. Especially, restrictions and inductions satisfy Mackey decomposition and Frobenius reciprocity, and so the map $H \mapsto \Omega(H, \mathfrak{X}_H)$ makes a so-called G-functor.

In Section 7, we give an application to the classical theory of representation of symmetric groups. The purpose of this section is to give an elementary and probably new proof for the fact that any ordinary character of a symmetric group can be written as a linear combination of permutation characters induced from Young subgroups.

There are some congruences in finite group theory that can be proved by the theory of Burnside rings, for example, Sylow's third theorem, Frobenius' theorem about the number of solutions of the equation $g^n=1$ on a finite group and Brown's theorem (cf. Example 8.1) about the Euler characteristic of nontrivial p-subgroups. See [Wa 70], [Gl 81], [DSY 90], [DY 90]. Using the same way to generalized Burnside rings, we obtain corresponding results, which are far generalized than the ordinary results, for a family of subgroups. For example, by observing the coefficients of standard basis in the identity element of a generalized Burnside ring, we can easily prove the following result:

THEOREM C. *Assume that for a prime p, $\mathfrak{X}$ satisfies the condition* $(\mathrm{C})_p$

$$S \in \mathfrak{X},\ g \in WS \text{ is a } p\text{-element} \Longrightarrow \overline{\langle g \rangle S} \in \mathfrak{X}.$$

Then for any $S \in \mathfrak{X}$,

$$\sum_{T \in \mathfrak{X}} \mu_{\mathfrak{X}}(S, T) \equiv 0 \qquad (\mathrm{mod}\ |WS|_p),$$

where $\mu_{\mathfrak{X}}$ is the Möbius function of the poset $\mathfrak{X}$ with the order relation by inclusion.

The proof will be given in Theorem 8.12 in Section 8. The above theorem implies some congruences (e.g. Corollary 2.2) in Brown-Thévenaz paper [BT 88]. To prove their theorem by generalized Burnside rings instead of Crapo complementation formula is an interesting problem.

In Section 9, we study Mackey functors and their representation category — the generalized Hecke category with coefficient in a generalized Burnside ring functor.

A generalized Burnside ring with respect to a family $\mathfrak{X}$ of subgroups of G is a typical example of the notion of *abstract Burnside rings* introduced in [Yo 87a]. In Appendix, we state a brief outline of the theory of abstract Burnside rings together with the theory of generalized Burnside rings.

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2 The Burnside ring of a finite group

In this section, we give a brief outline of the theory of the Burnside ring of a finite group. The details and further results (prime ideals, transfer theorems, Dress induction theorem, etc.) are found in [Ar 82], [Di 79], [Dr 71a], [Yo 83a], [Yo 90], etc.

Throughout this paper, G denotes a finite group and p denotes a prime (or 0, ∞ sometimes).

2.1 The set of G-isomorphism classes of finite left G-sets makes a commutative semi-ring with respect o disjoint union $X+Y$ and cartesian product $X\times Y$. Its Grothendieck ring is called the *Burnside ring* of G and is denoted by $\Omega(G)$, and so $\Omega(G)$ is an abelian group generated by the isomorphism classes $[X]$ of finite G-sets with relation $[X+Y]=[X]+[Y]$. A finite G-set is G-isomorphic to a disjoint union of homogeneous G-sets G/H. Furthermore, two G-sets G/H and G/K are G-isomorphic if and only if H and K are G-conjugate. Thus the Burnside ring $\Omega(G)$ is additively the free abelian group on the set $\{[G/H]\mid(H)\in C(G)\}$, where $C(G)$ is the set of the G-conjugacy classes (H) of subgroups H of G. Using this standard basis, the multiplication is given by

$$[G/H]\cdot[G/K]=\sum_{KgH\in K\backslash G/H}[G/gHg^{-1}\cap K]. \tag{1}$$

2.2 For any G-set X and a subgroup S of G, let X^S be the set of all S-fixed points. Then for a homogeneous G-set G/H,

$$(G/H)^S=\{gH\in G/H\mid S\subseteq {}^gH\}, \tag{2}$$

where

$${}^gH:=gHg^{-1}.$$

In particular,

$$(G/H)^S \neq \emptyset \iff S \leq_G H, \tag{3}$$

where $S \leq_G H$ means that H contains a G-conjugate of S.

For a subgroup S of G, we use the following symbols for the normalizer and the Weyl group:

$$NS := N_G(S),\ \ WS := W_G S := NS/S.$$

By (2), there are bijections among $(G/H)^H$, WH and the set of G-automorphisms of G/H:

$$(G/S)^S = WS \cong \mathrm{Aut}_G(G/S). \tag{4}$$

We can extend the map $X \mapsto |X^S|$ into a ring homomorphism

$$\varphi_S := \Omega(G) \longrightarrow \boldsymbol{Z} : [X] \mapsto |X^S|.$$

Using (2), we have that

$$\varphi_S([G/H]) = \frac{1}{|H|} \sum_{g \in G} \zeta(S, {}^g H), \tag{5}$$

where $\zeta(S, T) := 1$ if $S \leq T$, $:= 0$ otherwise.

The *ghost ring* is the direct product of some copies of the integer ring:

$$\tilde{\Omega}(G) := \prod_{(S) \in C(G)} \boldsymbol{Z}.$$

Then we have a ring homomorphism called a *Burnside homomorphism* as follows:

$$\varphi = (\varphi_S) : \Omega(G) \longrightarrow \tilde{\Omega}(G) : x \mapsto (\varphi_S(x)),$$

2.3 LEMMA. *The Burnside homomorphism $\varphi : \Omega(G) \longrightarrow \tilde{\Omega}(G)$ is an injective ring homomorphism and*

$$\mathrm{Coker}\ \varphi \cong \prod_{(S) \in C(G)} (\boldsymbol{Z}/|WS|\boldsymbol{Z}).$$

In particular, φ gives a ring isomorphism

$$1 \otimes \varphi : \boldsymbol{Q} \otimes_{\boldsymbol{Z}} \Omega(G) \xrightarrow{\cong} \boldsymbol{Q} \otimes_{\boldsymbol{Z}} \tilde{\Omega}(G). \tag{6}$$

This lemma was proved by Burnside. Refer to [Dr 71a], [Di 79] for the proof.

2.4 By the above lemma, we can regard $\Omega(G)$ a subring of $\tilde{\Omega}(G)$. So we often write

$$x(S) := \varphi_S(x) \ \text{ for } x \in \Omega(G),\ S \leq G. \tag{7}$$

Furthermore, the cokernel of φ is called an *obstruction group* and is written as

$$\mathrm{Obs}(G):=\prod_{(S)\in C(G)}(\boldsymbol{Z}/|WS|\boldsymbol{Z}).$$

2.5 For a prime p, let $\boldsymbol{Z}_{(p)}$ be the localization of $\boldsymbol{Z}$ at p:

$$\boldsymbol{Z}_{(p)}:=\{a/b|a\in\boldsymbol{Z},\ b\in\boldsymbol{Z}-p\boldsymbol{Z}\}\subseteq\boldsymbol{Q}$$

Put

$$\Omega(G)_{(p)}:=\boldsymbol{Z}_{(p)}\otimes_{\boldsymbol{Z}}\Omega(G),\ \tilde{\Omega}(G)_{(p)}:=\boldsymbol{Z}_{(p)}\otimes_{\boldsymbol{Z}}\tilde{\Omega}(G),$$
$$\mathrm{Obs}(G)_{(p)}:=\prod_{(S)\in C(G)}(\boldsymbol{Z}/|WS|_p\boldsymbol{Z})\cong\boldsymbol{Z}_{(p)}\otimes\mathrm{Obs}(G).$$

Then we can view $\Omega(G)_{(p)}$ (resp. $\tilde{\Omega}(G)_{(p)}$) as a subring of $\boldsymbol{Q}\otimes\Omega(G)$ (resp. $\boldsymbol{Q}\otimes\tilde{\Omega}(G)$). The Burnside homomorphism φ induces

$$\varphi^{(p)}:\Omega(G)_{(p)}\longrightarrow\tilde{\Omega}(G)_{(p)},$$

which has a cokernel isomorphic to $\mathrm{Obs}(G)_{(p)}$.

2.6 Furthermore, if there is no confusion, it is convenient to extend the above notation to $p=\infty$ and $p=0$:

$$\begin{array}{llll}
\Omega(G)_{(\infty)} := \Omega(G), & \tilde{\Omega}(G)_{(\infty)} := \tilde{\Omega}(G), \\
\mathrm{Obs}(G)_{(\infty)} := \mathrm{Obs}(G), & \varphi^{(\infty)} := \varphi, \\
\Omega(G)_{(0)} := \boldsymbol{Q}\otimes\Omega(G), & \tilde{\Omega}(G)_{(0)} := \boldsymbol{Q}\otimes\tilde{\Omega}(G), \\
\mathrm{Obs}(G)_{(0)} := 0, & \varphi^{(0)} := 1_{\boldsymbol{Q}}\otimes\varphi.
\end{array}$$

2.7 LEMMA (Cauchy-Frobenius). *Let X be a finite G-set. Then*

$$\sum_{g\in G}|X^{(g)}|=|G|\cdot|G\backslash X|\equiv 0 \pmod{|G|},$$

where $G\backslash X$ is the set of G-orbits in X.

2.8 We define the *Cauchy-Frobenius homomorphism* by

$$\psi^{(p)}:\tilde{\Omega}(G)_{(p)}\longrightarrow\mathrm{Obs}(G)_{(p)}$$
$$:\chi\mapsto\Big(\sum_{gS\in(WS)_p}\chi(\langle g\rangle S)\ \mathrm{mod}\ |WS|_p\Big), \tag{8}$$

where $(WS)_p$ is a Sylow p-subgroup of WS. We simply write

$$\psi:=\psi^{(\infty)}:\tilde{\Omega}(G)\longrightarrow\mathrm{Obs}(G)$$
$$:\chi\mapsto\Big(\sum_{gS\in WS}\chi(\langle g\rangle S)\ \mathrm{mod}\ |WS|\Big). \tag{9}$$

Here we interpret as

$$(WS)_\infty=WS,\ |WS|_\infty=|WS| \tag{10}$$

2.9 PROPOSITION (Fundamental Theorem for Burnside Rings). *The following sequence of abelian groups is exact*:

$$0 \longrightarrow \Omega(G)_{(p)} \xrightarrow{\varphi^{(p)}} \tilde{\Omega}(G)_{(p)} \xrightarrow{\psi^{(p)}} \mathrm{Obs}(G)_{(p)} \longrightarrow 0,$$

This proposition was essentially found by A. Dress in his unpublished work at first. The proof is found in [Dr 86], [Di 79, Chapter 1], [Yo 90].

2.10 REMARK. A. Drees ([DY 90]) pointed out that $\psi^{(p)} \neq 1 \otimes \psi$ for a prime p in general and this difference implies a Frobenius type congruence

$$\#\{p\text{-element of } G\} \equiv 0 \pmod{|G|_p}$$

which is a weak form of the Frobenius theorem on the number of solutions of the equation $g^n = 1$ on G. In fact, let $\chi \in \tilde{\Omega}(G)$ be an element of $\tilde{\Omega}(G)$ defined by

$$\chi(S) := \begin{cases} |G|_{p'} & \text{if } S \textit{ is a } p\text{-subgroup} \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

Then by the definition of the Cauchy-Frobenius homomorphism, we have that $\psi^{(p)}(\chi) = 0$ and $\psi^{(q)}(\chi) = 0$, where q is a prime distinct to p. Thus the fundamental theorem implies that

$$\chi \in \Omega(G)_{(p)} \cap \bigcap_{q \neq q} \Omega(G)_{(q)} = \Omega(G).$$

Applying the Cauchy-Frobenius homomorphism ψ to χ, we have that

$$\phi(\chi)_1 = \#\{p\text{-element of } G\} \equiv 0 \pmod{|G|_p},$$

as required.

2.11 By the isomorphism $1 \otimes \varphi : \boldsymbol{Q} \otimes \Omega(G) \cong \boldsymbol{Q} \otimes \tilde{\Omega}(G) = \prod_{(s)} \boldsymbol{Q}$ (cf. (6)), there is an element $e_H \in \boldsymbol{Q} \otimes \Omega(G)$ for each $H \leq G$ such that

$$\varphi_S(e_H) = \begin{cases} 1 & \text{if } (S) = (H) \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

Clearly, $\{e_H \mid (H) \in C(G)\}$ is the set of primitive idempotents of $\boldsymbol{Q} \otimes \Omega(G)$. About idempotents of Burnside rings of finite groups, refer to [Gl 81], [Yo 83a].

2.12 In order to present the primitive idempotent e_H by the standard basis, we need the Möbius function of the poset of subgroups of G. In general, the *Möbius function* $\mu_P : P \times P \longrightarrow \boldsymbol{Z}$ of a finite poset is defined inductively as follows:

$\mu_P(x, x)=1$; $\mu_P(x, y)=0$ if $x \not\leq y$;
$\sum_{t \leq y} \mu_P(x, t)=0$ if $x<y$.

Let μ denote the Möbius function of the poset of subgroups of G.

2.13 LEMMA. $e_H=\frac{1}{|NS|}\sum_{D \leq H}|D|\mu(D, H)[G/D]$.

2.14 For a finite group S, we denote by S^p the smallest normal subgroup of S with S/S^p a p-group. The group S is called *p-perfect* if $S^p=S$. So S is p-perfect if and only if S has no proper normal subgroup of index p.

For $p=\infty$, let S^∞ denote the last term of the derived sequence of S, so that ∞-perfectness stands for the usual perfectness.

2.15 Let Q be a p-perfect subgroup of G. Define an idempotent e_Q^p of $\boldsymbol{Q}\otimes\Omega(G)$ by

$$e_Q^p:=\sum_{\substack{(H)\in C(G)\\ :(H^p)=(Q)}} e_H,$$

where the summation is taken over $(H)\in C(G)$ such that H^p is G-conjugate to Q. By (12), e_Q^p has the following value at a subgroup S:

$$\varphi_S(e_Q^p)=\begin{cases}1 & \text{if } (S^p)=(Q)\\ 0 & \text{otherwise.}\end{cases} \tag{13}$$

2.16 LEMMA. *For a p-perfect subgroup Q of G, the idempotent e_Q^p of $\boldsymbol{Q}\otimes\Omega(G)$ belongs to $\Omega(G)_{(p)}$. Conversely, any idempotent of $\Omega(G)_{(p)}$ has the form e_Q^p for a p-perfect subgroup Q.*

2.17 LEMMA. *Let $\chi\in\boldsymbol{Q}\otimes\tilde{\Omega}(G)$. Then*

$$\chi=\sum_{(D)\in C(G)}\frac{1}{|WD|}\left(\sum_{H\leq G}\mu(D, H)\chi(H)\right)\ [G/D].$$

This follows immediately from the idempotent formula (Lemma 2.13). Note that $\chi\cdot e_H=\chi(H)\cdot e_H$.

2.18 COROLLARY. *For a p-perfect subgroup Q,*

$$e_Q^p:=\frac{1}{|NQ|}\sum_{\substack{H\leq NQ\\ :H^p=Q}}\sum_{D\leq H}|D|\mu(D, H)[G/D],$$

2.19 LEMMA. *Fet n be a divisor of $|G|$. Let χ_n be an element of $\tilde{\Omega}(G)$ defined by*

$$\chi_n(S):=\begin{cases}|G|/n & \text{if } |S| \text{ divides } n\\ 0 & \text{otherwise.}\end{cases}$$

Then χ_n *is in* $\Omega(G)$.

This result was first proved by Wagner [Wa 70]. The element χ_n is called a *Frobenius element.* Another short proof of this lemma is found in Dress-Siebeneicher-Yoshida [DSY 90].

2.20 Applying the fundamental theorem (Proposition 2.9 and Lemma 2.17 to the Frobenius element χ_n, we have some congruences, for example, Frobenius theorem as follows:

COROLLARY (Frobenius). *If* n *is a divisor of* $|G|$, *then*

$$\#\{g\in G | g^n=1\}\equiv 0 \pmod{n}.$$

Refer to Dress-Siebeneicher-Yoshida [DSY 90].

2 The generalized Burnside ring and the fundamental theorem

3.1 Let $\mathfrak{X}$ be a family of subgroups of a finite group G such that if $H \in\mathfrak{X}$, then ${}^gH := gHg^{-1}\in\mathfrak{X}$ for any $g\in G$. Throughout this paper, $\mathfrak{X}$ denotes such a family. Let $\Omega(G, \mathfrak{X})$ be the subgroup of $\Omega(G)$ generated by elements $[G/H]$ for $H\in\mathfrak{X}$. Then $\Omega(G, \mathfrak{X})$ is a free abelian group with basis

$$\{[G/H] | (H)\in C(G),\ H\in\mathfrak{X}\}.$$

A $(G, \mathfrak{X})$-set X is a finite G-set in which the stabilizer of every element belongs to $\mathfrak{X}$. Thus $\Omega(G, \mathfrak{X})$ is the Grothendieck group of the category of $(G, \mathfrak{X})$-sets and G-maps. Note that $\Omega(G, \mathfrak{X})$ is a subring of $\Omega(G)$ if and only if $\mathfrak{X}$ is closed under intersection and $G\in\mathfrak{X}$.

3.2 Let $C(\mathfrak{X})$ be the set of G-conjugacy classes of subgroups belonging to $\mathfrak{X}$. A *ghost ring* $\tilde{\Omega}(G, \mathfrak{X})$ is the direct product of $|C(\mathfrak{X})|$ copies of the integer ring $\boldsymbol{Z}$.

$$\tilde{\Omega}(G, \mathfrak{X}) := \prod_{(S)\in C(\mathfrak{X})} \boldsymbol{Z}.$$

For any $S\in\mathfrak{X}$, let

$$\varphi_S : \Omega(G, \mathfrak{X})\longrightarrow \boldsymbol{Z} : [X] \mapsto |X^S| \tag{1}$$

be the restriction of $\varphi_S : \Omega(G)\longrightarrow \boldsymbol{Z}$ into $\Omega(G, \mathfrak{X})$. Thus we have an additive homomorphism called the *Burnside homomorphism* with respect to $\mathfrak{X}$

$$\varphi := (\varphi_S)_{(S)} : \Omega(G, \mathfrak{X})\longrightarrow\tilde{\Omega}(G, \mathfrak{X}) : x \mapsto (\varphi_S(x)).$$

3.3 LEMMA. *The Burnside homomorphism* $\varphi : \Omega(G, \mathfrak{X})\longrightarrow\tilde{\Omega}(G, \mathfrak{X})$ *is* *an injective additive homomorphism with a cokernel*

$$\operatorname{Coker}\varphi\cong\prod_{(S)\in C(\mathfrak{X})}(\boldsymbol{Z}/|WS|\boldsymbol{Z}).$$

PROOF. The matrix corresponding to $\varphi:\Omega(G,\mathfrak{X})\longrightarrow\tilde{\Omega}(G,\mathfrak{X})$ is a square matrix and under some suitable rearrangement of $C(\mathfrak{X})$, this matrix becomes a triangular matrix of which diagonal constituents $\{|WS|\mid S\in\mathfrak{X}\}$. See (2), (3), (4) of Section 2. This proves the lemma. □

3.4 COROLLARY. *$\boldsymbol{Q}\otimes_{Z}\Omega(G,\mathfrak{X})$ has a unique ring structure isomorphic to $Q\otimes\tilde{\Omega}(G,\mathfrak{X})$ via the Burnside homomorphism $1\otimes\varphi:\boldsymbol{Q}\otimes\Omega(G,\mathfrak{X})\xrightarrow{\cong}Q\otimes\tilde{\Omega}(G,\mathfrak{X})$.*

PROOF. By the lemma, $1\otimes\varphi$ is an isomorphism of $\boldsymbol{Q}$-vector spaces. □

3.5 REMARK. For any element x of $\Omega(G,\mathfrak{X})$ (or $\boldsymbol{Q}\otimes\Omega(G,\mathfrak{X})$) and any S in $\mathfrak{X}$, we often write

$$x(S):=\varphi_S(x). \tag{2}$$

Then the above corollary means that elements of $\Omega(G,\mathfrak{X})$ are determined by their values at subgroups in $\mathfrak{X}$, that is, for $x,\ y\in\Omega(G,\mathfrak{X})$,

$$x=y\Longleftrightarrow x(S)=y(S)\ \text{for all}\ S\in\mathfrak{X}.$$

3.6 **Condition** $(\mathrm{C})_p$ Let p be a prime or ∞. For a subgroup H of G, let

$$\bar{H}:=\cap\{S\in\mathfrak{X}|H\subseteq S\}. \tag{3}$$

(We put $\bar{H}:=G$ if there is no element $U\in\mathfrak{X}$ containing H.) We consider the following condition:

$$(\mathrm{C})_p \qquad gS\in(WS)_p,\ S\in\mathfrak{X}\implies\overline{\langle g\rangle S}\in\mathfrak{X}.$$

where $(WS)_p$ is a Sylow p-subgroup of WS. For $p=\infty$, we interpret $(\mathrm{C})_p$ as follows:

$$(\mathrm{C})_\infty \qquad gS\in WS,\ S\in\mathfrak{X}\implies\overline{\langle g\rangle S}\in\mathfrak{X}.$$

Clearly, for a prime q such that WS is a q'-group for every $S\in\mathfrak{X}$, in particular, for a prime q which does not divide $|G|$, the above condition $(\mathrm{C})_q$ is valid.

3.7 LEMMA. (a) *Assume that the condition $(\mathrm{C})_p$ holds for a prime p. Then*

$$S\in\mathfrak{X},\ P/S\ \textit{is a } p\textit{-subgroup of}\ WS\implies\bar{P}\in\mathfrak{X}.$$

(b) *Assume that the condition* $(\mathrm{C})_\infty$ *holds. Then*

$$S\in\mathfrak{X},\ H/S \text{ is a solvable subgroup of } WS \Longrightarrow \bar{H}\in\mathfrak{X}.$$

(c) $$(\mathrm{C})_\infty \Longleftrightarrow (\mathrm{C})_p \text{ for all prime } p.$$

PROOF. First we claim that under the condition $(\mathrm{C})_p$,

$$N\trianglelefteq P\leq G,\ \bar{N}\in\mathfrak{X},\ P/N \text{ is a cyclic } p\text{-group} \Longrightarrow \bar{P}\in\mathfrak{X}. \tag{4}$$

In fact, let g be an element with $P=\langle g\rangle N$. Then g normalizes $\bar{N}$, and so by the assumption $(\mathrm{C})_p$,

$$P=\langle g\rangle N\subseteq\overline{\langle g\rangle\bar{N}}\in\mathfrak{X}.$$

On the other hand, if $P\subseteq T\in\mathfrak{X}$, then T contains $\bar{N}$ and also $\langle g\rangle\bar{N}$, whence T contains furthermore $\overline{\langle g\rangle\bar{N}}$. Thus the above claim (4) holds.

We will prove (a) by induction on $|P|$. Let N be a normal subgroup of P such that N contains S and P/N is cyclic of prime order. Then by the assumption of induction, we have that $\bar{N}\in\mathfrak{X}$. Thus (4) implies that $\bar{P}\in\mathfrak{X}$, proving (a). If we interpret a "∞-group" as a solvable group, the proof of (a) gives the proof of (b). Finally we will prove (c). Assume that the condition $(\mathrm{C})_p$ holds for all prime p. Let $S\in\mathfrak{X}$ and let C/S be a cyclic subgroup of WS. We will prove that $\bar{C}\in\mathfrak{X}$ by induction on $|C|$. Let N/S be a normal subgroup of C/S of prime index p. Then by the induction assumption, we have that $\bar{N}\in\mathfrak{X}$. Thus by the condition $(\mathrm{C})_p$ and the above claim (4), we have that $\bar{C}\in\mathfrak{X}$, proving (c). □

3.8 LEMMA. *Let x be an element of $\Omega(G,\mathfrak{X})$ and H a subgroup of G such that $\bar{H}\in\mathfrak{X}$. Then*

$$x(H)=x(\bar{H}),$$

where $x(H)$ denotes the image of $x\in\Omega(G,\mathfrak{X})\subseteq\Omega(G)$ by $\varphi_H:\Omega(G)\longrightarrow\boldsymbol{Z}$.

PROOF. We may assume that $x=[G/T]$, $T\in\mathfrak{X}$. Then by (2) in Section 2,

$$\begin{aligned} x(H)&=\#\{gT\in G/T\mid H\subseteq{}^gT\}\\ &=\#\{gT\in G/T\mid \bar{H}\subseteq{}^gT\}\\ &=x(\bar{H}), \end{aligned}$$

proving the lemma. □

3.9 We define the *obstruction groups* by

$$\mathrm{Obs}(G,\mathfrak{X}):=\prod_{(S)\in C(\mathfrak{X})}(\boldsymbol{Z}/|WS|\boldsymbol{Z}),$$

$$\mathrm{Obs}(G, \mathfrak{X})_{(p)} := \prod_{(S)\in C(\mathfrak{X})} (\boldsymbol{Z}/|WS|_p\boldsymbol{Z}).$$

Furthermore we define *Cauchy-Frobenius homomorphisms* by

$$\begin{aligned}\psi=\psi^{(\infty)}:\ \tilde{\Omega}(G, \mathfrak{X}) &\longrightarrow \qquad \mathrm{Obs}(G, \mathfrak{X}) \\ :\ (\chi(S))_{(S)} &\longmapsto \Big(\sum_{gS\in WS}\chi(\overline{\langle g\rangle S}) \bmod |WS|\Big)_{(S)},\end{aligned}\tag{5}$$

$$\begin{aligned}\psi^{(p)}:\ \tilde{\Omega}(G, \mathfrak{X})_{(p)} &\longrightarrow \qquad \mathrm{Obs}(G, \mathfrak{X})_{(p)} \\ :\ (\chi(S))_{(S)} &\longmapsto \Big(\sum_{gS\in (WS)_p}\chi(\overline{\langle g\rangle S}) \bmod |WS|_p\Big)_{(S)}\end{aligned}\tag{6}$$

where as before, $(WS)_p$ is a Sylow p-subgroup of WS.

3.10 THEOREM (Fundamental theorem). *Let p be a prime or ∞. Then under the condition* $(\mathrm{C})_p$, *the following sequence is exact*:

$$0\longrightarrow\Omega(G, \mathfrak{X})_{(p)}\xrightarrow{\varphi^{(p)}}\tilde{\Omega}(G, \mathfrak{X})_{(p)}\xrightarrow{\psi^{(p)}}\mathrm{Obs}(G, \mathfrak{X})_{(p)}\longrightarrow 0\tag{7}$$

In particular, under $(\mathrm{C})_\infty$, *the following sequence is exact*:

$$0\longrightarrow\Omega(G, \mathfrak{X})\xrightarrow{\varphi}\tilde{\Omega}(G, \mathfrak{X})\xrightarrow{\psi}\mathrm{Obs}(G, \mathfrak{X})\longrightarrow 0\tag{8}$$

PROOF. First of all, it follows from Lemma 3.3 that $\varphi^{(p)}=1\otimes\varphi$ is injective and its cokernel is isomorphic to $\mathrm{Obs}(G, \mathfrak{X})_{(p)}\cong \boldsymbol{Z}_{(p)}\otimes\mathrm{Obs}(G, \mathfrak{X})$. Furthermore, $\psi^{(p)}$ is surjective because the matrix of $\psi^{(p)}$ is a unipotent triangular matrix. Thus $\mathrm{Ker}\,\psi^{(p)}$ and $\mathrm{Im}\,\varphi^{(p)}$ have the same index $\prod_{(S)\in C(\mathfrak{X})}|WS|_p$, whence it will suffice to show that $\psi^{(p)}\varphi^{(p)}=0$. Let $x\in\Omega(G, \mathfrak{X})$. Then for any $S\in\mathfrak{X}$ and $gS\in(WS)_p$,

$$x(\overline{\langle g\rangle S})=x(\langle g\rangle S).$$

by the condition $(\mathrm{C})_p$ and Lemma 3.8. Now by the Cauchy-Frobenius lemma and this identity, the S-component of $\psi^{(p)}\varphi^{(p)}(x)$ is

$$\begin{aligned}\sum_{gS\in(WS)_p} x(\overline{\langle g\rangle S}) &= \sum_{gS\in(WS)_p} x(\langle g\rangle S) \\ &\equiv 0 \pmod{|WS|_p}.\end{aligned}$$

Thus $\psi^{(p)}\varphi^{(p)}=0$, and so the sequence (7) is exact. The exactness of the sequence (8) is proved by the same way. □

3.11 THEOREM. (a) *Under the condition* $(\mathrm{C})_p$, $\Omega(G, \mathfrak{X})_{(p)}$ *has a unique ring structure such that* $\varphi^{(p)}:\Omega(G, \mathfrak{X})_{(p)}\longrightarrow\tilde{\Omega}(G, \mathfrak{X})_{(p)}$ *is a ring homomorphism.*

(b) *In particular, under the condition* $(\mathrm{C})_\infty$, $\Omega(G, \mathfrak{X})$ *has a unique ring structure such that* $\varphi:\Omega(G, \mathfrak{X})\longrightarrow\tilde{\Omega}(G, \mathfrak{X})$ *is a ring homomorphism. Furthermore, for a prime p, the two ring structures on* $\boldsymbol{Z}_{(p)}\otimes\Omega(G, \mathfrak{X})=\Omega(G, \mathfrak{X})_{(p)}$ *defined by* (a) *and* (b) *coincide.*

PROOF. In order to prove that the existence of a ring structure on $\Omega(G, \mathfrak{X})_{(p)}$, it will suffice to show that $\mathrm{Im}\,\varphi^{(p)}$ is a subring of $\tilde{\Omega}(G, \mathfrak{X})_{(p)}$, because $\varphi^{(p)}$ is injective. Take $x=[X]$, $y=[Y]$, where X and Y are $(G, \mathfrak{X})$-sets. Then by Lemma 3.8, for any subgroup H of G with $\bar{H}\in\mathfrak{X}$,

$$\varphi(x)(H)=x(\bar{H})=|X^H|.$$

Thus

$$\begin{aligned}\psi^{(p)}(\varphi^{(p)}(x)\cdot\varphi^{(p)}(y))_S &= \sum_{gS\in(WS)_p}(\varphi^{(p)}(x)\cdot\varphi^{(p)}(y))(\overline{\langle g\rangle S})\\ &= \sum_{gS\in(WS)_p}\varphi^{(p)}(x)(\overline{\langle g\rangle S})\cdot\varphi^{(p)}(y)(\overline{\langle g\rangle S})\\ &= \sum_{gS\in(WS)_p}x(\langle g\rangle S)\cdot y(\langle g\rangle S)\\ &= \sum_{gS\in(WS)_p}|X^{<g>S}|\cdot|Y^{<g>S}|\\ &= \sum_{gS\in(WS)_p}|((X\times Y)^S)^{<g>}|\\ &\equiv 0 \mod |WS|_p.\end{aligned}$$

The last congruence follows from the Cauchy-Frobenius lemma. This proves that $\varphi(x)\cdot\varphi(y)$ belongs to $\mathrm{Ker}\,\psi^{(p)}=\mathrm{Im}\,\varphi^{(p)}$, that is, the image of $\varphi^{(p)}$ is closed under multiplication.

Next, in order to prove the existence of an identity element, it will suffice to show that the identity element 1 of $\tilde{\Omega}(G, \mathfrak{X})$ belongs to the image of $\varphi^{(p)}$. But

$$\psi^{(p)}(1)_S=\sum_{gS\in(WS)_p}1\equiv 0 \mod |WS|_p,$$

and so by the fundamental theorem, we conclude that the identity element 1 of $\tilde{\Omega}(G, \mathfrak{X})$ is contained in $\mathrm{Im}\,\varphi^{(p)}=\mathrm{Ker}\,\psi^{(p)}$. Thus $\Omega(G, \mathfrak{X})$ has an identity element and $\varphi^{(p)}$ maps the identity element to 1.

The above proof is valid when $p=\infty$, too □

3.12 DEFINITION. Let R be a commutative ring. The R-module $R\otimes_Z\Omega(G, \mathfrak{X})$ is called a *generalized Burnside ring* provided it has a ring structure with identity element such that the Burnside homomorphism

$$1\otimes\varphi : R\otimes\Omega(G, \mathfrak{X})\longrightarrow R\otimes\tilde{\Omega}(G, \mathfrak{X})$$

is an injective ring homomorphism. Usually $1\otimes\varphi$ is simply written as φ.

By Corollary 3.11, if the condition $(C)_p$ in 3.6 holds, R is p-torsion free and $|G|_{p'}\cdot 1_R$ is invertible in R, then $R\otimes\Omega(G, \mathfrak{X})$, particularly $\Omega(G, \mathfrak{X})_{(p)}$, is a generalized Burnside ring. Furthermore, $\boldsymbol{Q}\otimes\Omega(G, \mathfrak{X})$ is always a generalized Burnside ring by Corollary 3.4. An interesting open question is that $\Omega(G, \mathfrak{X})$ becomes a generalized Burnside ring under what condition on $\mathfrak{X}$.

3.13 LEMMA. *Assume that* $|G|\cdot 1_R$ *is not a zero divisor in* R *and that* $R\otimes\Omega(G,\mathfrak{X})$ *is a generalized Burnside ring. Let* $S\in\mathfrak{X}$ *and* $x\in R\otimes\Omega(G,\mathfrak{X})$. *Then the multiplication of* x *and* $[G/S]$ *in this generalized Burnside ring has the form*

$$[G/S]\cdot x=\varphi_S(x)\cdot[G/S]+\sum_{(D)<(S)} m(D)[G/D], \tag{9}$$

where (D) *runs over conjugacy classes such that* D *is* G*-conjugate to a proper subgroup of* S.

PROOF. Decompose the multiplication $[G/S]\cdot x$ into a summation of transitive $(G,\mathfrak{X})$-sets:

$$[G/S]\cdot x=\coprod_{(D)\in C(\mathfrak{X})} m(D)[G/D]. \tag{10}$$

Let E be any maximal element of $\mathfrak{X}$ under the condition that $m(E)\neq 0$. Then applying φ_E to (10), it follows from (3) of Section 2 that

$$\varphi_E([G/S])\cdot\varphi_E(x)=m(E)\cdot|WE|\neq 0,$$

and so $\varphi_E([G/S]\neq 0$. Again by (3) of Section 2, we have that $E\leq_G S$. So in the above summation, only the terms $m(D)\cdot[G/D]$ such that $D\leq_G S$ appear. Applying φ_S again on (10), we have that

$$\varphi_S([G/S])\cdot\varphi_S(x)=m(S)\cdot\varphi_S([G/S]).$$

Since $\varphi_S([G/S])=|WS|\cdot 1_R\neq 0$ (see (4) in Section 2), we have that $m(S)=\varphi_S(x)$, proving the lemma. □

3.14 Let $K(\mathfrak{X})$ be the ideal of the ordinary Burnside ring $\Omega(G)$ defined by

$$K(\mathfrak{X}):=\{x\in G|x(S)=0 \quad \text{for all } S\in\mathfrak{X}\}.$$

Then φ is factorized as follows:

$$\varphi:\Omega(G,\mathfrak{X})\xrightarrow{\lambda}\Omega(G)/K(\mathfrak{X})\xrightarrow{\mu}\tilde{\Omega}(G,\mathfrak{X}),$$

where

$$\begin{aligned}\lambda:&\quad x \longmapsto x+K(\mathfrak{X})\\ \mu:&\ y+K(\mathfrak{X})\longmapsto (y(S))_{(S)\in C(\mathfrak{X})}.\end{aligned}$$

Since μ is an injective ring homomorphism, we have that $\Omega(G,\mathfrak{X})$ is a generalized Burnside ring if and only if $\Omega(G,\mathfrak{X})$ is isomorphic to a subring of $\Omega(G)/K(\mathfrak{X})$ via λ.

To characterize a family $\mathfrak{X}$ for which $\lambda:\Omega(G,\mathfrak{X})\longrightarrow\Omega(G)/K(\mathfrak{X})$ is an isomorphism is another interesting open problem.

3.15 EXAMPLE. We will first give some examples of generalized Burnside rings which are obtained from the ordinary Burnside rings.

(a) $\boldsymbol{Q}\otimes\Omega(G,\mathfrak{X})$ is a trivial generalized Burnside ring for any family $\mathfrak{X}$, which is isomorphic to the direct product of some copies of $\boldsymbol{Q}$. See Corollary 3.4.

(b) Assume that $\mathfrak{X}$ is closed under intersection and that $G\in\mathfrak{X}$. Then the condition $(\mathrm{C})_\infty$ trivially holds, and so $\Omega(G,\mathfrak{X})$ is a generalized Burnside ring by Corollary 3.11. On the other hand, $\Omega(G,\mathfrak{X})$ is a subring of $\Omega(G)$ by (1) in Section 2. Two ring structures on $\Omega(G,\mathfrak{X})$ coincide.

(c) Assume the following condition on $\mathfrak{X}$:

$$S\in\mathfrak{X},\ gS\in(WS)_p \implies \langle g\rangle S\in\mathfrak{X}.$$

Then the condition $(\mathrm{C})_p$ holds trivially, and so $\Omega(G,\mathfrak{X})_{(p)}$ is a generalized Burnside ring. This ring is isomorphic to $\Omega(G)_{(p)}/K(\mathfrak{X})_{(p)}$, where

$$K(\mathfrak{X})_{(p)}:=\{x\in\Omega(G)\mid x(S)=0 \text{ for all } S\in\mathfrak{X}\}.$$

To prove this, let x be an element of $\Omega(G)$ and put

$$\chi:=(x(S))_{(S)\in C(\mathfrak{X})}\in\tilde{\Omega}(G,\mathfrak{X}),$$

where $x(S)$ is the image of x by $\Omega(G,\mathfrak{X})\hookrightarrow\Omega(G)\xrightarrow{\varphi_S}\boldsymbol{Z}$. Then by the fundamental theorem of the Burnside ring (or the Cauchy-Frobenius lemma), we have that for any $S\in\mathfrak{X}$,

$$\begin{aligned}\psi^{(p)}(\chi)_S&\equiv\sum_{gS\in(WS)_p}\chi(\overline{\langle g\rangle S})\\&\equiv\sum_{gS\in(WS)_p}\chi(\langle g\rangle S)\\&\equiv 0 \pmod{|WS_p|}.\end{aligned}$$

Here $\overline{\langle g\rangle S}=\langle g\rangle S$ by the assumption. Thus by the fundamental theorem of the generalized Burnside ring, we have that χ belongs to $\Omega(G,\mathfrak{X})$. The correspondence $x\mapsto\chi$ defines a linear map ρ which makes the following diagram commutative:

$$\begin{array}{ccc} \Omega(G)_{(p)} & \xrightarrow{\rho} & \Omega(G, \mathfrak{X})_{(p)} \\ \varphi \downarrow & & \downarrow \varphi \\ \tilde{\Omega}(G)_{(p)} & \xrightarrow{\text{proj.}} & \tilde{\Omega}(G, \mathfrak{X})_{(p)}. \end{array}$$

By an easy argument, we can prove that ρ preserves multiplication and its kernel is $K(\mathfrak{X})_{(p)}$. See Proposition 6.3 and Corollary 6.4.

(d) Let e_Q^p be a primitive idempotent of $\Omega(Q)_{(p)}$ corresponding to a p-perfect subgroup Q of G(cf. 2.15, 2.16), and let

$$\mathfrak{X} := \{H \leq G \mid (H^p) = (Q)\}.$$

Then $\mathfrak{X}$ satisfies the condition $(\mathrm{C})_p$, and so $\Omega(G, \mathfrak{X})_{(p)}$ is a generalized Burnside ring. This ring is isomorphic to the ring $e_Q^p\,\Omega(G)_{(p)}$ with identity element e_Q^p. In order to prove this isomorphism, check that the Burnside homomorphisms for $\Omega(G)$ and $\Omega(G, \mathfrak{X})$ give injective ring homomorphisms $\varphi' : e_Q^p\,\Omega(G)_{(p)} \longrightarrow \tilde{\Omega}(G, \mathfrak{X})_{(p)}$ and $\varphi : \Omega(G, \mathfrak{X})_{(p)} \longrightarrow \tilde{\Omega}(G, \mathfrak{X})_{(p)}$ both of which cokernels are isomorphic to $\mathrm{Obs}(G, \mathfrak{X})_{(p)}$, and that there is a linear map from $\Omega(G, \mathfrak{X})_{(p)}$ to $e_Q^p\Omega(G)_{(p)}$ commutative with the above two maps.

3.16 EXAMPLE. Let S_n be the symmetric group of degree n and $\mathfrak{Y}$ the set of Young subgroups in S_n. Then $\mathfrak{Y}$ is closed under intersection and it contains G. Thus $\Omega(G, \mathfrak{Y})$ is a subring of $\Omega(G)$ and the ring structure of the generalized Burnside ring on $\Omega(G, \mathfrak{Y})$ is coincident with this subring structure. A. Dress proved that this generalized Burnside ring $\Omega(S_n, \mathfrak{Y})$ is isomorphic to the representation ring (the ordinary character ring) $R(S_n)$ (cf. [Dr 86]).

3.17 EXAMPLE. (a) Let p be a prime and let r be a non-negative integer. Let $P = C_{p^r}$ denote a cyclic group of order p^r. Then P has a unique subgroup P_i of order p^i for each $0 \leq i \leq r$. For any $0 \leq s,\ i \leq r$, we have that

$$\varphi_{P_s}([P/P_i]) = \begin{cases} p^{r-i} & \text{if } s \leq i \\ 0 & \text{otherwise.} \end{cases}$$

Let I be any subset of $\{0, 1, \cdots, r\}$ and let

$$\mathfrak{X} := \{P_i \mid i \in I\} \subseteq \mathrm{Sub}(P).$$

Consider the generalized Burnside ring $\boldsymbol{Q} \otimes \Omega(P, \mathfrak{X})$. In this ring, multiplication is given by the followin formula :

$$[P/P_i]\cdot[P/P_j]=\begin{cases}p^{r-j}[P/P_i] & \text{if } i\leq j\\ p^{r-i}[P/P_j] & \text{if } j\leq i\end{cases}$$

This formula is proved by comparing the values of the both sides at any P_s. See also Corollary 4.6. Note that the coefficients are always integers. Similarly, the identity element of $\boldsymbol{Q}\otimes\Omega(P,\mathfrak{X})$ is given as follows:

$$1_{\boldsymbol{Q}\otimes\Omega(P,\mathfrak{X})}=\frac{1}{p^{r-\max(I)}}[P/P_{\max(I)}],$$

where $\max(I)$ denotes a maximal element of I. See Corollary 4.4. Thus the necessary and sufficient condition that $\Omega(P,\mathfrak{X})$ (or $\Omega(P,\mathfrak{X})_{(p)}$) becomes a generalized Burnside ring, that is, it contains this identity element, is that $r\in I$. Clearly this condition is equivalent to the familiar $(\mathrm{C})_p$.

(b) Let $C:=C_n$ be a cyclic group of order n. Then C has a unique subgroup C_d of order d for any divisor d of n. For any divisors d, s, we have that

$$\varphi_{C_s}([C/C_d])=\begin{cases}n/d & \text{is } s \text{ divides } d\\ 0 & \text{otherwise.}\end{cases}$$

Let D be a set of divisors of n and let

$$\mathfrak{D}:=\{C_d|d\in D\}\subseteq\mathrm{Sub}(C).$$

Consider the generalized Burnside ring $\boldsymbol{Q}\otimes\Omega(C,\mathfrak{D})$. In this ring, multiplication is given by the following formula:

$$[C/C_a]\cdot[C/C_d]=\sum_{d\in D}\frac{nd}{ab}\Big(\sum_{h|(a,b)}\mu_D(d,h)\Big)[C/C_d],$$

where h runs over divisors of the greatest common divisor of a and b, and μ_D is the Möbius function of the poset D with order relation defined by divisor relation. This formula is proved by comparing the values of the both sides at any C_s. But such a direct proof is complicated, so we should use Corollary 4.6. Similarly, the identity element of $\boldsymbol{Q}\otimes\Omega(C,\mathfrak{D})$ is given as follows:

$$1_{\boldsymbol{Q}\otimes\Omega(C,\mathfrak{D})}=\sum_{d\in D}\frac{d}{n}\Big(\sum_{h\in D}\mu_{\mathfrak{D}}(d,h)\Big)[\mathrm{C}/\mathrm{C}_d].$$

See Corollary 4.4. The author does not know that the necessary and sufficient condition that $\Omega(C,\mathfrak{D})$ (or $\Omega(C,\mathfrak{D})_{(p)}$) becomes a generalized Burnside ring is $n\in D$.

An interesting problem is that under what condition there is a ring homomorphism from $\Omega(C,\mathfrak{D})$ to $\Omega(G,\mathfrak{X})$. For the ordinary Burnside rings, there is such a map from $\Omega(C_{|G|})$ to $\Omega(G)$ (cf. [DSY 90]).

4 Primitive idempotents

4.1 By Corollary 3.4, $\boldsymbol{Q}\otimes\Omega(G,\mathfrak{X})$ has a ring structure isomorphic to $\boldsymbol{Q}\otimes\tilde{\Omega}(G,\mathfrak{X})=\prod_{(S)\in C(\mathfrak{X})}\boldsymbol{Q}$ via the Burnside homomorphism $1\otimes\varphi$. Thus as in the case of the ordinary Burnside rings, there is an element e_H of $\boldsymbol{Q}\otimes\Omega(G,\mathfrak{X})$ for each $H\in\mathfrak{X}$ such that for any $S\in\mathfrak{X}$,

$$\varphi_S(e_H)=\begin{cases}1 & \text{if } (S)=(H)\\ 0 & \text{otherwise.}\end{cases} \tag{1}$$

Clearly, the set of primitive idempotents of $\boldsymbol{Q}\otimes\Omega(G,\mathfrak{X})$ consists of all e_H, $(H)\in C(\mathfrak{X})$.

In order to obtain an explicit formula for e_H, let

$$\mu_{\mathfrak{X}}:\mathfrak{X}\times\mathfrak{X}\longrightarrow\boldsymbol{Z}$$

be the Möbius function on the poset $\mathfrak{X}$ with the order relation by inclusion (cf. paragraph 2.12). Define functions $\zeta,\ \delta:\mathfrak{X}\times\mathfrak{X}\longrightarrow\boldsymbol{Z}$ by $\zeta(S,T):=1$ if $S\subseteq T$, and $:=0$ otherwise; $\delta(S,T):=1$ if $S=T$, and $:=0$ otherwise. Then by the definition of the Möbius function, we have that

$$\mu_{\mathfrak{X}}(S,S)=1;\quad \mu_{\mathfrak{X}}(S,T)=0 \text{ if } S\nsubseteq T \tag{2}$$

$$(\mu_{\mathfrak{X}}*\zeta)(S,T)=\sum_{U\in\mathfrak{X}}\mu_{\mathfrak{X}}(S,U)\zeta(U,T)=\delta(S,T), \tag{3}$$

$$(\zeta_{\mathfrak{X}}*\mu)(S,T)=\sum_{U\in\mathfrak{X}}\zeta(S,U)\mu_{\mathfrak{X}}(U,T)=\delta(S,T), \tag{4}$$

See [Ai 79], [St 86].

4.2 THEOREM. $e_H=\dfrac{1}{|NS|}\displaystyle\sum_{D\in\mathfrak{X}}|D|\mu_{\mathfrak{X}}(D,H)[G/D]$

PROOF. Let S be an element of $\mathfrak{X}$. By (2) of Section 2, we have that

$$\begin{aligned}\varphi_S([G/D])&=\#\{gD\in G/D\mid S\subseteq{}^gD\}\\ &=\frac{1}{|D|}\sum_{g\in G}\zeta(S,{}^gD)\\ &=\frac{1}{|D|}\sum_{g\in G}\zeta({}^gS,D).\end{aligned}$$

Thus φ_S maps the right hand side of the equation of this theorem to

$$\frac{1}{|NH|}\sum_{D\in\mathfrak{X}}|D|\mu_{\mathfrak{X}}(D,H)\varphi_S([G/D])=\frac{1}{|NH|}\sum_{D\in\mathfrak{X}}\sum_{g\in G}\zeta({}^gS,D)\mu_{\mathfrak{X}}(D,H)$$

$$=\frac{1}{|NH|}\sum_{g\in G}\delta({}^{g}S, H)$$
$$=\begin{cases}1 & \text{if } (S)=(H),\\ 0 & \text{otherwise.}\end{cases}$$

This equals $\varphi_S(e_H)$ by (1). Thus the proposition follows from the injectivity of $\varphi=(\varphi_S)$. □

4.3 COROLLARY (Standard expansion). *Let* $\chi\in\boldsymbol{Q}\otimes\tilde{\Omega}(G,\mathfrak{X})$. *Then*

$$\chi=\sum_{(D)\in C(\mathfrak{X})}\frac{1}{|WD|}\Big(\sum_{H\in\mathfrak{X}}\mu_{\mathfrak{X}}(D, H)\chi(H)\Big)[G/D].$$

PROOF. We identify $\boldsymbol{Q}\otimes\Omega(G,\mathfrak{X})$ with a subring of $\boldsymbol{Q}\otimes\tilde{\Omega}(G,\mathfrak{X})$. For any $H\in\mathfrak{X}$,

$$\chi\cdot e_H=\chi(H)\cdot e_H.$$

In fact, the values of the both sides at any $S\in\mathfrak{X}$ is $\chi(H)$ if $(S)=(H)$ and is 0 otherwise. Thus we have that

$$\begin{aligned}\chi&=\sum_{(H)\in C(\mathfrak{X})}\chi\cdot e_H\\&=\sum_{H\in\mathfrak{X}}\frac{1}{|G:NH|}\chi(H)\cdot e_H\\&=\frac{1}{|G|}\sum_{H\in\mathfrak{X}}\chi(H)\sum_{D\in\mathfrak{X}}|D|\mu_{\mathfrak{X}}(D, H)[G/D]\\&=\frac{1}{|G|}\sum_{D\in\mathfrak{X}}|D|\Big(\sum_{H\in\mathfrak{X}}\mu_{\mathfrak{X}}(D, H)\chi(H)\Big)[G/D]\\&=\sum_{(D)\in C(\mathfrak{X})}\frac{1}{|WD|}\Big(\sum_{H\in\mathfrak{X}}\mu_{\mathfrak{X}}(D, H)\chi(H)\Big)[GKD].\end{aligned}$$

Note that each G-conjugacy class (D) contains $|G:ND|$ conjugations of D. □

4.4 COROLLARY *The identity of the ring* $\boldsymbol{Q}\otimes\Omega(G,\mathfrak{X})$ *is given by*

$$1_{\boldsymbol{Q}\otimes\Omega(G,\mathfrak{X})}=\sum_{(D)\in C(\mathfrak{X})}\frac{1}{|WD|}\Big(\sum_{H\in\mathfrak{X}}\mu_{\mathfrak{X}}(D, H)\Big)[G/D]$$

PROOF. Trivial by Corollary 4.3. □

4.5 REMARK. In general, the *Euler characteristic* $\chi(P)$ of a finite poset P is defined by

$$\chi(P):=\sum_{x,y\in P}\mu_P(x, y).$$

so that it equals the usual Euler characteristic of the geometric realization of the order complex of P. Using this concept, the coefficients in the identity element of $\boldsymbol{Q}\otimes\Omega(G,\mathfrak{X})$ can be written by the Euler characteristics as follows. Let $\mathfrak{X}_{>D}$ be the subposet $\{S\in\mathfrak{X}\mid S>D\}$ of $\mathfrak{X}$. Then by the definition of Möbius function, we have that

$$\begin{aligned}\sum_{S\in\mathfrak{X}}\mu_{\mathfrak{X}}(D,S)&=1-\sum_{R,S\in\mathfrak{X}_{>D}}\mu_{\mathfrak{X}}(R,S)\\&=1-\chi(\mathfrak{X}_{>D}).\end{aligned}$$

4.6 COROLLARY. *Let $A, B\in\mathfrak{X}$. Then the multiplication in $\boldsymbol{Q}\otimes\Omega(G,\mathfrak{X})$ is given by*

$$[G/A]\cdot[G/B]=\sum_{(D)}\sum_{A'\sim A}\sum_{B'\sim B}\frac{|WA|\cdot|WB|}{|WD|}\Big(\sum_{H\subseteq A'\cap B'}\mu_{\mathfrak{X}}(D,H)\Big)[G/D],$$

where (D) runs over $C(\mathfrak{X})$, A', B' runs over G-conjugations of A and B, respectively, and H runs over elements of $\mathfrak{X}$ contained in A' and B'.

PROOF. Let $a:=[G/A]$, $b:=[G/B]$. Then by Corollary 4.3, we have that

$$a\cdot b=\sum_{(D)\in C(\mathfrak{X})}\frac{1}{|WD|}\Big(\sum_{H\in\mathfrak{X}}\mu_{\mathfrak{X}}(D,H)a(H)b(H)\Big)[G/D].$$

By (1) and (5) in Section 2,

$$\begin{aligned}a(H)\cdot b(H)&=|(G/A\times G/B)^H|\\&=\sum_{AgB\in A\backslash G/B}|(G/A\cap{}^gB)^H|\\&=\sum_{g\in G}\frac{|A\cap{}^gB|}{|A|\cdot|B|}\cdot\frac{1}{|A\cap{}^gB|}\sum_{g'\in G}\zeta(H,{}^{g'}(A\cap{}^gB))\\&=\frac{1}{|A|\cdot|B|}\sum_{g,g'\in G}\zeta(H,{}^gA\cap{}^{g'}B)\\&=\sum_{A'\sim A}\sum_{B'\sim B}|WA|\cdot|WB|\zeta(H,A'\cap B'),\end{aligned}$$

where $\zeta(H,K):=1$ if $H\subseteq K$ and $:=0$ otherwise, and in the summation $\sum_{A'\sim A}$, A' runs over G-conjugations of A. Now the expansion of $[G/A]\cdot[G/B]$ in $\Omega(G,\mathfrak{X})$ follows from the above two equalities. □

4.7 **The equivalence relation $\sim_p$** We assume the condition $(\mathrm{C})_p$, where p is a prime or ∞, so that by the Corollary 3.11, $\Omega(G,\mathfrak{X})$ has the structure of a generalized Burnside ring. We study the primitive idempotents of the generalized Burnside ring $\Omega(G,\mathfrak{X})$.

Let $\sim_p$ be the equivalence relation on $C(\mathfrak{X})$ generated by the relation

$$(\overline{\langle g\rangle S})\sim_p(S)\ \text{for}\ S\in\mathfrak{X},\ gS\in(WS)_p, \tag{5}$$

where $(WS)_p$ means a Sylow p-subgroup $((WS)_\infty:=WS)$. This definition can be lifted to $\mathfrak{X}$, that is, $S \sim_p T$ if and only if $(S) \sim_p (T)$. An easy induction argument (cf. the proof of Lemma 3.7 (c)) shows that

$$S \sim_\infty T \Longleftrightarrow \begin{array}{l} S=S_1 \sim_{p_1} S_2 \sim \cdots \sim_{p_{n-1}} S_n \sim_{p_n} T \\ \text{for some } S_i \in \mathfrak{X} \text{ and primes } p_i. \end{array} \tag{6}$$

4.8 LEMMA. *Assume that* $(\mathrm{C})_p$ *holds for a prime* p. *Then*

$$S \sim_p T \Longrightarrow \varphi_S(x) \equiv \varphi_T(x) \pmod{p} \text{ for all } x \in \Omega(G, \mathfrak{X})$$

PROOF. We prove that $\varphi_S([X]) \equiv \varphi_T([X]) \pmod{p}$ for a $(G, \mathfrak{X})$-set X. We may assume that $T=\overline{\langle g \rangle S}$ for some p-element $gS \in WS$. Then

$$|X^T| = |X^{\langle g \rangle S}|.$$

(See Lemma 3.8 of Section 3.) Since S is a normal subgroup of $\langle g \rangle S$ of p-power index, it follows from the basic principle of finite permutation groups that

$$|X^{\langle g \rangle S}| \equiv |X^S| \pmod{p}.$$

This proves that $\varphi_S([X]) \equiv \varphi_T([X]) \pmod{p}$. □

4.9 For an element $Q \in \mathfrak{X}$, we define an idempotent of $\boldsymbol{Q} \otimes \Omega(G, \mathfrak{X})$ by

$$e_Q^p := \sum_{\substack{(H) \in C(\mathfrak{X}) \\ H \sim_p S}} e_H, \tag{7}$$

so that

$$e_Q^p(S) = \begin{cases} 1 & \text{is } S \sim_p Q \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

4.10 REMARK. We can not choose a p-perfect subgroup Q in the primitive idempotent e_Q^p as in the case of ordinary Burnside ring $\Omega(G)$ (cf. 2.15) because Q^p does not belong to $\mathfrak{X}$ in general. But we can choose Q such that WQ is a p'-group. Furthermore, if p is a prime, then such (Q) is uniquely determined for each primitive idempotent of $\Omega(G, \mathfrak{X})_{(p)}$. See Proposition 5.13 for the proof of this fact.

4.11 LEMMA. $e_Q^p = \sum_{(D) \in C(\mathfrak{X})} \frac{1}{|WD|} \left(\sum_{H \sim_p Q} \mu_{\mathfrak{X}}(D, H) \right) [G/D]$.

PROOF. By Corollary 4.3. □

4.12 THEOREM. *The element* e_Q^p *is a primitive idempotent of* $\Omega(G, \mathfrak{X})_{(p)}$, *and conversely any primitive idempotent of* $\Omega(G, \mathfrak{X})_{(p)}$ *has this form. Thus the set of primitive idempotents of* $\Omega(G, \mathfrak{X})_{(p)}$ *is bijectively* *corresponding to the equivalence classes of the equivalence relation* $\sim_p$ *in* $C(\mathfrak{X})$.

PROOF. The idempotent e_Q^p depends only on the equivalence class containing Q. We will first show that e_Q^p belongs to $\Omega(G, \mathfrak{X})_{(p)}$. The S-component of $\psi^{(p)}(e_Q^p)$, where $\psi^{(p)}$ is the Cauchy-Frobenius homomorphism (cf. 3.9) is equal to

$$\begin{aligned}\psi_S^{(p)}(e_Q^p) &= \sum_{gS\in(WS)_p} e_Q^p(\overline{\langle g\rangle S})\\ &=\#\{gS\in(WS)_p \mid \overline{\langle g\rangle S} \sim_p Q\}\\ &=\#\{gS\in(WS)_p \mid S \sim_p Q\}\\ &=\begin{cases}|WS|_p & \text{is } S\sim_p Q\\ 0 & \text{otherwise}\end{cases}\\ &\equiv 0 \mod |WS|_p.\end{aligned}$$

Thus by the fundamental theorem, we have that $e_Q^p\in\operatorname{Ker}\psi^{(p)}=\operatorname{Im}\varphi^{(p)}$, and so $e_Q^p\in\Omega(G, \mathfrak{X})_{(p)}$. Hence e_Q^p is an idempotent of $\Omega(G, \mathfrak{X})_{(p)}$.

Next let e be a primitive idempotent of $\Omega(G, \mathfrak{X})_{(p)}$. Then again by the fundamental theorem,

$$\psi^{(p)}(e)_S=\sum_{gS\in(WS)_p} e(\overline{\langle g\rangle S})\equiv 0 \mod |WS|_p.$$

However, the value $e(\overline{\langle g\rangle S})=1$ or 0, and so

$$e(\overline{\langle g\rangle S})=e(S) \text{ for all } gS\in(WS)_p.$$

This means that e is constant on each $\sim_p$-equivalence class. Thus e is a summation of distinct idempotents of the form e_Q^p. By the fact that each $e_Q^p\in\Omega(G, \mathfrak{X})_{(p)}$ which has been already proved, we conclude that any primitive idempotent coincides with one of e_Q^p's. The theorem is proved. □

4.13 COROLLARY. (a) *Assume that* $(\mathrm{C})_p$ *holds for a prime* p. *Then*

$$S\sim_p T \Longleftrightarrow \varphi_S(x)\equiv\varphi_T(x) \pmod p \text{ for all } x\in\Omega(G, \mathfrak{X}).$$

(b) $(S)=(T) \Longleftrightarrow \varphi_S=\varphi_T$.

RROOF. (a) By Lemma 4.8, it remains only to show the if-part. The element $x:=|G|_{p'}e_S^p$ belongs to $\Omega(G,\mathfrak{X})$ by the idempotent formula and Theorem 4.12. By (8),

$$\varphi_T(x)\equiv\varphi_S(x)=|G|_{p'}\not\equiv 0 \pmod p.$$

Thus again by (8), we have that $S\sim_p T$.

(b) Assume that $\varphi_S=\varphi_T$, so that

$$0 \neq |WS| = \varphi_S([G/S]) = \varphi_T([G/S]),$$

and so $T \subseteq_G S$. By the symmetry, we have that $S \subseteq_G T$. Hence $(S) = (T)$. The converse is trivial. □

4.14 REMARK. As was stated before, when p is a prime, each $\sim_p$-equivalence class in $\mathfrak{X}$ contains a "defect subgroup" D, unique up to G-conjugation, such that $|WD|$ is prime to p. The proof will be give in the next section (Proposition 5.13).

4.15 PROPOSITION. *Assume that the condition* $(\mathrm{C})_p$ *holds. Let* e_Q^p *be a primitive idempotent of* $\Omega(G, \mathfrak{X})_{(p)}$ *corresponding to a subgroup* $Q \in \mathfrak{X}$. *Then the Burnside homomorphism induces the following ring homomorphism*:

$$\varphi_Q^{(p)} : e_Q^p \, \Omega(G, \mathfrak{X})_{(p)} \longrightarrow \prod_{(S) \sim p(Q)} \boldsymbol{Z}_{(p)}.$$

Furthermore, the cokernel of this map is isomorphic to

$$\mathrm{Coker}(\varphi_Q^p) \cong \prod_{(S) \sim p(Q)} \boldsymbol{Z}_{(p)}.$$

PROOF. This is clear from the fundamental theorem and (8). □

5 Prime ideals

In this section, we decide prime ideals of the generalized Burnside ring $\Omega(G, \mathfrak{X})$ by the method of Dress [Dr 71a]. For a commutative ring R, let $\mathrm{Spec}(R)$ denote the set of prime ideals of R. Usually, $\mathrm{Spec}(R)$ is equipped with Zariski topology. In this section p denotes a prime or 0, and so $p\boldsymbol{Z}$ is a prime ideal of $\boldsymbol{Z}$. If we hope to avoid distinguishing $p = 0$ from other primes, it is convenient to interpret as $\Omega(G, \mathfrak{X})_{(0)} = \boldsymbol{Q} \otimes \Omega(G, \mathfrak{X})$, $n_0 = 1$, $n_{0'} = n$, etc.

5.1 For any $S \in \mathfrak{X}$ and p, a prime or 0, define a subgroup of $\Omega(G, \mathfrak{X})$ by

$$\mathfrak{p}(S, p) := \varphi_S^{-1}(p\boldsymbol{Z}) = \{x \in \Omega(G, \mathfrak{X}) \mid \varphi_S(x) \equiv 0 \pmod{p}\}. \tag{1}$$

If $\Omega(G, \mathfrak{X})$ has a structure of a generalized Burnside ring together with the Burnside homomorphism $\varphi = (\varphi_S) : \Omega(G, \mathfrak{X}) \longrightarrow \tilde{\Omega}(G, \mathfrak{X})$, then the S-component of φ

$$\varphi_S : \Omega(G, \mathfrak{X}) \longrightarrow \boldsymbol{Z}$$

is a surjective ring homomorphism, and so $\mathfrak{p}(S, p)$ is a prime ideal of $\Omega(G, \mathfrak{X})$. Conversely, by the going-up theorem (or another theorem like this) about integral extension of commutative rings, prime ideals of $\Omega(G, \mathfrak{X})$ have this form. Using the going-up theorem is the second way by which Dress [Dr 71a] proved that each prime ideal of the Burnside ring of a finite group has the form $\mathfrak{p}(S, p)$. In this section, we will study prime ideals by the first way of Dress.

5.2 We assume that $\Omega(G, \mathfrak{X})$ is a generalized Burnside ring, that is, $\Omega(G, \mathfrak{X})$ has a ring structure such that the Burnside homomorphism φ: $\Omega(G, \mathfrak{X}) \longrightarrow \tilde{\Omega}(G, \mathfrak{X})$ is an injective ring homomorphism.

A *support* supp $\mathfrak{p}$ of a prime ideal $\mathfrak{p}$ of $\Omega(G, \mathfrak{X})$ is a pair $((S), p\boldsymbol{Z}) \in C(\mathfrak{X}) \times \mathrm{Spec}(\boldsymbol{Z})$ such that $p\boldsymbol{Z} = \mathfrak{p} \cap \boldsymbol{Z}$ and $S \in \mathfrak{X}$ is a minimal subject to $[G/S] \notin \mathfrak{p}$.

5.3 LEMMA *Let* $((S), p\boldsymbol{Z}) := \mathrm{supp}\, \mathfrak{p}$ *be a support of a prime ideal* $\mathfrak{p}$ *of* $\Omega(G, \mathfrak{X})$. *Then for any* $x \in \Omega(G, \mathfrak{X})$,

$$x \equiv \varphi_S(x) \cdot 1 \mod \mathfrak{p}.$$

PROOF. The decomposition of $x \cdot [G/S]$ into a sum of transitive $(G, \mathfrak{X})$-sets has the following form:

$$x \cdot [G/S] = \varphi_S(x) \cdot [G/S] + \sum_{D<S} m(D)[G/D],$$

where D runs over proper subgroups of S(cf. Lemma 3.13). Since $[G/D] \in \mathfrak{p}$ for any proper subgroup D of S, we have that

$$x \cdot [G/S] \equiv \varphi_S(x) \cdot [G/S] \mod \mathfrak{p},$$

which implies that $x \equiv \varphi_S(x) \cdot 1 \mod \mathfrak{p}$ because [G/S] is not contained in the prime ideal $\mathfrak{p}$. □

5.4 COROLLARY. *A support* $((S), p\boldsymbol{Z})$ *of a prime ideal* $\mathfrak{p}$ *of* $\Omega(G, \mathfrak{X})$ *is uniquely determined. Furthermore, WS is a* p'*-group if* p *is a prime.*

PROOF. Let $((T), p\boldsymbol{Z})$ be another support of $\mathfrak{p}$. Applying Lemma 5.3 to $x = [G/T]$, we have that

$$\mathfrak{p} \not\ni [G/T] \equiv \varphi_S([G/T]) \cdot 1 \mod \mathfrak{p},$$

and so $\varphi_S([G/T]) \neq 0$, which implies that $S \subseteq_G T$ by (3) in Section 2. By the minimality of T, we conclude that $(S) = (T)$. □

5.5 THEOREM. *Assume that* $\Omega(G, \mathfrak{X})$ *is a generalized Burnside ring. Let* $\mathfrak{S}$ *be the set of pairs* $((S), p\boldsymbol{Z}) \in C(\mathfrak{X}) \times \mathrm{Spec}(\boldsymbol{Z})$ *such that* $(|WS|, p) = 1$ *if* p *is a prime. Then the map*

$$\begin{aligned} f: \quad \mathfrak{S} \quad &\longrightarrow \mathrm{Spec}(\Omega(G, \mathfrak{X})) \\ : ((S), p\boldsymbol{Z}) &\longmapsto \mathfrak{p}(S, p)(=\varphi_S^{-1}(p\boldsymbol{Z})) \end{aligned}$$

is a bijection. Furthermore, the inverse g of f is given by $g: \mathfrak{p} \longmapsto \mathrm{supp}\ \mathfrak{p}$.

PROOF. By Lemma 5.4, the map g from $\mathrm{Spec}(\Omega(G, \mathfrak{X}))$ to $\mathfrak{X}$ is well-defined. Let $((S), p\boldsymbol{Z}) \in \mathfrak{S}$, and put $\mathfrak{p} := \mathfrak{p}(S, p)$. Since $\varphi_S([G/S]) = |WS| \cdot 1 \notin \mathfrak{p}$, we have that $[G/S] \notin \mathfrak{p}$. If D is a proper subgroup of S, then $\varphi_S[G/D] = 0$, and so $[G/D] \in \mathfrak{p}$, whence S is a minimal subject to $[G/S] \notin \mathfrak{p}$. Thus $((S), p\boldsymbol{Z})$ is a support of $\mathfrak{p}(S, p)$. This means $gf = \mathrm{id}$. Next let $\mathfrak{p}$ be any prime ideal of $\Omega(G, \mathfrak{X})$ with support $((S), p\boldsymbol{Z})$. By Lemma 5.3, an element $x \in \Omega(G, \mathfrak{X})$ belongs to $\mathfrak{p}$ if and only if $\varphi_S(x) \in p\boldsymbol{Z}$. Thus $\mathfrak{p} = \mathfrak{p}(S, p)$, proving that $fg = \mathrm{id}$. □

5.6 LEMMA. $\mathfrak{p}(S, p) \subseteq \mathfrak{p}(T, q)$ *if and only if* $\mathfrak{p}(S, q) = \mathfrak{p}(T, q)$ *and* $p = 0$ *or* q.

PROOF. Assume that $\mathfrak{p}(S, p) \subseteq \mathfrak{p}(T, q)$. Then there is a canonical surjection:

$$\boldsymbol{Z}/p\boldsymbol{Z} \cong \Omega(G, \mathfrak{X})/\mathfrak{p}(S, p) \longrightarrow \Omega(G, \mathfrak{X})/\mathfrak{p}(T, q) \cong \boldsymbol{Z}/q\boldsymbol{Z}.$$

Thus $p = 0$ or q, and $\mathfrak{p}(S, q) = \mathfrak{p}(T, q)$. □

5.7 It still remains the problem when $\mathfrak{p}(H, p) = \mathfrak{p}(K, p)$ happens for $H, K \in \mathfrak{X}$. We can solve this problem under the condition $(\mathrm{C})_\infty$ in 3.6. So assume $(\mathrm{C})_\infty$, that is,

$$S \in \mathfrak{X},\ gS \in (WS)_\infty \implies \overline{\langle g \rangle S} \in \mathfrak{X},$$

where $\bar{H}$ for a subgroup H of G is the intersection of all subgroup S in $\mathfrak{X}$ containing H. Under this condition, $\Omega(G, \mathfrak{X})$ becomes a generalized Burnside ring by Theorem 3.11. Furthermore, as in the paragraph 4.7, for a prime p (or $p = \infty$), let $\sim_p$ be the equivalence relation on $C(\mathfrak{X})$, and also on $\mathfrak{X}$, generated by the relation

$$(\overline{\langle g \rangle S}) \sim_p (S) \text{ for all } X \in \mathfrak{X},\ gS \in (WS)_p.$$

We extend this relation to $p = 0$ by

$$(S) \sim_0 (T) \Longleftrightarrow (S) = (T). \tag{2}$$

5.8 THEOREM. *Assume the condition* $(\mathrm{C})_\infty$ *and let* $\sim_p$ *be the equivalence relations defined as above. Then*

$$\mathfrak{p}(S, p) = \mathfrak{p}(T, q) \Longleftrightarrow p = q \text{ and } (S) \sim_p (T).$$

PROOF. Assume that $\mathfrak{p}:=\mathfrak{p}(S, p)=\mathfrak{p}(T, q)$. Comparing the characteristics of the residue rings, we have that $p=q$. Let

$$f_Q:=\begin{cases}|G|_{p'}e_Q^{\mathfrak{F}} & \text{if } p \text{ is a prime,}\\ |G|e_Q & \text{if } p=0,\end{cases}$$

where e_Q(resp. $e_Q^{\mathfrak{F}}$) is the primitive idempotent of $\boldsymbol{Q}\otimes\Omega(G, \mathfrak{X})$ (resp. $\Omega(G, \mathfrak{X})_{(p)}$) corresponding to a subgroup Q. Then by Theorem 4.2, 4.12, (1), (8) in Section 4, we have that f_Q is contained in $\Omega(G, \mathfrak{X})$ and that for any $H\in\mathfrak{X}$,

$$\begin{aligned} f_Q\notin\mathfrak{p}(H, p) &\iff \varphi_H(f_Q)\not\equiv 0 \pmod p\\ &\iff \varphi_H(f_Q)=|G|_{p'}\\ &\iff H\sim_p Q. \end{aligned}$$

Thus we have that $S\sim_p T$. Conversely, it follows immediately from Corollary 4.13 that if $S\sim_p T$, then $\mathfrak{p}(S, p)=\mathfrak{p}(T, p)$. □

5.9 COROLLARY. *$\mathfrak{p}(S, p)\subseteq\mathfrak{p}(T, q)$ if and only if $p=0$ or q, and $(S)\sim_q(T)$.*

PROOF. This result follows from the theorem and Lemma 5.6. □

5.10 COROLLARY. *Let $\sim_\infty$ be the equivalence class on $\mathfrak{X}$ defined in* 4.7. *Then there is a bijective correspondence between the set of connected components of* Spec$(\Omega(G, \mathfrak{X}))$ *and the set $\mathfrak{X}/\sim_\infty$ of $\sim_\infty$-equivalence classes.*

PROOF. This follows trivially from the above corollary and (6) in Section 4. (*Another Proof* : By Theorem 4.12). □

5.11 REMARK. (a) There is a proof of Theorem 5.8 without using the knowledge about idempotents. So by Corollary 5.10, we know that the number of primitive idempotents of $\Omega(G, \mathfrak{X})_{(p)}$ equals $|\mathfrak{X}/\sim_\infty|$. This is a classical method by which A. Dress and T. tom Dieck proved that the number of primitive idempotents of the Burnside ring $\Omega(G)$ equals the number of conjugacy classes of perfect subgroups of G. See tom Dieck's book [Di 79].

(b) We can study Spec$(\Omega(G, \mathfrak{X})_{(p)})$ and more generally Spec$(R\otimes\Omega(G, \mathfrak{X}))$. (Refer to [Dr 71a].) For example, the following theorem holds. We can prove it easily and similarly to Theorem 5.8. So the proof is omitted.

5.12 THEOREM. *Assume the condition* $(\mathrm{C})_p$ *for a prime p. Then there is a bijective correspondence between* Spec$(\Omega(G, \mathfrak{X})_{(p)})$ *and $\sim_p$-*

equivalence classes.

5.13 PROPOSITION. *Let p be a prime and assume that the condition $(\mathrm{C})_p$ holds. Then each $\sim_p$-equivalence class in $\mathfrak{X}$ contains a unique $(D)\in C(\mathfrak{X})$ such that WD is a p'-group. In particular, maximal subgroups belonging to a $\sim_p$-equivalence class are conjugate each other in G Thus the set $\{(D)\in C(\mathfrak{X})\mid WD$ is a p'-group$\}$ is a complete set of representatives of $\sim_p$-equivalence classes.*

PROOF. This result follows from Theorem 5.5 and 5.8 under the assumption $(\mathrm{C})_\infty$. But because in this paper we omitted to prove the result for prime ideals of $\Omega(G, \mathfrak{X})_{(p)}$ corresponding to these two theorems, we will give a direct proof of this proposition. Let $\mathfrak{C}$ be a $\sim_p$-equivalence class in $\mathfrak{X}$, and let D be a maximal element of $\mathfrak{C}$. Then by $(\mathrm{C})_p$, we have that WD is a p'-group. Let S be an element of $\mathfrak{C}$ such that WS is a p'-group. Then by Lemma 4.8,

$$\varphi_D(x)\equiv\varphi_S(x) \pmod{p}$$

for all $x\in\Omega(G, \mathfrak{X})$. In particular, applying this to $x=[G/S]$, we have that

$$\varphi_D([G/S])\equiv\varphi_S([G/S])=|WS|\not\equiv 0 \pmod{p},$$

and so $D\leq_G S$. The maximality of D implies that D and S are G-conjugate each other. □

5.14 The above proposition is a purely group-theoretic statement. However the author does not know any application of this fact to finite group theory.

We will give an example. Let $\mathfrak{X}=\mathrm{Sub}(G)$, the lattice of all subgroups of G. In this case, $S\sim_p T$ if and only if $(S^p)=(T^p)$. Thus there is a p-perfect subgroup Q in each equivalence class. Let D be a subgroup such that D/Q is a Sylow p-subgroup of WQ. Then D satisfies the condition of the proposition.

Clearly, such a subgroup D does not exist if $p=\infty$, because WQ does not possess maximal solvable subgroups unique up to G-conjugation. So for a generalized Burnside ring, we have not yet had a good set of representatives of $\sim_\infty$-equivalence classes of subgroups in $\mathfrak{X}$

6 Transfer-Induction theorems

In this section, we will study some functorial properties of generalized Burnside rings. Similarly as in the case of usual Burnside rings, there exist restrictions, inductions, inflations, fixed-point maps between generalized Burnside rings, and using such maps, we can prove some transfer theorems.

6.1 For any finite group H, we denote by $\mathrm{Sub}(H)$ the lattice of subgroups of H. The group H acts on this lattice by conjugation. When a family $\mathfrak{Y}$ of subgroups of H is closed under H-conjugation, we write it as

$$\mathfrak{Y} \subseteq_H \mathrm{Sub}(H).$$

Through this section, $\mathfrak{X}$ denotes such a family of subgroups of G, that is,

$$\mathfrak{X} \subseteq_G \mathrm{Sub}(G).$$

We define a family $\cap\mathfrak{X} \subseteq_G \mathrm{Sub}(G)$ by

$$\cap\mathfrak{X} := \{S_1 \cap \cdots \cap S_n | n \geq 0,\ S_i \in \mathfrak{X}\}.$$

This family contains G.

As before, we define

$$\bar{K} := \cap\{S \in \mathfrak{X} | K \subseteq S\}.$$

(We will use this symbol only for $\mathfrak{X}$; we do not use for other families.) Furthermore, p denotes a prime, 0 or ∞ and we write $M_{(p)}$ for $\boldsymbol{Z}_{(p)} \otimes M$ for any abelian group M as usual.

The condition $(\mathrm{C})_p$ (cf. 3.6) and the equivalence relation $\sim_p$ (cf. 4.7) play again essential roles in this section. The condition $(\mathrm{C})_p$ states that if $S \in \mathfrak{X}$, $gS \in (WS)_p$, then $\overline{\langle g \rangle S} \in \mathfrak{X}$, and $\sim_p$ is generated by the relation $(\overline{\langle g \rangle S}) \sim_p (S)$ for $S \in \mathfrak{X}$, $gS \in (WS)_p$.

6.2 Let H be a subgroup of G and let $\mathfrak{Y} \subseteq_H \mathrm{Sub}(H) \cap \mathfrak{X}$. Then we have two linear maps as follows:

$$\begin{array}{rlcl}
\mathrm{ind}^G : & \Omega(H, \mathfrak{Y})_{(p)} & \longrightarrow & \Omega(G, \mathfrak{X})_{(p)} \\
: & [H/T] & \longmapsto & [G/T]; \\
\mathrm{res}^{\sim}{}_H : & \tilde{\Omega}(G, \mathfrak{X})_{(p)} & \longrightarrow & \tilde{\Omega}(H, \mathfrak{Y})_{(p)} \\
: & (\chi(S))_{(S) \in C(\mathfrak{X})} & \longmapsto & (\chi(T))_{(T) \in C(\mathfrak{Y})}
\end{array}$$

The map $\mathrm{res}^{\sim}$ is a ring homomorphism. We are interesting in the case where ind induces a linear map between the ghost rings and the case where res induces a ring homomorphism between the generalized Burnside rings. Such maps res and ind are called a *restriction* and an *induction*, respectively.

6.3 PROPOSITION. *Under the above notation, assume that*

$$T\in\mathfrak{Y},\ hT\in(W_HT)_p \implies \overline{\langle h\rangle T}\in\mathfrak{Y}. \qquad (1)$$

Then the restriction of res˜ *into* $\Omega(G,\mathfrak{X})_{(p)}$ *gives a map*

$$\mathrm{res}_H:\Omega(G,\mathfrak{X})_{(p)}\longrightarrow\Omega(H,\mathfrak{Y})_{(p)}.$$

If furthermore $\Omega(G,\mathfrak{X})_{(p)}$ *is a generalized Burnside rings, then* res *is a ring homomorphism.*

PROOF. By the assumption, the family $\mathfrak{Y}$ satisfies the condition $(\mathrm{C})_p$ in 3.6. Thus $\Omega(H,\mathfrak{Y})_{(p)}$ is a generalized Burnside ring and furthermore the fundamental theorem (Theorem 3.10) holds. Let $x=[X]\in\Omega(G,\mathfrak{X})_{(p)}$ for a $(G,\mathfrak{X})$-set X. Then

$$\theta:=\mathrm{res}\tilde{}_H\ \varphi^{(p)}(x)=(|X^T|)_{(T)\in C(\mathfrak{Y})}.$$

It will suffice to show that θ is contained in the image of $\varphi^{(p)}$. Let $T\in\mathfrak{Y}$. By the Cauchy-Frobenius theorem (Lemma 2.7) and Lemma 3.8,

$$\begin{aligned}\psi^{(p)}(\theta)_T&=\sum_{hT\in(W_HT)_p}\theta(\overline{\langle h\rangle T})\\&=\sum_{hT\in(W_HT)_p}x(\overline{\langle h\rangle T})\\&=\sum_{hT\in(W_HT)_p}x(\overline{\langle h\rangle T})\\&=\sum_{hT\in(W_HT)_p}|(X^T)^{\langle h\rangle T}|\\&\equiv 0 \mod |W_HT|_p.\end{aligned}$$

Thus $\theta\in\mathrm{Ker}\,\psi^{(p)}=\mathrm{Im}\,\varphi^{(p)}$, and so by the fundamental theorem, there exists a unique $x'\in\Omega(H,\mathfrak{Y})$ such that $\varphi^{(p)}(x')=\theta$. The correspondence $x\longmapsto x'$ gives the desired homomorphism

$$\mathrm{res}_H:\Omega(G,\mathfrak{X})_{(p)}\longrightarrow\Omega(H,\mathfrak{Y})_{(p)}.$$

If $\Omega(G,\mathfrak{X})_{(p)}$ is a generalized Burnside ring, then two Burnside homomorphisms $\varphi^{(p)}$'s with respect to G and H are injective ring homomorphisms and they are commutaive with the restriction maps res_H and $\mathrm{res}\tilde{}_H$. Since $\mathrm{res}\tilde{}_H$ is a ring homomorphism, res_H is also a ring homomorphism. □

6.4 COROLLARY. *Assume that the following condition holds:*

$$\langle g\rangle S\in\mathfrak{X} \text{ for any } S\in\mathfrak{X},\ gS\in(WS)_p.$$

Then $\Omega(G,\mathfrak{X})_{(p)}$ *is a generalized Burnside ring and there exists a ring homomorphism* ρ *from* $\Omega(G)$ *to* $\Omega(G,\mathfrak{X})$ *which makes the following diagram commutative:*

$$\begin{array}{ccc}\Omega(G) & \xrightarrow{\rho} & \Omega(G,\mathfrak{X})\\ \varphi\downarrow & & \varphi\downarrow\\ \tilde{\Omega}(G) & \xrightarrow{\text{proj.}} & \tilde{\Omega}(G,\mathfrak{X}).\end{array}$$

PROOF. The condition $(\mathrm{C})_p$ holds by the assumption. We can apply Proposition 6.3 to $\mathfrak{X}\subseteq\mathrm{Sub}(G)$. □

6.5 COROLLARY. *Assume that $\mathfrak{X}$ satisfies the condition $(\mathrm{C})_p$. Then there is a surjective ring homomorphism*

$$\rho:\Omega(G,\cap\mathfrak{X})_{(p)}\longrightarrow\Omega(G,\mathfrak{X})_{(p)} \tag{2}$$

such that $\rho(x)(S)=x(S)$ for $S\in\mathfrak{X}$. Thus the generalized Burnside ring $\Omega(G,\mathfrak{X})_{(p)}$ is isomorphic to the factor ring of $\Omega(G,\cap\mathfrak{X})$, the subring of $\Omega(G)$, by the ideal

$$\{x\in\Omega(G,\cap\mathfrak{X})_{(p)}\mid x(T)=0 \textit{ if } T\in\cap\mathfrak{X}-\mathfrak{X}\}. \tag{3}$$

PROOF. The family $\cap\mathfrak{X}$, together with $\mathfrak{X}$, satisfies the assumption of the Proposition 6.3. Thus we have a map ρ which has the desired property. To prove the surjectivity of ρ, we restrict ρ into $\Omega(G,\mathfrak{X})\subseteq\Omega(G,\cap\mathfrak{X})$, so that we have identity map, and so ρ is a split epimorphism. □

6.6 COROLLARY. *Assume that $(\mathrm{C})_p$ holds for $\mathfrak{X}$. Let $H\in\cap\mathfrak{X}$ and $\mathfrak{X}_H:=\mathfrak{X}\cap\mathrm{Sub}(H)$. Then there is a ring homomorphism*

$$\mathrm{res}_H:\Omega(G,\mathfrak{X})_{(p)}\longrightarrow\Omega(H,\mathfrak{X}_H)_{(p)}$$

such that $\mathrm{res}_H(x)(T)=x(T)$ for $T\in\mathfrak{X}_H$. If $\mathfrak{X}$ is closed under intersection, then

$$\mathrm{res}_H([G/S])=\sum_{HgS\in H\backslash G/S}[H/H\cap{}^gS].$$

PROOF. The first part follows immediately from Proposition 6.3. The remainder and be easily proved from the fact that $\Omega(G,\mathfrak{X})$ is the subring of $\Omega(G)$. □

6.7 PROPOSITION. *Assume that the family $\mathfrak{X}$ satisfies the condition $(\mathrm{C})_p$ and that $\mathfrak{X}=\mathfrak{X}_1\dot{\cup}\mathfrak{X}_2$ is a disjoint union of two families $\mathfrak{X}_i$ closed with respect to $\sim_p$. Then there is a ring isomorphism*

$$\rho:\Omega(G,\mathfrak{X})_{(p)}\cong\Omega(G,\mathfrak{X}_1)_{(p)}\times\Omega(G,\mathfrak{X}_2)_{(p)} \tag{4}$$

such that $\rho(x)(S_1, S_2)=(x(S_1)x(S_2))$ *for* $S_i\in\mathfrak{X}$.

PROOF. Applying Proposition 6.3 to $\mathfrak{X}_i$, we have a ring homomorphism

$$\mathrm{res}_i:\Omega(G,\mathfrak{X})_{(p)}\longrightarrow\Omega(G,\mathfrak{X}_i)_{(p)}.$$

Thus we have a commutative diagram

$$\begin{array}{ccc}\Omega(G,\mathfrak{X})_{(p)} & \xrightarrow{\rho} & \Omega(G,\mathfrak{X}_1)_{(p)}\times\Omega(G,\mathfrak{X}_2)_{(p)}\\ \varphi\downarrow & & \downarrow\varphi'\\ \tilde{\Omega}(G,\mathfrak{X})_{(p)} & \xrightarrow{\cong} & \tilde{\Omega}(G,\mathfrak{X}_1)_{(p)}\times\tilde{\Omega}(G,\mathfrak{X}_2)_{(p)},\end{array}$$

where $\varphi':=\varphi_1\times\varphi_2$. Since φ and φ' are both injective and since their cokernels are both isomorphic to $\mathrm{Obs}(G,\mathfrak{X})_{(p)}$, we have that $\rho=\mathrm{res}_1\times\mathrm{res}_2$ is an isomorphism. □

6.8 COROLLARY. *Assume that the condition* $(\mathrm{C})_p$ *holds for* $\mathfrak{X}$. *Let* e *be an idempotent of* $\Omega(G,\mathfrak{X})_{(p)}$ *and let*

$$\mathfrak{X}_1:=\{S\in\mathfrak{X}\mid e(S)=1\}.$$

Then

$$e\cdot\Omega(G,\mathfrak{X})_{(p)}\cong\Omega(G,\mathfrak{X}_1)_{(p)}.$$

PROOF. Let $e_1:=e$, $e_2:=1-e$, and let $\mathfrak{X}_2:=\{S\in\mathfrak{X}\mid e(S)=0\}$. Then the families $\mathfrak{X}_i$, $i=1, 2$ satisfy $(\mathrm{C})_p$ by Theorem 4.12 and (8) in Section 4. Let ρ be the isomorphism in Proposition 6.7. Since $\rho(e_i)(S_j)=\delta_{ij}$ for $S_i\in\mathfrak{X}_j$,

$$\rho(e_i\Omega(G,\mathfrak{X})_{(p)})\subseteq\Omega(G,\mathfrak{X}_i)_{(p)}.$$

Furthermore since $\Omega(G,\mathfrak{X})_{(p)}=e_1\,\Omega(G,\mathfrak{X})_{(p)}\oplus e_2\,\Omega(G,\mathfrak{X})_{(p)}$, we have that ρ induces ring isomorphisms

$$e_i\,\Omega(G,\mathfrak{X})_{(p)}\longrightarrow\Omega(G,\mathfrak{X}_i)_{(p)},\ i=1,2.$$

□

6.9 COROLLARY. *Assume* $(\mathrm{C})_p$. *Let* $\mathfrak{C}_1, \mathfrak{C}_2,\cdots$ *be the* $\sim_p$*-equivalence classes in* $\mathfrak{X}$. *Then the restriction maps induce the following ring isomorphism*:

$$\Omega(G,\mathfrak{X})_{(p)}\xrightarrow{\cong}\prod_i\Omega(G,\mathfrak{C}_i)_{(p)}.$$

PROOF. This follows from above two corollaries. □

6.10 For any $H\in\cap\mathfrak{X}$, we define a family $\mathfrak{X}_H$ by

$$\mathfrak{X}_H:=\mathfrak{X}\cap\mathrm{Sub}(H)=\{T\in\mathfrak{X}\mid T\subseteq H\}.$$

Assume that

$\mathfrak{X}$ satisfies $(\mathrm{C})_p$ for p a prime 0, or ∞.

Then by Corollary 6.6, for any pair $H\leq K$ in $\cap\mathfrak{X}$, there exist maps as follows:

$$\begin{aligned}
\mathrm{res}_H^K &: \Omega(K,\mathfrak{X}_K)_{(p)} \longrightarrow \Omega(H,\mathfrak{X}_H)_{(p)}\\
\mathrm{ind}_H^K &: \Omega(H,\mathfrak{X}_H)_{(p)} \longrightarrow \Omega(K,\mathfrak{X}_K)_{(p)}\\
&: \ [H/T] \longmapsto [G/T].
\end{aligned}$$

Furthermore, for $H\in\cap\mathfrak{X}$ and $g\in G$, the *conjugation* map is defined by

$$\begin{aligned}
\mathrm{con}_H^g &: \Omega(H,\mathfrak{X}_H)_{(p)} \longrightarrow \Omega({}^gH,\mathfrak{X}_{{}^gH})_{(p)}\\
&: \ [H/S] \longmapsto [{}^gH/{}^gS],
\end{aligned}$$

where ${}^gH:=gHg^{-1}$. By the definition of restriction and conjugation, we have that

$$\mathrm{res}_H^K(x)(S) = x(S), \quad x\in\Omega(K,\mathfrak{X}_K)_{(p)},\ S\in\mathfrak{X}_K; \tag{5}$$
$$\mathrm{con}_H^g(y)({}^gT) = y(T), \quad y\in\Omega(H,\mathfrak{X}_H)_{(p)},\ T\in\mathfrak{X}_H. \tag{6}$$

About induction maps, we can calculate their values at subgroups by the following lemma:

6.11 LEMMA. *Let H, $K\in\cap\mathfrak{X}$ with $H\subseteq K$. Then under* $(\mathrm{C})_p$,

$$\mathrm{ind}_H^K(y)(T)=\frac{1}{|H|}\sum_{\substack{g\in K\\ :T\subseteq{}^gH}} y(g^{-1}Tg)$$

for any $y\in\Omega(H,\mathfrak{X}_H)$ and $T\in\mathfrak{X}_K$. In particular, ind *can be extended to*

$$\mathrm{ind}^{\sim}:\tilde{\Omega}(H,\mathfrak{X}_H)\longrightarrow\tilde{\Omega}(K,\mathfrak{X}_K) \tag{7}$$

PROOF. Let $y=[H/T]$, $T\in\mathfrak{X}_H$, and $S\in\mathfrak{X}_K$. Then

$$\begin{aligned}
\varphi_S(\mathrm{ind}^K([H/T])&=\varphi_S([K/T])\\
&=\#\{kT\in K/T\mid S\subseteq{}^kT\}\\
&=\sum_{kH\in K/H}\#\{kT\in H/T\mid S\subseteq{}^{kh}T\}\\
&=\frac{1}{|H|}\sum_{k\in K}\#\{hT\in H/T\mid {}^kS\subseteq{}^hT\}
\end{aligned}$$

$$=\frac{1}{|H|}\sum_{k\in K}([H/T])({}^kS)$$
$$=\frac{1}{|H|}\sum_{k\in K}y({}^kS).$$

This proves the lemma. □

6.12 LEMMA (Mackey decomposition). *Let $H, K, L\in\cap\mathfrak{X}$ with $H, K\subseteq L$ and let $x\in\Omega(G, \mathfrak{X}_H)$. Then*

$$\mathrm{res}_K\cdot\mathrm{ind}^L(x)=\sum_{KgH\in K\backslash L/H}\mathrm{ind}^K\mathrm{res}_{{}^gH\cap K}\mathrm{con}^g(x). \tag{8}$$

PROOF. We put simply

$${}^gx:=\mathrm{con}^g(x).$$

Then for any $T\in\mathfrak{X}_K$,

$$\begin{aligned}\mathrm{res}_K\cdot\mathrm{ind}^L(x)(T)&=\mathrm{ind}^L(x)(T)\\&=\frac{1}{|H|}\sum_{\substack{g\in L\\:{}^gT\subseteq H}}x({}^gT)\\&=\frac{1}{|H|}\sum_{\substack{g\in L\\:T\subseteq{}^gH}}x(g^{-1}Tg)\\&=\frac{1}{|H|}\sum_{\substack{g\in L\\:T\subseteq{}^gH}}{}^gx(T)\\&=\sum_{KgH}\sum_{\substack{k\in K/K\cap{}^gH\\:T\subseteq{}^k({}^gH\cap K)}}{}^gx(k^{-1}Tk)\\&=\sum_{KgH}\mathrm{ind}^K\mathrm{res}_{K\cap{}^gH}\mathrm{con}^g(x)(T),\end{aligned}$$

as required. □

6.13 LEMMA (Frobenius reciprocity). *Let $H, K\in\cap\mathfrak{X}$ with $H\subseteq K$, $x\in\Omega(H, \mathfrak{X}_H)$ and $y\in\Omega(K, \mathfrak{X}_K)$. Then*

$$\mathrm{ind}^K(x)\cdot y=\mathrm{ind}^K(x\cdot\mathrm{res}_H(y)).$$

PROOF. Let $T\in\mathfrak{X}_K$. *Then*

$$\begin{aligned}(\mathrm{ind}^K(x)\cdot y)(T)&=\frac{1}{|H|}\sum_{\substack{k\in K\\:{}^gT\subseteq H}}x({}^kT)y(T)\\&=\frac{1}{|H|}\sum_{\substack{k\in K\\:{}^gT\subseteq H}}(x\cdot\mathrm{res}_H(y))({}^kT)\\&=\mathrm{ind}^K(x\cdot\mathrm{res}_H(y))(T).\end{aligned}$$

□

6.14 Lemma. *Let $H\in\mathfrak{X}$ and $x\in\Omega(G,\mathfrak{X})_{(p)}$. Then*

$$\mathrm{ind}^G\mathrm{res}_H(x)=\mathrm{ind}^G(1_H)\cdot x,$$

where 1_H denotes the identity element of $\Omega(H,\mathfrak{X}_H)_{(p)}$.

Proof. For any $S\in\mathfrak{X}$,

$$\begin{aligned}\mathrm{ind}^G\mathrm{res}_H(x)(S)&=\frac{1}{|H|}\sum_{\substack{g\in G\\ :{}^gS\subseteq H}}\mathrm{res}_H(x)({}^gS)\\ &=\frac{1}{|H|}\sum_{\substack{g\in G\\ :{}^gS\subseteq H}}x({}^gS)\\ &=\frac{1}{|H|}\sum_{\substack{g\in G\\ :{}^gS\subseteq H}}x(S)\\ &=(\mathrm{ind}^G(1_H)\cdot x)(S).\end{aligned}$$

□

6.15 The above lemmas show that the correspondence $H(\in\cap\mathfrak{X})\longmapsto\Omega(H,\mathfrak{X}_H)_{(p)}$ together with induction, restriction and conjugation makes a G-functor. Here a G-functor $\boldsymbol{a}$ with respect to a family $\mathfrak{X}$ is a family $(\boldsymbol{a}(H),\ \mathrm{ind}_H^K,\ \mathrm{res}_H^K,\ \mathrm{con}_H^g)$ consisting of abelian groups $\boldsymbol{a}(H)$ ($\mathrm{H}\in\mathfrak{X}$) and three kinds of linear maps (induction or corestiction, restriction, and conjugation) as follows:

$$\begin{aligned}&\mathrm{ind}_H^K:\Omega(H,\mathfrak{X}_H)\longrightarrow\Omega(K,\mathfrak{X}_K)\ :x\longmapsto x^{\uparrow K};\\ &\mathrm{mes}_H^K:\Omega(K,\mathfrak{X}_K)\longrightarrow\Omega(H,\mathfrak{X}_H)\ :y\longmapsto y_{\downarrow H};\\ &\mathrm{con}_H^g:\Omega(H,\mathfrak{X}_H)\longrightarrow\Omega({}^gH,\mathfrak{X}^g{}_H):x\longmapsto {}^gx,\end{aligned}$$

where $H,\ K$ are elements of $\cap\mathfrak{X}$ with $H\subseteq K$ and $g\in G$. Furthermore, these maps must satisfy some axioms, that is, the transitivity: $\mathrm{ind}_K^L\ \mathrm{ind}_H^K=\mathrm{ind}_H^L$, etc., the commutativity of conjugations with induction and restriction, and Mackey decomposition like Lemma 6.12. The precise definition (for the usual G-functors) is found in Green [Gr 71] or [Yo 80].

6.16 Similarly as in the case of the usual G-functors, we can define some concepts, for example, pairings, Green functors (ie. a G-functor with ring structure), modules on Green functors, morphisms between G-functors, the action of a generalized Burnside ring on a G-functors, relative projectivity, Dress induction theorem, the stable element theorem, Araki's excision theorem, the representability of G-functors, Hecke rings with coefficient in a generalized Burnside ring (span rings), and so on.

However, such a theory can be made much more like the ordinary theory of G-functors. So here we state only a few results.

6.17 For an $H\in\cap\mathfrak{X}$, we denote the idempotent of $\Omega(H,\mathfrak{X}_H)_{(p)}$ corresponding to $D\in\mathfrak{X}_H$ by $e^p_{H,D}$ if we need to state H clearly.

We introduce the following symbols:

$$\begin{aligned} {}^g y &:= \mathrm{con}^g_H(y) && \text{for } g\in G,\ y\in\Omega(H,\mathfrak{X}_H)_{(p)}\,;\\ H(g) &:= {}^gH\cap H && \text{for } g\in G\,;\\ x_{\downarrow H} &:= \mathrm{res}^K_H(x) && \text{for } H\le K\in\mathfrak{X},\ x\in\Omega(K,\mathfrak{X}_K)_{(p)}\,;\\ y^{\uparrow K} &:= \mathrm{ind}^K_H(y) && \text{for } H\le K\in\mathfrak{X},\ y\in\Omega(H,\mathfrak{X}_H)_{(p)}\,;\\ \Omega(H,\mathfrak{X}_H)^G_{(p)} &:= \{y\in\Omega(H,\mathfrak{X}_H)_{(p)} | y_{\downarrow H(g)}={}^gy_{\downarrow H(g)}\}\subseteq\Omega(H,\mathfrak{X}_H). \end{aligned}$$

6.18 THEOREM (Stable element theorem). *Assume* $(\mathrm{C})_p$ *for* p *a prime,* 0 *or* ∞. *Let* D *be an element of* $\mathfrak{X}$ *such that* WD *is a* p'*-group. Then the restriction map induces the following ring isomorphism*:

$$e^p_{G,D}\cdot\Omega(G,\mathfrak{X})_{(p)}\cong\mathrm{res}_D(e^p_{G,D})\cdot\Omega(D,\mathfrak{X}_D)^G_{(p)}. \tag{9}$$

The inverse is given by $y\longmapsto\mathrm{ind}^G(y)$.

PROOF. Put

$$e:=e^p_{G,D},\quad f:=\mathrm{res}_D(e^p_{G,D}).$$

Define maps ρ and τ as follows:

$$\begin{aligned} \rho\ &: e\cdot\Omega(G,\mathfrak{X})_{(p)} \longrightarrow f\cdot\Omega(D,\mathfrak{X}_D)^G_{(p)}\\ &: \quad x \longmapsto \mathrm{res}_D(x)\\ \tau\ &: f\cdot\Omega(\mathrm{D},\mathfrak{X}_D)^G_{(p)} \longrightarrow e\cdot\Omega(G,\mathfrak{X})_{(p)}\\ &: \quad \mathrm{y} \longmapsto \mathrm{ind}_D(y). \end{aligned}$$

By the Frobenius reciprocity, these maps are well defined and ρ is a ring homomorphism. By Lemma 6.14,

$$\tau\cdot\rho(x)=e\cdot\mathrm{ind}^G\,\mathrm{res}_D(x)=e\cdot[G/D]\cdot x.$$

Take any $S\in\mathfrak{X}$. If $S\not\sim_p D$, then

$$(e\cdot[G/D])(S)=e(S)\cdot[G/D](S)=0\,;$$

and if $S\sim_p D$, then

$$(e\cdot[G/D])(S)=e(D)\cdot[G/D](D)=|WD|\not\equiv 0 \pmod p$$

by Lemma 4.13. Thus $\varphi(e\cdot[G/D])$ is invertible in $\varphi(e)\cdot\tilde{\Omega}(G,\mathfrak{X})_{(p)}=\prod_{(S)\sim_p(D)}\boldsymbol{Z}_{(p)}$. See Proposition 4.15. Since $\varphi(e)\cdot\tilde{\Omega}(G,\mathfrak{X})_{(p)}$ is an integral extension of $e\cdot\Omega(G,\mathfrak{X})_{(p)}$, we have that $e\cdot[G/D]$ is also invertible in

$e\cdot\Omega(G,\mathfrak{X})_{(p)}$(cf. Proposition 4.15), and so it is also a unit of $e\cdot\Omega(G,\mathfrak{X})_{(p)}$. Thus $x\longmapsto\tau\cdot\rho(x)$ is an isomorphism.

Next let $y\in f\cdot\Omega(D,\mathfrak{X}_D)^G_{(p)}$. Then by Mackey decomposition and Lemma 6.14, we have that

$$\begin{aligned}\rho\cdot\tau(y)&=\mathrm{res}_D(e\cdot\mathrm{ind}^G(y))\\&=f\cdot\sum_{DgD}\mathrm{ind}^D\,\mathrm{res}_{D(g)}({}^gy)\\&=f\cdot\sum_{DgD}\mathrm{ind}^D\,\mathrm{res}_{D(g)}(y)\\&=f\cdot\sum_{DgD}\mathrm{ind}^D(1_{D(g)})y\\&=f\cdot\mathrm{res}_D([G/D])\cdot y\\&=\mathrm{res}_D(e\cdot[G/D])\cdot y.\end{aligned}$$

As is proved above, $e\cdot[G/D]$ is a unit of $e\cdot\Omega(G,\mathfrak{X})_{(p)}$, and so $\mathrm{res}_D(e\cdot[G/D])$ is a unit of $f\cdot\Omega(D,\mathfrak{X}_D)$. Thus $\rho\cdot\tau$ is also an isomorphism. Hence ρ gives an isomorphism. It is easily checked that the above proof is valid even in the case where $p=0$. □

6.19 Remark. In the case of the usual Burnside rings,

$$\mathrm{res}_D(e^p_{G,D})=e^p_{DD} \qquad (10)$$

holds. In fact, for $T\leq D$,

$$\begin{aligned}\mathrm{res}_D(e^p_{G,D})(T)&=e^p_{G,D}(T)\\&=\begin{cases}1 & \text{if } D^p \text{ is } G\text{-conjugate to } T^p\\0 & \text{otherwise}\end{cases}\\&=\begin{cases}1 & \text{if } D^p \text{ is } D\text{-conjugate to } T^p\\0 & \text{otherwise}\end{cases}\\&=e^p_{D,D}(T)\end{aligned}$$

But the author does not know whether (10) holds in a generalized Burnside ring. Moreover, there remains the problem whether there is an isomorphism as in (9) for $p=\infty$.

In the case of the ordinary Burnside ring of a finite group, there is an important isomorphism:

$$e^p_{G,Q}\cdot\Omega(G)_{(p)}\cong e^p_{N,Q}\cdot\Omega(N)_{(p)}, \qquad (11)$$

where Q is a p-perfect subgroup of G and $N:=NQ$. S. Araki ([Ar 82]) pointed out that this isomorphism is still valid even if $p=\infty$, that is,

$$e^\infty_{G,Q}\cdot\Omega(G)\cong e^\infty_{N,Q}\cdot\Omega(N). \qquad (12)$$

But in the present case, a $\sim_p$-equivalence class may not contain a p-

perfect subgroup Q, and so there is no isomorphisms corresponding to (11), (12).

6.20 Assume the condition $(\mathrm{C})_p$ holds for $\mathfrak{X}$. Let $\mathfrak{Y}$ be a subset of $\mathfrak{X}$ closed under G-conjugation. Define

$$\mathfrak{H}_p\mathfrak{Y} := \{S\in\mathfrak{X} \mid S \sim_p T \text{ for } \exists \mathrm{T}\in\mathfrak{Y}\}; \tag{13}$$
$$K(G, \mathfrak{Y}) := \{x\in\Omega(G, \mathfrak{X}) \mid x(T)=0 \ \forall T\in\mathfrak{Y}\} \tag{14}$$
$$I(G, \mathfrak{H}_p\mathfrak{Y}) := \sum\{\mathrm{ind}^G(\Omega(H, \mathfrak{X}_H)) \mid H\in\mathfrak{H}_p\mathfrak{Y}\}. \tag{15}$$

By the Frobenius reciprocity and the fact that the restriction map is a ring homomorphism, $K(G, \mathfrak{Y})$ and $I(G, \mathfrak{H}_p\mathfrak{Y})$ are both ideals of $\Omega(G, \mathfrak{X})$.

Then the following Dress induction theorem holds:

6.21 THEOREM (Dress induction theorem [Dr 71a]). *Let p be a prime, 0 or ∞.*

$$|G|_{p'}\Omega(G, \mathfrak{X})\subseteq K(G, \mathfrak{Y})+I(G, \mathfrak{H}_p\mathfrak{Y}), \tag{16}$$

In particular,

$$\Omega(G, \mathfrak{X})=K(G, \mathfrak{Y})+I(G,\mathfrak{H}\mathfrak{Y}), \tag{17}$$

where

$$\mathfrak{H}\mathfrak{Y}:=\{S\in\mathfrak{X} \mid S \sim_\infty T \text{ for } \exists T\in\mathfrak{Y}\}. \tag{18}$$

PROOF. Present the identity element of $\Omega(G, \mathfrak{X})_{(p)}$ as the sum of primitive idempotents:

$$1=\sum_{(D)\in C(\mathfrak{X})/\sim_p} e^p_{G,D},$$

where (D) runs over a complete set of representatives of $\sim_p$-equivalence classes on $C(\mathfrak{X})$. The idempotent formula (Theorem 4.2) implies that $|G|_{p'}\, e^p_{G,D}$ belongs to $\Omega(G, \mathfrak{X})$. Thus by (8),

$$\begin{aligned} |G|_{p'}\, e^p_{G,D}\notin K(G, \mathfrak{X}) &\Longleftrightarrow |G|_{p'}\, e^p_{G,D}(T)\neq 0 \text{ for some } T\in\mathfrak{Y} \\ &\Longleftrightarrow T \sim_p D \text{ for some } T\in\mathfrak{Y} \\ &\Longleftrightarrow D\in\mathfrak{H}_p\mathfrak{Y}. \end{aligned}$$

On the other hand, when $D\in\mathfrak{H}_p\mathfrak{Y}$, Lemma 4.11 implies that $|G|_{p'}\, e^p_{G,D}\in \mathrm{ind}^G(\Omega(D, \mathfrak{X}_D))$. Thus

$$|G|_{p'}\in K(G, \mathfrak{Y})+I(G, \mathfrak{H}_p\mathfrak{Y}).$$

Since $K(G, \mathfrak{Y})$ and $I(G, \mathfrak{H}_p\mathfrak{Y})$ are both ideals of $\Omega(G, \mathfrak{X})$, the theorem is proved. This proof is valid also when $p=\infty$. □

6.22 REMARK. (a) The above Dress induction theorem is useful even for $\boldsymbol{Q}\otimes\Omega(G,\mathfrak{X})$. For example, applying Dress induction theorem via the canonical ring homomorphism π from $\Omega(G)$ to the representation ring $R(G)$, we have Artin's induction theorem.

(b) Similarly, applying (17) to the representation ring $R(G)$, we have a so-called hyper-elementary induction theorem (cf [Yo 83a]). In the case where G is a symmetric group S_n and $\mathfrak{X}=\mathfrak{Y}=\mathfrak{H}\mathfrak{Y}$, the set of Young subgroups, (17) holds because any Young subgroup is $\sim_\infty$-equivalent to G.

(c) The above proof of Dress induction theorem is a modification of the proof in [Yo 83a]. We can prove it by Dress' way using prime ideals, too (cf. [Dr 71a]).

6.23 Let N be a normal subgroup of G, For the family $\mathfrak{X}$, define a family $\mathfrak{X}/N$ of G/N by

$$\mathfrak{X}/N := \{S/N \mid N\subseteq S\in\mathfrak{X}\}.$$

Then we have the following maps:

$$\begin{array}{llcl}
\mathrm{fix} & : \quad \Omega(G,\mathfrak{X})_{(p)} & \longrightarrow & \Omega(G/N,\mathfrak{X}/N)_{(p)} \\
 & : \quad [G/A] & \longmapsto & \begin{cases}[(G/N)/(A/N)] & \text{if } N\subseteq A \\ 0 & \text{otherwise};\end{cases} \\
\mathrm{fix}^{\sim} & : \quad \tilde{\Omega}(G,\mathfrak{X})_{(p)} & \longrightarrow & \tilde{\Omega}(G/N,\mathfrak{X}/N)_{(p)} \\
 & : \quad (\chi(S))_{(S)\in C(\mathfrak{X})} & \longmapsto & (\chi(S))_{(S/N)\in C(\mathfrak{X}/N)}; \\
\mathrm{inf} & : \Omega(G/N,\mathfrak{X}KN)_{(p)} & \longrightarrow & \Omega(G,\mathfrak{X})_{(p)} \\
 & : [(G/N)/(S/N)] & \longmapsto & [G/S].
\end{array}$$

Clearly, fix$^{\sim}$ is an extension of fix, and if $\Omega(G,\mathfrak{X})_{(p)}$ is a generalized Burnside ring, then fix$^{\sim}$ and fix are ring homomorphisms.

6.24 PROPOSITION. *Assume the following condition*:

$$S\in\mathfrak{X} \implies \overline{SN}\in\mathfrak{X}. \tag{19}$$

Then inf $:\Omega(G/N)_{(p)}\longrightarrow\Omega(G,\mathfrak{X})_{(p)}$ *can be extended to*

$$\begin{array}{lcl}
\mathrm{inf}^{\sim}: \tilde{\Omega}(G/N,\mathfrak{X}/N)_{(p)} & \longrightarrow & \tilde{\Omega}(G,\mathfrak{X})_{(p)} \\
: (\theta(S/N))_{(S/N)\in\mathfrak{X}/N} & \longmapsto & (\theta(\overline{SN}/N))_{(S)\in\mathfrak{X}}.
\end{array}$$

If furthermore $\Omega(G,\mathfrak{X})_{(p)}$ is a generalized Burnside ring, then inf *is a ring homomorphism.*

PROOF. To prove that inf$^{\sim}$ is an extension of inf, let $S/N\in\mathfrak{X}/N$ and $T\in\mathfrak{X}$. Then by Lemma 3.8,

$$\begin{aligned}\varphi_T(\inf([(G/N)/(S/N)])) &= \varphi_T([G/S]) = [G/S](T)\\ &= [G/S](\overline{TN})\\ &= [(G/N)/(S/N)](\overline{TN}/N)\\ &= \inf^{\sim}([G/N)/(S/N)])_T.\end{aligned}$$

Thus $\inf^{\sim}$ is an extension of inf. Since $\inf^{\sim}$ is clearly a ring homomorphism, the remainder follows. □

6.25 The maps $\inf^{\sim}$ and inf are called *inflations*. The maps $\mathrm{fix}^{\sim}$ and fix are called *fixed-point homomorphisms*.

Under the condition (19), we have another map which is called an *orbit homomorphism*:

$$\begin{aligned}\mathrm{orb}: \Omega(G,\mathfrak{X})_{(p)} &\longrightarrow \Omega(G/N, \mathfrak{X}/N)_{(p)}\\ : \ [G/S] &\longmapsto [G/\overline{SN}].\end{aligned}$$

This map can not be extended to a linear map from $\tilde{\Omega}(G, \mathfrak{X})_{(p)}$ in general.

In the case of the ordinary Burnside rings, we have a multiplicative induction map

$$\mathrm{jnd}: \Omega(H) \longmapsto \Omega(G).$$

See [Yo 90]. However, it is an open question that under what conditions such multiplicative induction maps between generalized Burnside rings exist.

6.26 COROLLARY. *Let N be a normal subgroup of G, and assume that the condition* (19) *holds. Let S be an elemnt of $\mathfrak{X}$ which does not contain N. Then*

$$\sum_{T\in\mathfrak{X}} \mu_{\mathfrak{X}}(S, T) = 0.$$

PROOF. Extend the maps $\inf^{\sim}$ and $\mathrm{fix}^{\sim}$ to $\boldsymbol{Q}\otimes\Omega(G, \mathfrak{X}) \cong \boldsymbol{Q}\otimes\tilde{\Omega}(G, \mathfrak{X})$. Then these maps are ring homomorphisms and they preserve an identity element

$$1 = \sum_{(S)\in C(\mathfrak{X})} \frac{1}{|\mathrm{WS}|} \sum_{T\in(\mathfrak{X})} \mu_{\mathfrak{X}}(S, T)[G/S].$$

Thus the statement follows directly from the following equality:

$$\inf^{\sim}\, \mathrm{fix}^{\sim}([G/S]) = \begin{cases}[G/S] & \text{if } N\subseteq S\\ 0 & \text{otherwise.}\end{cases}$$

□

6.27 As an example for this corollary, assume that $\mathfrak{X}$ is a family of p-subgroups of G satisfying the condition $(\mathrm{C})_p$ and that S is a p-subgroup. Then the assumption of the corollary holds. In the meaning of Brown-Thévenaz paper [BT 88], such an N is a cone point of the poset $\mathfrak{X}$. Furthermore, if we use the notation of Euler characteristics, the above equality means that

$$\chi(\mathfrak{X}_{>S})=1.$$

See 4.5

6.28 Let $(G, \mathfrak{X})$ and $(H, \mathfrak{Y})$ be pairs of a finite group and a family of subgroups closed under conjugation. Assume that the condition $(\mathrm{C})_p$ holds for both pairs. In this section, we studied maps between generalized Burnside rings induced from group homomorphisms. But under some conditions, a map $\alpha: \mathfrak{X} \longrightarrow \mathfrak{Y}$ which may not necessarily come from a group homomorphism can induce a linear map from $\Omega(H, \mathfrak{Y})_{(p)}$ to $\Omega(G, \mathfrak{X})_{(p)}$.

Assume that $\alpha({}^gS)$ and $\alpha(S)$ are H-conjugate for all $S \in \mathfrak{X}$ and $g \in G$. Then α induces a ring homomorphism

$$\begin{aligned} \tilde{\alpha}: \tilde{\Omega}(H, \mathfrak{Y})_{(p)} &\longrightarrow \tilde{\Omega}(G, \mathfrak{X})_{(p)} \\ :(\theta(T))_{(T) \in \mathfrak{Y}} &\longmapsto (\theta(\alpha(S)))_{(S) \in \mathfrak{X}} \end{aligned}$$

Furthermore assume that $\tilde{\alpha}$ induces a linear map

$$\bar{\alpha}: \mathrm{Obs}(H, \mathfrak{Y})_{(p)} \longrightarrow \mathrm{Obs}(G, \mathfrak{X})_{(p)}.$$

This condition can be written in term of α, but such a formula is complicated and not practical for a non-abelian H. If we use the second fundamental theorem (Theorem 8.3) which will be stated in the next section, we obtain a simpler formula but containing Möbius functions.

Anyway, there is no useful criterion by which we can decide whether α induce a linear map

$$\alpha^*: \Omega(H, \mathfrak{Y}) \longrightarrow \Omega(G, \mathfrak{X}).$$

We will only give two examples of unusual maps between generalized Burnside rings.

6.29 Example. The first extraordinary example is the *Frobenius-Wielandt homomorphism* $f: \Omega(G_{|G|}) \longrightarrow \Omega(G)$, where C_n denotes a cyclic group of order n, which was given in Dress-Siebeneicher-Yoshida [DSY 90]. The map f maps a transitive $C_{|G|}$-set $[C_{|G|}/C_n]$ to a *Frobenius element* χ_n, where

$$\chi_n(S)=\begin{cases}|G|/n & \text{is } n \text{ divides } |S|\\ 0 & \text{otherwise.}\end{cases}$$

Although the map

$$\alpha : \mathrm{Sub}(G) \longrightarrow \mathrm{Sub}(C_{|G|}) : H \longmapsto C_{|H|} \ (\leq C_{|G|})$$

does not come from any group homomorphism, α induces the ring homomorphism f. The proof of this fact needs either the Frobenius theorem:

$$\{g \in G | g^n = 1\} \equiv 0 \pmod{n}$$

for a divisor n of $|G|$ or another argument about the *Wielandt element* $\wedge^n(G/1)$.

Furthermore, if there is a ring homomorphism $\rho : \Omega(G) \longmapsto \Omega(G, \mathfrak{X})$, for example, if $\langle g \rangle S \in \mathfrak{X}$ for any $S \in \mathfrak{X}$, $g \in WS$, then we have a ring homomorphism

$$f : \Omega(C_{|G|}) \longrightarrow \Omega(G, \mathfrak{X}).$$

In the next section, we will apply this homomorphism to get some congruences of Frobenius type.

6.20 Example. Another interesting and well-known example of unusual ring homomorphisms between generalized Burnside rings appears together with the character ring of a symmetric group. The details will be treated in the next section. See also [Dr 86], [JK 81]. Let S_n be the symmetric group of degree n and $\mathfrak{Y}$ the set of Young subgroups in S_n. Then the generalized Burnside ring $\Omega(S_n, \mathfrak{Y})$ is isomorphic to the representation ring (the ordinary character ring) $R(S_n)$ (Refer to [Dr 86]). On the other hand, there is a canonical ring homomorphism

$$\pi : \Omega(S_n) \longrightarrow R(S_n)$$

which assigns to the class of each S_n-set $[X]$ the associated permutation character π_X. Thus we have a surjective ring homomorphism from $\Omega(S_n)$ to $\Omega(S_n, \mathfrak{Y})$. But this map does not come from the canonical inclusion $\iota : \mathfrak{Y} \hookrightarrow \mathrm{Sub}(S_n)$, because $\pi_X(g) \neq [X](\overline{\langle g \rangle})$ for $X = S_n/H$ with $H \notin \mathfrak{Y}$.

7 Symmetric groups

7.1 Let $G := S_n$ be the symmetric group of degree n. Let $\mathfrak{Y}$ be the set of Young subgroups, where $Y(\pi)$ is called a Young subgroup with respect to a partition $\pi = \{\pi_1, \pi_2, \cdots\}$ of $\{1, \cdots, n\}$ if

$$Y(\pi)=\{\sigma\in G|\sigma(\pi_i)=\pi_i\}.$$

Thus $Y(\pi)$ is isomorphic to the direct product of the symmetric groups $\mathrm{Sym}(\pi_i)$ on π_i, $i=1,2,\cdots$. Then the family $\mathfrak{Y}$ is closed under G-conjugation and intersection, and G itself belongs to $\mathfrak{Y}$, and so we have a generalized Burnside ring $\Omega(G,\mathfrak{Y})$ as a subring of $\Omega(G)$. Note that two Young subgroups $Y(\pi)$ and $Y(\pi')$are conjugate if and only if the types of π and π' are coincident.

Let $\mathfrak{C}$ be a set of all cyclic subgroups of G. Then the map

$$\alpha:\mathfrak{C}\longrightarrow\mathfrak{Y}:C\longmapsto\bar{C}:=\cap\{Y\in\mathfrak{Y}|C\subseteq Y\}$$

induces an isomorphism

$$\tilde{\alpha}:\tilde{\Omega}(G,\mathfrak{Y})\xrightarrow{\cong}\tilde{\Omega}(G,\mathfrak{C}),$$

and $g\longmapsto\langle g\rangle$ induces a bijection between the set of conjugacy classes of G and $C(\mathfrak{C})$.

Thus we have a commutative diagram as follows:

$$\begin{tikzcd}
\Omega(G,\mathfrak{Y}) \arrow[r, hook] \arrow[d, "\varphi"'] & \Omega(G) \arrow[r, "\pi"] & R(G) \arrow[d, "\nu"] \\
\tilde{\Omega}(G,\mathfrak{Y}) \arrow[r, "\cong"] & \tilde{\Omega}(G,\mathfrak{C}) \arrow[r, "\cong"] & \tilde{R}(G),
\end{tikzcd}$$

where $R(C)$ and $\tilde{R}(G)$ are the ring of virtual characters and the ring of integral valued class functions, respectively, the map ν is the canonical injection, the map $\pi:[X]\longmapsto\pi_X$ assigns to $[X]$ the permutation character π_X. Let π' be the restriction of π to $\Omega(G,\mathfrak{Y})$. Then the following well-known fact follows easily from our theory. Refer to [HK 81].

7.2 PROPOSITION (Classical result). *The above map*

$$\pi':\Omega(G,\mathfrak{Y})\longrightarrow R(G):[X]\longmapsto\pi_X$$

is an isomorphism of rings. Thus in particular, each irreducible character of G is an integral linear combination of permutation characters induced from Young subgroups.

PROOF. We identify $\tilde{\Omega}(G,\mathfrak{Y})$, $\tilde{\Omega}(G,\mathfrak{C})$ and $\tilde{R}(G)$, so that $\mathrm{Im}\,\varphi\subseteq\mathrm{Im}\,\nu$ by the commutativity of the above diagram. The injectivity of φ implies the injectivity of π'. So in order to prove the surjectivity of π', it will suffice to show that $|\mathrm{Cok}\,\varphi|=|\mathrm{Cok}\,\nu|$. Let $\lambda=1^{\lambda_1}\cdots n^{\lambda_n}$ be a partition and Y_λ the Young subgroup corresponding to λ. Then $|WY_\lambda|=\prod_i\lambda_i!$, and so

$$|\mathrm{Cok}\ \varphi| = \prod_{(Y)\in C(\mathfrak{Y})} |WY| = \prod_{\lambda}\prod_{i} \lambda_i\,!. \tag{1}$$

On the other hand, let X be the character table of G. If x_λ be an element of the conjugation class of G corresponding to a partition $\lambda = 1^{\lambda_1}\cdots n^{\lambda_n}$, then $|C_G(x_\lambda)| = \prod_i i^{\lambda_i}\,\lambda_i\,!$. Thus by the orthognal relation, we have that

$$\begin{aligned}|\mathrm{Cok}\ \nu|^2 &= (\det X)^2 = \prod_{(x)} |C_G(x)| \\ &= \prod_{\lambda}\prod_{i} i^{\lambda_i}\lambda_i\,!. \end{aligned} \tag{2}$$

Then by (1) and (2), we can easily prove that Cok φ and Cok ν have the same order. □

7.3 REMARK. As is shown by A. Dress [Dr 86], a necessary and sufficient condition for which a class function χ of $G = S_n$ is an irreducible character is that it is contained in the kernels of the Cauchy-Frobenius homomorphisms and its inner product is 1.

8 Applications to congruences

8.1 EXAMPLE. As is well-known, the theory of ordinary Burnside rings gives some congruences about subgroup lattices of a finite group, for example,

(1)Sylow's third theorem (Wagner [Wa 70]):

$$|\mathrm{Syl}_p(G)| \equiv 1 \pmod{p}.$$

(2)Its generalization by Frobenius (Wagner [Wa 70]): For a prime power divisor p^k of $|G|$,

$$\#\{H \leq G \mid |H| = p^k\} \equiv 1 \pmod{p}.$$

(3)Frobenius theorem (Wagner [Wa 70]): For a divisor n of $|G|$,

$$\#\{g \in G \mid g^n = 1\} \equiv 0 \pmod{n}.$$

(4)Brown's homological Sylow's theorem (Brown [Br 75], Quillen [Qu 78], Gluck [Gl 81], Yoshida [Yo 83a]):

$$\chi(\{p\text{-subgroup } (\neq 1) \text{ of } G\}) \equiv 1 \pmod{|G|_{p'}},$$

where χ denotes the Euler characteristic of the poset of nontrivial p-subgroups of G.

See [DSY 90], [BT 88] for other examples. In general, for a finite poset P, the Euler characteristic $\chi(P)$ of P is defined by

$$\chi(P) := \sum_{x,y\in P} \mu(x, y).$$

8.2 We will first rewrite the fundamental theorem in another form. Note that the assumption that $\Omega(G, \mathfrak{X})$ is a generalized Burnside ring is not necessary.

For any family $\mathfrak{X}$ of subgroups of G closed under G-conjugation, we define the *second Cauchy-Frobenius homomorphism* by

$$\begin{aligned} \psi' = (\psi'_S) : \tilde{\Omega}(G, \mathfrak{X}) &\longrightarrow \mathrm{Obs}(G, \mathfrak{X}) \\ \psi'_S(\chi) := \sum_{T\in\mathfrak{X}} \mu_{\mathfrak{X}}(S, T)\chi(T) &\quad \mathrm{mod}\ |WS|. \end{aligned}$$

8.3 THEOREM (Second fundamental theorem). *The following sequence of abelian groups is exact*:

$$0 \longrightarrow \Omega(G, \mathfrak{X}) \xrightarrow{\varphi} \tilde{\Omega}(G, \mathfrak{X}) \xrightarrow{\psi'} \mathrm{Obs}(G, \mathfrak{X}) \longrightarrow 0.$$

PROOF. This exact sequence follows from Lemma 2.3 and Corollary 4.3 by the similar way as in the proof of the fundamental theorem. □

8.4 REMARK. Under the condition $(\mathrm{C})_\infty$ of 3.6(or 8.8 as below), this theorem follows from the first fundamental theorem (3.10). In fact, we can present explicitly ψ_S as a linear combination of the second Cauchy-Frobenius maps. Furthermore, if there is an automorphism of $\mathrm{Obs}(G, \mathfrak{X})$, it makes another form of the Cauchy-Frobenius homomorphism with an exact sequence.

8.5 LEMMA. *Assume that the condition* $(\mathrm{C})_\infty$ *holds. Define endomorphisms* $\tilde{\psi}_S$ *and* $\tilde{\psi}'_S$ *of* $\tilde{\Omega}(G, \mathfrak{X})$ *as follows*:

$$\begin{aligned} \tilde{\psi}_S(\chi) &:= \sum_{gS\in WS} \chi(\overline{\langle g\rangle S}); \\ \tilde{\psi}'_S(\chi) &:= \sum_{T\in\mathfrak{X}} \mu_{\mathfrak{X}}(S, T)\chi(T). \end{aligned}$$

Then

$$\frac{\tilde{\psi}_S(\chi)}{|WS|} = \sum_{(R)\in C(\mathfrak{X})} |WS\backslash(G/R)^S| \cdot \frac{\tilde{\psi}'_R(\chi)}{|WR|}.$$

Thus one of the first and second fundamental theorems can be induced from another one.

PROOF. It will suffice to show this equality for $\chi = [G/A]$, where $A \in \mathfrak{X}$. By Corollary 4.3, we have that

$$\tilde{\psi}'_R([G/A])=\begin{cases}|WR| & \text{if } (A)=(R)\\ 0 & \text{otherwise.}\end{cases}$$

On the other hand, the Cauchy-Frobenius lemma implies that

$$\tilde{\psi}_S([G/A])=|WS|\cdot|WS\backslash(G/A)^S|.$$

Thus the equality of the lemma holds for $\chi=[G/A]$. □

8.6 REMARK. Comparing to the first fundamental theorem, the second fundamental theorem has two advantages. The first advantage is that the second fundamental theorem holds without the assumption $(\mathrm{C})_p$. (Under the assumption $(\mathrm{C})_p$, we have a p-local version of the second fundamental theorem, which I am going to study in another paper.) Next, we can obtain the explicit inverse image of any element by the second Cauchy-Frobenius homomorphism stated in the following lemma. Using this lemma, we can obtain an explicit formula for the homomorphism between obstruction groups induced by restriction maps. Excepting the case of the usual Burnside ring of an abelian group, the author has no such a simple formula for the ordinal Cauchy-Frobenius homomorphism like the second one in general.

8.7 LEMMA. *Let $\theta'=(\theta'(T) \bmod |WT|)$ be an element of* $\mathrm{Obs}(G, \mathfrak{X})$. *Define an element $\theta=(\theta(S))$ of $\tilde{\Omega}(G, \mathfrak{X})$ by*

$$\theta(T):=\sum_{R\supseteq T}\theta'(R),$$

where R runs over elements of $\mathfrak{X}$ containing T. Then $\psi'(\theta)=\theta'$.

PROOF. Define $\zeta(S, T):=1$ if $S\subseteq T$, $:=0$ otherwise for $S, T\in\mathfrak{X}$. Then for any $S\in\mathfrak{X}$,

$$\begin{aligned}\psi'_S(\theta)&=\sum_{T\in\mathfrak{X}}\mu_{\mathfrak{X}}(S, T)\theta(T)\\ &=\sum_{T, R\in\mathfrak{X}}\mu_{\mathfrak{X}}(S, T)\zeta(T, R)\theta'(R)\\ &=\sum_{R\in\mathfrak{X}}\delta(S, R)\theta'(R)\\ &=\theta'(S),\end{aligned}$$

where δ is Kronecker's delta. This proves the lemma. □

8.8 In this section, we will give some generalizations of the above congruences. We use frequently the notation related with the condition $(\mathrm{C})_p$ (cf. 3.6), where p is a prime or ∞ which makes $\Omega(G, \mathfrak{X})_{(p)}$ a generalized Burnside ring. So we write down them again. $\bar{H}$ denotes the intersection of all $S \in \mathfrak{X}$ containing H.

$(\mathrm{C})_p$ $\quad gS \in (WS)_p,\ S \in \mathfrak{X} \implies \overline{\langle g \rangle S} \in \mathfrak{X},$

where $(WS)_p$ is a Sylow p-subgroup of WS. The condition $(\mathrm{C})_\infty$ is interpreted as follows:

$(\mathrm{C})_\infty$ $\quad gS \in WS,\ S \in \mathfrak{X} \implies \overline{\langle g \rangle S} \in \mathfrak{X}.$

The equivalenc relation $\sim_p$ is generated by the relation

$$S \sim_p \overline{\langle g \rangle S} \quad \text{for any } S \in \mathfrak{X},\ gS \in (WS)_p.$$

We interpret as $(WS)_\infty = WS$.

Furthermore, remember the Cauchy-Frobenius homomorphisms. Their S-components are given by

$$\psi_S^{(p)}(\chi) = \sum_{gS \in (WS)_p} \chi(\overline{\langle g \rangle S}) \pmod{|WS|_p},$$
$$\psi_S^{(\infty)}(\chi) = \sum_{gS \in WS} \chi(\overline{\langle g \rangle S}) \pmod{|WS|}.$$

In the following proposition, we list some technics proving congruences like ones of 8.1.

8.9 PROPOSITION. *Let $S \in \mathfrak{X}$ and $x \in \Omega(G, \mathfrak{X})$. Then the following hold*:

(a) $$\sum_{T \in \mathfrak{X}} \mu_{\mathfrak{X}}(S, T) x(T) \equiv 0 \pmod{|WS|}.$$

(b) *Under the condition $(\mathrm{C})_p$ for a prime p,*

$$\sum_{gS \in (WP)_p} x(\overline{\langle g \rangle S}) \equiv 0 \pmod{|WS|_p}.$$

(c) *Under the condition $(\mathrm{C})_\infty$,*

$$\sum_{gS \in WS} x(\overline{\langle g \rangle S}) \equiv 0 \pmod{|WS|}.$$

PROOF. (a) follows from the second fundamental theorem (Theorem 8.3). (b) and (c) follow from the first fundamental theorem (Theorem 3.10). $\square$

8.10 COROLLARY. *Assume the condition $(\mathrm{C})_p$. Let $x \in \Omega(G, \mathfrak{X})$ and $D, S \in \mathfrak{X}$. Then*

$$\sum_{T \sim_p D} \mu_{\mathfrak{X}}(S, T) x(T) \equiv 0 \pmod{|WS|_p},$$

where T runs over all subgroups T in $\mathfrak{X}$ such that $T \sim_p D$.

PROOF. Let $e_D^{\mathfrak{F}}$ be the idempotent of $\Omega(G, \mathfrak{X})_{(p)}$ corresponding to D. Then $e_D^{\mathfrak{F}}(S)=1$ if $S \sim_p D, =0$ otherwise. By the idempotent formula (4.11) or from the fundamental theorem, $|G|_{p'}\ e_D^{\mathfrak{F}}$ belongs to $\Omega(G, \mathfrak{X})$. Thus we can apply (a) of the above proposition to $|G|_{p'}\ e_D^{\mathfrak{F}} \cdot x \in \Omega(G, \mathfrak{X})$. $\square$

8.11 COROLLARY. *Let p be a prime or ∞. Define a linear map $\psi'^{(p)}$ by*

$$\psi'^{(p)} = (\psi'^{(p)}_S) : \tilde{\Omega}(G, \mathfrak{X})_{(p)} \longrightarrow \mathrm{Obs}(G, \mathfrak{X})_{(p)},$$

where

$$\psi'^{(p)}_S(\chi) = \sum_{T \sim_p S} \mu_{\mathfrak{X}}(S, T)\chi(T) \mod |WS|_p.$$

Then the following sequence of abelian groups is exact:

$$0 \longrightarrow \Omega(G, \mathfrak{X})_{(p)} \xrightarrow{\varphi^{(p)}} \tilde{\Omega}(G, \mathfrak{X})_{(p)} \xrightarrow{\psi'^{(p)}} \mathrm{Obs}(G, \mathfrak{X})_{(p)} \longrightarrow 0.$$

PROOF. The fact that $\psi'^{(p)}\varphi^{(p)}=0$ follows from Comollary 8.10. Furthermore, $\psi'^{(p)}$ is surjective. Thus this corollary follows from the first fundamental theorem (Theorem 3.10) or Lemma 3.3. $\square$

8.12 THEOREM. *Let S be any element of $\mathfrak{X}$. Assume that the condition* (C)$_p$ *holds. Then*

$$\sum_{T \in \mathfrak{X}} \mu_{\mathfrak{X}}(S, T) \equiv 0 \pmod{|WS|_p}.$$

In particular, if the condition (C)$_\infty$ *holds, then*

$$\sum_{T \in \mathfrak{X}} \mu_{\mathfrak{X}}(S, T) \equiv 0 \pmod{|WS|}.$$

PROOF. Under (C)$_p$, the module $\Omega(G, \mathfrak{X})$ becomes a generalized Burnside ring with an *identity element* 1. So the theorem follows from Proposition 8.9 (a). Another short proof is given by Corollary 4.4. $\square$

8.13 COROLLARY (Brown-Thévenaz[BT 88], Corollary 2.2). *Let H be a proper subgroup of G such that $G=H \cdot O^p(G)$, where $O^p(G)$ is the smallest normal subgroup of G by which quotient group $G/O^p(G)$ is a p-group. Let μ be the Möbius function of the subgroup lattice of G. Then*

$$\mu(H, G) \equiv 0 \pmod{|WH|_p}.$$

PROOF. Let $\mathfrak{X}$ be the family of proper subgroups of G containing some conjugates of H. Then $\mathfrak{X}$ satisfies the condition (C)$_p$. Furthermore, $\mu_{\mathfrak{X}}(H, T)=\mu(H, T)$ for any $T \in \mathfrak{X}$, and so the therem yields that

$$\mu(H, G)=-\sum_{T\in\mathfrak{X}}\mu(H, T)\equiv 0 \pmod{|WH|_p}.$$

□

In their paper [BT 88], Brown and Thévenaz called a subgroup H such that $G=HO^p(G)$ a *p-perfect* mod H.

8.14 THEOREM. *Assume the condition* $(C)_p$. *Then for any* $D, S\in\mathfrak{X}$,

$$\sum_{T\sim_p D}\mu_{\mathfrak{X}}(S, T)\equiv 0 \pmod{|WS|_p}.$$

PROOF. This congruence follows from Corollary 4.11 or Corollary 8.10. □

8.15 THEOREM (Weak Frobenius theorem). *Assume the condition* $(C)_\infty$. *Let p be a prime and $D, S\in\mathfrak{X}$. Then*

$$\#\{gS\in WS \mid \overline{\langle g\rangle S}\sim_p D\}\equiv 0 \pmod{|WS|_p}.$$

PROOF. This proof is based on the idea of Dress ([DY 90]). Let $e_D^{\mathfrak{X}}$ be the idempotent corresponding to D, so that $e_D^{\mathfrak{X}}(S)=1$ if $S\sim_p D, =0$ otherwise. By Lemma 4.11, $|G|_{p'}e_D^{\mathfrak{X}}$ belongs to $\Omega(G, \mathfrak{X})$. Thus by applying the Cauchy-Frobenius homomorphism $\psi^{(\infty)}$ to this element, we have that

$$\begin{aligned}\sum_{gS\in WS}|G|_{p'}\, e_D^{\mathfrak{X}}(\overline{\langle g\rangle S}) &= |G|_{p'}\#\{gS\in WS \mid \langle g\rangle S\sim_p D\}\\ &\equiv 0 \pmod{|WS|}.\end{aligned}$$

□

8.16 THEOREM. *Let p be a prime or ∞ and n a divisor of $|G|$. Assume that the family $\mathfrak{X}$ satisfies the following condition*:

$$S\in\mathfrak{X},\ gS\in(WS)_p \implies \langle g\rangle S\in\mathfrak{X}. \tag{1}$$

Let $S, D\in\mathfrak{X}$. Then the following hold:

(a) $$\sum_{|T|\,|\,n}\mu_{\mathfrak{X}}(S, T)\equiv 0 \mod \frac{n|WS|}{|G|},$$

where T runs over subgroups in $\mathfrak{X}$ of which order is divisible by n.

(b) $$\sum_{\substack{T\sim_p D\\ |T|\,|\,n}}\mu_{\mathfrak{X}}(S, T)\equiv 0 \mod \frac{n_p|WS|_p}{|G|_p}.$$

PROOF. Under the above assumption (1), there is a ring homomorphism $\rho:\Omega(G)\longrightarrow\Omega(G, \mathfrak{X})$ given by Corollary 6.4. On the other hand, let

χ_n be the Frobenius element of $\Omega(G)$ (cf. Lema 2.19), that is, $\chi_n(S)=|G|/n$ if n divides $|G|$, and $=0$ otherwise. Apply Proposition 8.9 and Corollary 8.10 to $\rho(\chi_n)$, so (a) and (b) follow. $\square$

8.17 REMARK. The author expects a congruence which contains both of Theorem 8.15 and Theorem 8.16. Theorem 8.15 is a generalization of the following weak Frobenius theorem:

$$\#\{p\text{-element of } G\}\equiv 0 \pmod{|G|_p}.$$

See Remark 2.10. Theorem 8.15 needed only the condition $(\mathrm{C})_\infty$. Under the assumption $(\mathrm{C})_\infty$, we have not yet obtained a generalization of the Frobenius theorem which states that if n a divisor of $|G|$, then

$$\#\{g\in G\,|\,g^n=1\}\equiv 0 \pmod{n}.$$

On the other hand, the congruences in Theorem 8.16 which were proved under the stronger assumption (1) imply the Frobenius theorem.

9 Mackey functors and the generalized Hecke ring

In this section, we will study Mackey functors and the generalized Hecke ring with respect to a family $\mathfrak{X}$ of a finite group G closed under intersection. It is possible to make a similar theory to this section under the assumption $(C)_\infty$ or $(C)_p$. But we will not consider the general case for the two reasons that the theory under the condition $(\mathrm{C})_p$ is complicated to state although it is essentially equal to the theory in this section and that such a general theory has only a few interesting applications. Because the proofs of many results are similar as in usual generalized Burnside rings and abstract Burnside rings (cf. [Yo 87a]), we give only outline of them. Refer to [Dr 73], [Yo 87b] for the theory of Mackey functors.

9.1 Throughout this section, we assume that $\mathfrak{X}$ is a family of subgroups of G satisfying the following condition:

$$S\in\mathfrak{X},\ g\in G \implies {}^gS\in\mathfrak{X}, \tag{1}$$
$$S,\ T\in\mathfrak{X} \implies S\cap T\in\mathfrak{X}. \tag{2}$$

Remember that a $(G,\mathfrak{X})$-set X is a G-set such that the stabilizer G_x for each $x\in X$ is a member of $\mathfrak{X}$. We denote by $\boldsymbol{Set}_f(G,\mathfrak{X})$ the category of $(G,\mathfrak{X})$-sets and G-maps. This category is a subcategory of the category $\boldsymbol{Set}_f^G$ of finite G-sets and closed under disjoint union and finite limits by the above assumption. Of course, the Grothendieck ring of $\boldsymbol{Set}_f(G,\mathfrak{X})$ with respect to disjoint union and cartesian product is coincident with the generalized Burnside ring $\Omega(G, \mathfrak{X})$.

9.2 We use the above notation and terminology. A $(G, \mathfrak{X})$-set *over* a $(G, \mathfrak{X})$-set X is a G-map

$$\alpha: A \longrightarrow X$$

from a $(G, \mathfrak{X})$-set A. A morphism between $(G, \mathfrak{X})$-sets $\alpha: A \longrightarrow X$ and $\beta: B \longrightarrow X$ over X is defined by a G-map f from A to B such that $\alpha = \beta f$. The category of $(G, \mathfrak{X})$-sets over X is denoted by $\boldsymbol{Set}_f(G, \mathfrak{X})/X$. (Sometimes, it is called a comma category.) This category has finite coproducts:

$$(A \longrightarrow X) + (B \longrightarrow X) = (A + B \longrightarrow X)$$

and an initial object $\emptyset \longrightarrow X$.

As an example, let $X = G/H$ for $H \in \mathfrak{X}$. Then we have that

$$\boldsymbol{Set}_f(G, \mathfrak{X})/(G/H) \cong \boldsymbol{Set}_f(H, \mathfrak{X}_H), \tag{3}$$

where $\mathfrak{X}_H$ is the set of members of $\mathfrak{X}$ contained in H. In fact, if $\alpha: A \longrightarrow G/H$ is an object of $\boldsymbol{Set}_f(G, \mathfrak{X})/(G/H)$, that is, a G-map, then $\alpha^{-1}(H) \subseteq A$ is an H-subset of A and the stabilizer of any element of $\alpha^{-1}(H)$ belongs to $\mathfrak{X}_H$. Conversely, if B is an object of $\boldsymbol{Set}_f(H, \mathfrak{X}_H)$, then $A := G \times_H B \longrightarrow G/H$ is an object of $\boldsymbol{Set}_f(G, \mathfrak{X})/(G/H)$. These assignments are both functorial and inverses each other.

9.3 Let $\Omega_{\mathfrak{X}}(X)$ be the Grothendieck group of $\boldsymbol{Set}_f(G, \mathfrak{X})/X$ with respect to coproduct. Then $\Omega_{\mathfrak{X}}(X)$ is generated by elements of the form $[G/S \longrightarrow X]$ with $S \in \mathfrak{X}$. Note that $\alpha: G/S \longrightarrow X$ and $\beta: G/T \longrightarrow X$ are isomorphic to each other as $(G, \mathfrak{X})$-sets over X if and only if there exists an element g of G such that $S = {}^gT (:= gTg^{-1})$ and $\alpha(S) = g\beta(T)$, and so the rank of $\Omega_{\mathfrak{X}}(X)$ is given by

$$\operatorname{rank} \Omega_{\mathfrak{X}}(X) = \sum_{(S) \in C(\mathfrak{X})} |WS \backslash X^S|.$$

By an easy calculation, we have that this number is equal to

$$\operatorname{rank} \Omega_{\mathfrak{X}}(X) = \sum_{x \in G \backslash X} |C(\mathfrak{X} \cap \operatorname{Sub}(G_x))|,$$

where x runs over a complete set of representatives of G-orbits of X and $C(\mathfrak{X} \cap \operatorname{Sub}(G_x))$ denotes the set of G_x-conjugacy classes of subgroups S of G_x with $S \in \mathfrak{X}$.

9.4 We define the *ghost ring* $\tilde{\Omega}_{\mathfrak{X}}(X)$ as the product of some copies of the integer ring $\boldsymbol{Z}$ as follows:

$$\tilde{\Omega}_{\mathfrak{X}}(X) := \prod_{(S)\in C(\mathfrak{X})} \prod_{x\in WS\backslash X^S} \boldsymbol{Z},$$

where x runs over a set of complete representatives of WS-orbits in X^S. Furthermore we define the *obstruction group* $\mathrm{Obs}_{\mathfrak{X}}(X)$ by

$$\mathrm{Obs}_{\mathfrak{X}}(X) := \prod_{(S)\in C(\mathfrak{X})} \prod_{x\in WS\backslash X^S} (\boldsymbol{Z}/|(WS)_x|\boldsymbol{Z}).$$

The *Burnside homomorphism* φ and the *Cauchy-Frobenius* map ψ are defined by

$$\begin{aligned} \varphi=(\varphi_{S,x}) &: \Omega_{\mathfrak{X}}(X) \longrightarrow \tilde{\Omega}_{\mathfrak{X}}(X) \\ &: [A \xrightarrow{\varphi} X] \longmapsto (|A^S \cap \alpha^{-1}(x)|)_{(S,x)}, \\ \psi=(\psi_{S,x}) &: \tilde{\Omega}_{\mathfrak{X}}(X) \longrightarrow \mathrm{Obs}_{\mathfrak{X}}(X) \\ &: (\chi(S,x))_{(S,x)} \longmapsto \Big(\sum_{gS\in(WS)_x} \chi(\overline{\langle g\rangle S}, x)\Big). \end{aligned}$$

Under these notation, the following fundamental theorem for the generalized Burnside ring over X holds:

9.5 THEOREM. *The following sequence of abelian groups is exact*:

$$0 \longrightarrow \Omega_{\mathfrak{X}}(X) \xrightarrow{\varphi} \tilde{\Omega}_{\mathfrak{X}}(X) \xrightarrow{\psi} \mathrm{Obs}_{\mathfrak{X}}(X) \longrightarrow 0.$$

PROOF. The proof is similar as in the fundamental theorem of generalized Burnside ring (Theorem 3.10). So the proof is omitted. See also the proof of the fundamental theorem for abstract Burnside rings in [Yo 87a]. □

9.6 THEOREM. $\Omega_{\mathfrak{X}}(X)$ *has a unique ring structure such that the Burnside homomorphism φ is a ring homomorphism.*

PROOF. This theorem is a corollary of the fundamental theorem and proved by a similar way as in Theorem 3.11. □

9.7 We call the above ring $\Omega_{\mathfrak{X}}(X)$ the *generalized Burnside ring of G with respect to $\mathfrak{X}$ over X.* Clearly, for the terminal object $\mathbf{1}$, we have that the $\Omega_{\mathfrak{X}}(\mathbf{1})$ is isomorphic to the generalized Burnside ring $\Omega(G, \mathfrak{X})$.

9.8 As in Section 6, we put $\mathfrak{X}_H := \mathfrak{X} \cap \mathrm{Sub}(H)$ for $H \in \mathfrak{X}$. Furthermore, we simply put

$$\Omega(H, \mathfrak{X}) := \Omega(H, \mathfrak{X}_H).$$

For any $g \in G$ and $H \in \mathfrak{X}$, the *conjugation* map is defined by

$$\begin{aligned}\mathrm{con}^g : \Omega(H, \mathfrak{X}) &\longrightarrow \Omega({}^gH, \mathfrak{X})\\ : [H/T] &\longmapsto [{}^gH/{}^gT].\end{aligned}$$

9.9 LEMMA. *There are ring isomorphisms as follows:*

$$\begin{aligned}\Omega_{\mathfrak{X}}(X) &\cong (\bigoplus_{x \in X} \Omega(G_x, \mathfrak{X}))^G \\ &\cong \bigoplus_{x \in G\backslash X} \Omega(G_x, \mathfrak{X}),\end{aligned} \tag{4}$$

where $(\cdots)^G$ *denotes the G-fixed point set with respect to the G-action defined by G-conjugation. In particular,*

$$\Omega_{\mathfrak{X}}(G/H) \cong \Omega(H, \mathfrak{X}). \tag{5}$$

The isomorphism (4) *is given by*

$$f : [\alpha : A \longrightarrow X] \longmapsto (\alpha^{-1}(x)).$$

PROOF. The map f defined above can be extended to the ghost rings:

$$\begin{aligned}\tilde{f} :\ \tilde{\Omega}_{\mathfrak{X}}(X) &\longrightarrow (\oplus_{x \in X} \tilde{\Omega}(G_x, \mathfrak{X}))^G \\ : (\chi(S, x))_{S,x} &\longmapsto ((\chi(S, x)_S)_x).\end{aligned}$$

This map is clearly a ring homomorphism. Furthermore, f and $\tilde{f}$ are commutative with the Burnside homomorphisms, and so f is also a ring homomorphism. The inverse image $f^{-1}(([A_x])_x)$ of a G-fixed element $([A_x])_x$ of $\oplus_x \Omega(G_x, \mathfrak{X})$ is given by the natural G-map $[\coprod_x A_x \longrightarrow X]$. Finally, G permutes the direct summands $\Omega(G_x, \mathfrak{X})$, $x \in X$, the second isomorphism is trivial. □

9.10 Let $\lambda : X \longrightarrow Y$ be a G-map between $(G, \mathfrak{X})$-sets X and Y. Then we have a linear map

$$\begin{aligned}\lambda_* :\ \Omega_{\mathfrak{X}}(X) &\longrightarrow \Omega_{\mathfrak{X}}(Y) \\ : [A \longrightarrow X] &\longmapsto [A \longrightarrow X \longrightarrow Y].\end{aligned} \tag{6}$$

This map can be extended to

$$\begin{aligned}\lambda_* :\ \tilde{\Omega}_{\mathfrak{X}}(X) &\longrightarrow \tilde{\Omega}_{\mathfrak{X}}(Y) \\ : (\theta(S, x))_{S,x} &\longmapsto (\sum_{x \in \lambda^{-1}(y)^T} \theta(T, x))_{T,y}.\end{aligned} \tag{7}$$

In fact, let $\theta = \varphi([A \xrightarrow{\alpha} X]) \in \tilde{\Omega}_{\mathfrak{X}}(X)$, so that $\theta(S, x) = |\alpha^{-1}(x)^S|$. On the other hand, for any $T \in \mathfrak{X}$ and $y \in Y^T$,

$$\begin{aligned}(\lambda_*\theta)(T, y) &= |(\lambda\alpha)^{-1}(y)^T| \\ &= |\alpha^{-1}\lambda^{-1}(y)^T| \\ &= \sum_{x\in\lambda^{-1}(y)^T} |\alpha^{-1}(x)^T| \\ &= \sum_{x\in\lambda^{-1}(y)^T} \theta(T, x).\end{aligned}$$

When $X=G/H$, $Y=G/K$ and $\lambda: G/H \longrightarrow G/K$ is a G-map induced by inclusion $H\subseteq K$, the above map λ_* coincides with the induction map $\mathrm{ind}_H^K: \Omega(G, \mathfrak{X}_H) \longrightarrow \Omega(K, \mathfrak{X}_K)$ defined in 6.2.

9.11 As in the last paragraph, let $\lambda: X \longrightarrow Y$ be a G-map between $(G, \mathfrak{X})$-sets X and Y. Then there is a ring homomorphism with inverse direction:

$$\begin{aligned}\lambda^*:\ & \tilde{\Omega}_{\mathfrak{X}}(Y) \longrightarrow \tilde{\Omega}_{\mathfrak{X}}(X) \\ & : (\chi(S, y))_{S,y} \longmapsto (\chi(S, \lambda(x)))_{S,x}.\end{aligned}$$

By the fundamental theorem, the restriction of λ^* into $\Omega_{\mathfrak{X}}(Y)$ gives a ring homomorphism

$$\lambda^*: \Omega_{\mathfrak{X}}(Y) \longrightarrow \Omega_{\mathfrak{X}}(X).$$

When $X=G/H$, $Y=G/K$ and $\lambda: G/H \longrightarrow G/K$ is a G-map induced by inclusion $H\subseteq K$, the above map λ^* coincides with the restriction map in Proposition 6.3.

9.12 LEMMA. *Let X, Y be $(G, \mathfrak{X})$-sets. Then the canonical injections $X \hookrightarrow X+Y \hookleftarrow Y$ induce a bi-product diagram*

$$\Omega_{\mathfrak{X}}(X) \rightleftarrows \Omega_{\mathfrak{X}}(X+Y) \leftrightarrows \Omega_{\mathfrak{X}}(Y).$$

Similarly, $\tilde{\Omega}_{\mathfrak{X}}(X+Y)$ is a bi-product of $\tilde{\Omega}_{\mathfrak{X}}(X)$ and $\tilde{\Omega}_{\mathfrak{X}}(Y)$.

PROOF. The statement that $\tilde{\Omega}_{\mathfrak{X}}(X+Y) \cong \tilde{\Omega}_{\mathfrak{X}}(X) \oplus \tilde{\Omega}_{\mathfrak{X}}(Y)$ follows from the commutativity of λ_* and λ^* with the Burnside homomorphisms. (REMARK: the fibre products of $X \longrightarrow X+Y$ and $Y \longrightarrow X+Y$ is $\emptyset \longrightarrow X+Y$ and the fibre product of $X \longrightarrow X+Y$ and itself is also $X \longrightarrow X+Y$. Thus this lemma follows from the following Mackey decomposition, too.) $\square$

9.13 LEMMA (Mackey decomposition). *Let*

$$\begin{array}{ccc} W & \xrightarrow{p} & X \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & Z \end{array}$$

be a pull-back diagram of $(G, \mathfrak{X})$*-sets and G-maps. Then the maps given in* 9.10 *and* 9.11 *make the following diagram commutative.*

$$\begin{array}{ccc} \Omega_{\mathfrak{X}}(W) & \xrightarrow{p^*} & \Omega_{\mathfrak{X}}(X) \\ {\scriptstyle q^*}\uparrow & & \uparrow{\scriptstyle f^*} \\ \Omega_{\mathfrak{X}}(Y) & \xrightarrow{g^*} & \Omega_{\mathfrak{X}}(Z) \end{array}$$

PROOF. We have to check that

$$f^* g_* = p_* q^*. \tag{8}$$

Let $\theta = (\theta(T, y))_{T,y}$ be an element of $\Omega_{\mathfrak{X}}(Y)$ viewed as an element of $\tilde{\Omega}_{\mathfrak{X}}(Y)$. Then we have that

$$f^* g_*(\theta) = \Big(\sum_{y \in (g^{-1}f(x))^T} \theta(T, y)\Big)_{T,x} \tag{9}$$

$$p_* q^*(\theta) = \Big(\sum_{w \in p^{-1}(x)^T} \theta(T, q(w))\Big)_{T,x}. \tag{10}$$

Let $T \in \mathfrak{X}$ and $x \in X^T$. We may assume that $W = X \times_Z Y$ and p, q are projections into X, Y. Thus

$$\begin{aligned} \sum_{w \in p^{-1}(x)^T} \theta(T, q(w)) &= \sum_{\substack{y \in Y^T \\ :(x,y) \in W}} \theta(T, y) \\ &= \sum_{\substack{y \in (g^{-1}f(x))^T \\ :(x,y) \in W}} \theta(T, y) \\ &= \sum_{y \in (g^{-1}f(x))^T} \theta(T, y). \end{aligned}$$

Here note that if $y \in g^{-1}f(x)^T$, then $(x, y) \in W$. Thus the equality (8) holds by (9). □

9.14 LEMMA (Frobenius reciprocity). *Let* $\lambda: X \longrightarrow Y$ *be a G-map between* $(G, \mathfrak{X})$*-sets. Let* $a \in \Omega_{\mathfrak{X}}(X)$ *and* $b \in \Omega_{\mathfrak{X}}(Y)$. *Then*

$$\lambda_*(a) \cdot b = \lambda_*(a \cdot \lambda^*(b)). \tag{11}$$

Similar equation holds for elements of $\tilde{\Omega}_{\mathfrak{X}}(X)$ *and* $\tilde{\Omega}_{\mathfrak{X}}(Y)$, *too.*

PROOF. It will suffice to show that the equation (11) for $a \in \tilde{\Omega}_{\mathfrak{X}}(X)$ and $b \in \tilde{\Omega}_{\mathfrak{X}}(Y)$ holds. Let $T \in \mathfrak{X}$ and $y \in Y^T$. Then we have that

$$\begin{aligned} \lambda_*(a \cdot \lambda^*(b))(T, y) &= \sum_{x \in \lambda^{-1}(y)^T} a(T, x) \cdot b(T, \lambda(x)) \\ &= \sum_{x \in \lambda^{-1}(y)^T} a(T, x) \cdot b(T, y) \\ &= \lambda_*(a)(T, y) \cdot b(T, y). \end{aligned}$$

This proves the lemma. $\square$

9.15 PROPOSITION. *The correspondence* $X \longmapsto \Omega_{\mathfrak{X}}(X)$ *together with the maps defined in* 9.10 *and* 9.11 *makes a Green functor on the category of* $(G, \mathfrak{X})$*-sets.*

PROOF. By Lemma 9.12 and Lemma 9.13, the above correspondence is a Mackey functor. Furthermore, $\Omega_{\mathfrak{X}}(X)$ has a ring structure and Frobenius reciprocity (Lemma 9.14), and so we have a Green functor. (See [Gr 70], [Dr 73], [Yo 80] for the definition of Mackey functors and Green functors.) $\square$

9.16 We now construct a category which represents Mackey functors. Let $\boldsymbol{Hec}(G, \Omega_{\mathfrak{X}})$ be the category defined by the following data. An object of $\boldsymbol{Hec}(G, \Omega_{\mathfrak{X}})$ is a $(G, \mathfrak{X})$-set. A hom-set is defined and denoted by

$$\Omega_{\mathfrak{X}}(X, Y) := \Omega_{\mathfrak{X}}(X \times Y).$$

Finally, the composition

$$\Omega_{\mathfrak{X}}(Y, Z) \times \Omega_{\mathfrak{X}}(X, Y) \longrightarrow \Omega_{\mathfrak{X}}(X, Z)$$

is defined by the composition as follows. (We simply write $\Omega(XYZ)$ for $\Omega_{\mathfrak{X}}(X \times Y \times Z)$, etc.)

$$\Omega(X, Z) \times \Omega(X, Y) \xrightarrow{\pi_{23}^* \times \pi_{12}^*} \Omega(XYZ) \times \Omega(XYZ)$$
$$\xrightarrow{\text{multi}} \Omega(XYZ) \xrightarrow{\pi_{13*}} \Omega(X, Z)$$

where π_{ij} is the projection map from $(X \times Y \times Z)$ to the i, j-th components.

This category $\boldsymbol{Hec}(G, \Omega_{\mathfrak{X}})$ is called the *Hecke category with coefficient in* $\Omega_{\mathfrak{X}}$. Using only Mackey decomposition and Frobenius reciprocity, it is possible to prove that $\boldsymbol{Hec}(G, \Omega_{\mathfrak{X}})$ becomes really a category. We, however, omit the proof of the associative law because it is simply an easy diagram chase. The identity morphism of X in $\boldsymbol{Hec}(G, \Omega_{\mathfrak{X}})$ is $[\delta : X \longrightarrow X \times X]$, where δ is the diagonal G-map.

9.17 LEMMA. *The category of Mackey functors from* $\boldsymbol{Set}_f(G, \mathfrak{X})$ *is equivalent to the additive functor category* $[\boldsymbol{Hec}(G, \mathfrak{X})^{\mathrm{op}}, \boldsymbol{Mod}_k]$.

PROOF. Let $F : \boldsymbol{Hec}(G, \mathfrak{X})^{\mathrm{op}} \longrightarrow \boldsymbol{Mod}_k$ be an additive functor. For a $(G, \mathfrak{X})$-set X, let $M(X) := F(X)$. For a G-map $\lambda : X \longrightarrow Y$, define

$$\lambda^* := F(\langle 1_X, \lambda\rangle : X \longrightarrow X\times Y)$$
$$\lambda_* := F(\langle \lambda, 1_Y\rangle : X \longrightarrow Y\times X).$$

Then we have a Mackey functor M. Conversely, if M is a Mackey functor, then the assignment

$$X \longmapsto M(X),$$
$$[\langle\lambda, \mu\rangle : A \longrightarrow X\times Y] \longmapsto (M(Y) \xrightarrow{\mu^*} M(A) \xrightarrow{\lambda_*} M(X))$$

gives a contravariant functor from $\boldsymbol{Hec}(G, \mathfrak{X})$. □

9.18 For any finite group G and a commutative ring k, a *permutation kG-module* is defined to be a kG-module isomorphic to the free k-module kX with basis X, where X is a finite G-set. Let $\boldsymbol{Hec}(G, k)$ be the category of permutation kG-modules and kG-maps, so that there is a canonical embedding into the category of finitely generated kG-modules

$$\boldsymbol{Hec}(G, k) \longrightarrow \boldsymbol{Mod}_{kG}.$$

(Remember that the endomorphism ring $\mathrm{End}_{kG}(kX)$ is called the Hecker-ing.) We often use the matrix notation for morphisms in $\boldsymbol{Hec}(G, k)$. Using this notation, a morphism from kY to kX is a G-matrix $(a_{x,y})$, where a matrix $(a_{x,y})$ is called a *G-matrix* if $a_{gx,gy}=a_{x,y}$ for any $g\in G$. The G-map corresponding to the G-matrix $a_{x,y}$ is

$$f : kY \longrightarrow kX : y(\in Y) \longrightarrow \sum_{x\in X} a_{x,y}x.$$

The category $\boldsymbol{Hec}(G, k)$ is equivalent to the Hecke category corresponding to the Green functor $X \longmapsto \mathrm{Ext}^0(kX, k)$:

$$\boldsymbol{Hec}(G, k) \cong \boldsymbol{Hec}(\boldsymbol{Set}_f^G, \mathrm{Ext}^0(k[-], k)).$$

This category is a representation category for cohomological G-functors (cf [Yo 83]). See Yoshida [Yo 87b] for Hecke categories and its applicatin to the theory of block designs.

9.19 For each $S\in\mathfrak{X}$, define an additive functor Φ_S from $\boldsymbol{Hec}(G, \Omega_{\mathfrak{X}})$ to $\boldsymbol{Hec}(WS, \boldsymbol{Z})$ by

$$\Phi : X \longmapsto k[X^S]$$

and on hom-sets

$$\begin{aligned}\Phi_S :\quad & \Omega_{\mathfrak{X}}(X, Y) \longrightarrow (\boldsymbol{Z}[X^S\times Y^S])^{WS}\\ :\ & [\gamma : A \longrightarrow X\times Y] \longmapsto (|\gamma^{-1}(x, y)^S|)_{x,y}.\end{aligned}$$

We define an additive functor Φ by the product of Φ_S's:

$$\Phi=(\Phi_S):\boldsymbol{Hec}(G,\Omega_{\mathfrak{X}})\longrightarrow\prod_{(S)\in C(\mathfrak{X})}\boldsymbol{Hec}(WS,\boldsymbol{Z}).$$

We now have a fundamental theorem for the Hecke category $\boldsymbol{Hec}(G,\mathfrak{X})$ as follows:

9.20 THEOREM (Fundamental theorem for generalized Hecke categories). *The functor Φ is an embedding of categories. For each X, Y, the cokernel of*

$$\Phi:\Omega_{\mathfrak{X}}(X\times Y)\longrightarrow\prod_{(S)\in C(\mathfrak{X})}(\boldsymbol{Z}[X^S\times Y^S])^{WS}$$

is isomorphic to

$$\prod_{(S)\in C(\mathfrak{X})}\ \prod_{(x,y)\in WS\backslash X^S\times Y^S}(\boldsymbol{Z}/|(WS)_{x,y}|\boldsymbol{Z}).$$

PROOF. The injectivity of Φ and the statement on the cokernel of Φ follow from the fundamental theorem 9.5. The fact that Φ maps the composition in $\boldsymbol{Hec}(G,\Omega_{\mathfrak{X}})$ to the matrix multiplication follows from the definition of the composition in $\boldsymbol{Hec}(G,\Omega_{\mathfrak{X}})$. We have to prove that if $\Phi(X)\cong\Phi(Y)$, then $X\cong Y$ in $\boldsymbol{Hec}(G,\Omega_{\mathfrak{X}})$. The isomorphism $\Phi(X)\cong\Phi(Y)$ means that $\boldsymbol{Z}[X^S]\cong\boldsymbol{Z}[Y^S]$ as WS-modules for each $S\in\mathfrak{X}$, and so comparing the ranks, we have that $|X^S|=|Y^S|$. Thus $X\cong Y$ as G-sets, and hence $X\cong Y$ in $\boldsymbol{Hec}(G,\Omega_{\mathfrak{X}})$, too. $\square$

9.21 REMARK. It is not trivial to prove that the associativity for the compositions of morphisms in $\boldsymbol{Hec}(G,\Omega_{\mathfrak{X}})$ defined in 9.16 holds. However it is easier to prove that Φ preserves the composition. Thus we can prove the associative law in the Hecke category $\boldsymbol{Hec}(G,\Omega_{\mathfrak{X}})$ by the injectivity of Φ on hom-sets.

9.22 Let M be a Mackey functor from $\boldsymbol{Set}_f(G,\mathfrak{X})$, so that each component $M(X)$ is an $\Omega_{\mathfrak{X}}(X)$-module by

$$[\lambda:A\longrightarrow X]\cdot m:=\lambda_*\lambda^*(m).$$

Let H, $K\in\mathfrak{X}$. Then the hom-set $\Omega_{\mathfrak{X}}(G/H,G/K)$ is an abelian group generated by

$$\{[H,u,A,K]\,|\,u\in G,\ A\in\mathfrak{X},\ A\subseteq H\cap{}^uK\}$$

with relations

$$[H,huk,{}^hA,K]=[H,u,A,K]\quad\text{for } h\in H,\ k\in K.$$

Let $(\boldsymbol{a}, \mathrm{ind}, \mathrm{res}, \mathrm{con})$ be a G-functor defined on $\mathfrak{X}$. The action of $\Omega_{\mathfrak{X}}(G/H \times G/K)$ on $\boldsymbol{a}(K)$ is written as

$$\begin{aligned}[H, u, A, K] : \boldsymbol{a}(K) &\longrightarrow \boldsymbol{a}(H) \\ : \beta &\longmapsto \mathrm{ind}_H \mathrm{res}_A \mathrm{con}^u(\beta).\end{aligned}$$

The composition of these operators is given by

$$[H, u, A, K] \cdot [K, v, B, L] = \sum_{k \in {}^{u^{-1}}A \backslash K / B} [H, ukv, A \cap {}^{uk}B, L],$$

where k runs over a set of complete representatives of double cosets. (It is very complicated to check the associative law by the above multiplication law.)

9.23 The theory of Hecke categories gives some refinements of congruences is Section 8. As is suggested in Yoshida [Yo 85] (where a Hecke category with respect to the Burnside ring functor Ω_G of G is called a span category), such a congruences contains terms related with p-blocks and Brauer correspondence in modular representation theory. We will state these congruences in another paper. It is very important to be able to treat Hecke categories like as group rings.

For example, the center $Z(\mathfrak{C})$ of a category $\mathfrak{C}$ is the set of endo-natural transformations of the identity functor of $\mathfrak{C}$. So we have the concept of central idempotents of Hecke categories. Similarly, for a subgroup D, the correspondence $X \longmapsto X^D$ induces a functor

$$Br_D : \boldsymbol{Hec}(G, F \otimes \Omega_{\mathfrak{X}}) \longrightarrow \boldsymbol{Hec}(N_G(D), F \otimes \Omega_{\mathfrak{X}}).$$

This functor Br_D that is called a *Brauer functor* plays an essential role in the theory of generalized Hecke category and Mackey functors. This functor induces the *Brauer homomorphism* between the centers of Hecke categories under some conditions.

9.24 Concluding Remark. There are other algebraic structures related with generalized Burnside rings, for example, the unit group of a generalized Burnside ring (cf. [Yo 90]), the generalized monomial Burnside rings (cf. [Dr 71b]), the polynomial and power series rings with coefficient in a generalized Burnside ring (cf. [Yo 91]), and so on. We are especially interested in generalized monomial Burnside rings because the idempotent formula of these rings gives an explicit formula of Brauer induction theorem. See Snaith [Sn 88], [Yo 83a], [Bo 89].

Appendix

A Abstract Burnside rings

In this section, we will give a brief outline of the theory of the abstract Burnside ring of a finite category. The details are found in Yoshida [Yo 87].

Throughout this section Γ denotes a finite category.

A. 1 Let Γ be a finite category. This means that Γ has finite number of morphisms, and so in particular, Γ has finite number of objects and each hom-set $\mathrm{Hom}_\Gamma(a, b)$ is a finite set.

The (finite) set of objects of Γ is denoted by $\mathrm{Obj}(\Gamma)$ or Γ itself. The finite set of morphisms of Γ is denoted by $\mathrm{Mor}(\Gamma)$. For any objects $a, b \in \Gamma$, we put

$$\begin{aligned}
\Gamma(a, b) &:= \mathrm{Hom}_\Gamma(a, b), \\
\langle a, b\rangle &:= |\Gamma(a, b)|, \\
\mathrm{Epi}(a, b) &:= \{\text{epimorphism from } a \text{ to } b\} \\
\mathrm{Mon}(a, b) &:= \{\text{monomorphism from } a \text{ to } b\} \\
\mathrm{Epi}(\Gamma) &:= \{\text{epimorphism in } \Gamma\} \\
\mathrm{Mon}(\Gamma) &:= \{\text{monomorphism in } \Gamma\} \\
\mathrm{Iso}(\Gamma) &:= \{\text{isomorphism in } \Gamma\}.
\end{aligned}$$

For an object a of Γ, we denote by $\mathrm{Aut}\, a$ and $\mathrm{End}\, a$ the automorphism group and the monoid of endomorphisms of a, respectively. $\Gamma/\cong$ is the set of isomorphism classes of objects of Γ.

A. 2 Let $\Omega(\Gamma)$ be the free abelian group $\boldsymbol{Z}[\Gamma/\cong]$ with basis $\Gamma/\cong$. Define the ring $\tilde{\Omega}(\Gamma)$ by

$$\tilde{\Omega}(\Gamma) := \prod_{i \in \Gamma/\cong} \boldsymbol{Z},$$

where i runs over a complete set of representatives of isomorphism classes of Γ. Thus any element χ of $\tilde{\Omega}(\Gamma)$ is identified with the map $\chi : \Gamma \longrightarrow \boldsymbol{Z}$ such that

$$\chi(a) = \chi(b) \quad \text{if } a \cong b.$$

The *Burnside homomorphism* φ is a linear map defined by

$$\begin{aligned}
\varphi = (\varphi_i) : \ &\Omega(\Gamma) \longrightarrow \tilde{\Omega}(\Gamma) \\
&: x(\in \Gamma) \longmapsto (\langle i, x\rangle)_i .
\end{aligned}$$

Let R be a commutative ring. The R-module $R \otimes \Omega(\Gamma)$ is called to be an *abstract Burnside ring* provided

(a) $1_R \otimes \varphi : R \otimes \Omega(\Gamma) \longrightarrow R \otimes \tilde{\Omega}(\Gamma)$ is injective;
(b) The image $\mathrm{Im}(1_R \otimes \varphi)$ is a subring of $R \otimes \tilde{\Omega}(\Gamma)$.

When these conditions hold, $R \otimes \Omega(\Gamma)$ has a unique ring structure by which $1_R \otimes \varphi$ becomes a ring homomorphism.

A. 3 A *unique factorization system* (E, M) of Γ consists of two classes E and M of morphisms of Γ satisfying the following conditions:

(a) $E \cap M = \mathrm{Iso}(\Gamma)$
(b) E is closed under composition,
(c) M is closed under composition,
(d) Every morphism $f : a \longrightarrow b$ in Γ has an *(E, M)-factorization*:

$$f = (a \xrightarrow{e} im(f) \xrightarrow{m} b) \quad \text{with } e \in E,\ m \in M.$$

(e) If $f = (a \xrightarrow{e'} i' \xrightarrow{m'} b)$ is another (E, M)-factorization, then there exists a unique isomorphism $h' : im(f) \longrightarrow i'$ such that $e' = he$, $m = m'h$.

When Γ has a unique factorization system (E, M), we put

$$E(a, b) := \Gamma(a, b) \cap E,$$
$$M(a, b) := \Gamma(a, b) \cap M.$$

A. 4 HYPOTHESIS (F). The category Γ has a unique factorization system (E, M) such that

$$E \subseteq \mathrm{Epi}(\Gamma), \quad M \subseteq \mathrm{Mon}(\Gamma). \tag{1}$$

Note that under this hypothesis (F),

$$E(a, a) = M(a, a) = \mathrm{Ant}\ (a). \tag{2}$$

A. 5 THEOREM. *Assume the above hypothesis (F). Then the Burnside homomorphism $\varphi : \Omega(\Gamma) \longrightarrow \tilde{\Omega}(\Gamma)$ is injective and its cokernel*

$$\mathrm{Cok}(\varphi) \cong \prod_{i \in \Gamma/\cong} (\boldsymbol{Z}/|\mathrm{Aut}(i)|\boldsymbol{Z}).$$

A. 6 COROLLARY. *Under* (F), $\boldsymbol{Q} \otimes \Omega(\Gamma)$ *is an abstract Burnside ring which is isomorphic to* $\boldsymbol{Q} \otimes \tilde{\Omega}(\Gamma) \cong \boldsymbol{Q}^{\Gamma/\cong}$ *via* $1_{\boldsymbol{Q}} \otimes \varphi$

A. 7 In order to prove the above theorem, define $\Gamma/\cong \times \Gamma/\cong$-matrices H, L, D, U by

$H_{a,b} := (a, b),$
$L_{a,b} := |E(a, b)|/|\mathrm{Aut}\ b|,$
$D_{a,b} := |\mathrm{Aut}\ a|\delta_{ab},$
$U_{a,b} := |M(a, b)|/|\mathrm{Aut}\ a|.$

By (1), these matrices are all integral and by (d) and (e) of A. 3, we have that

$$H = LDU.$$

By (2), L(resp. U) is conjugate to a unipotent lower (resp. upper) triangular matrix. This proves the theorem.

A. 8 EXAMPLE. Let Γ be the category of transitive $(G, \mathfrak{X})$-sets and G-maps, where $\mathfrak{X}$ is a family of subgroups of G closed under G-conjugation. Then since all morphisms in Γ are surjective, Γ satisfies trivially the hypothesis(F) with $E = \mathrm{Mor}(\Gamma)$ and $M = \mathrm{Iso}(\Gamma)$. Applying Theorem A. 5 to this category gives Lemma 3.3.

A. 9 Next, the condition $(\mathrm{C})_\infty$ in (3.6) is rewritten as the following form:

HYPOTHESIS (C). For any $a \in \Gamma$ and any automorphism σ of a, there exists a coequalizer diagram:

$$a \underset{\sigma}{\overset{1}{\rightrightarrows}} a \longrightarrow a/\sigma.$$

A. 10 EXAMPLE. Let $\mathfrak{X}$ be a family of subgroups of G closed under G-conjugation and let Γ be the full subcatgory of finite G-sets consisting of G-sets of the form G/S, $S \in \mathfrak{X}$. Then any automorphism of $G/S \in \Gamma$ has the form

$$\sigma_g : G/S \longrightarrow G/S : xS \longmapsto xgS$$

for some $gS \in WS$.

Assume that $\mathfrak{X}$ satisfies the condition $(\mathrm{C})_\infty$ in (3.6), so that for any $S \in \mathfrak{X}$ and $gS \in WS$, there is the smallest element $\overline{\langle g \rangle S}$ of $\mathfrak{X}$ containing $\langle g \rangle X$. Then for any $gS \in WS$, $G/\overline{\langle g \rangle S}$ is a coequalizer of 1 and σ_g.

Conversely, we can check that if Γ satisfies the above hypothesis (C), then the family $\mathfrak{X}$ satisfies $(\mathrm{C})_\infty$.

A. 11 Define an *obstruction group* $\mathrm{Obs}(\Gamma)$ and the *Cauchy-Frobenius map* ψ by

$$\begin{aligned}
\mathrm{Obs}(\Gamma) &:= \prod_{i\in\Gamma/\cong} (\boldsymbol{Z}/|\mathrm{Aut}\ i|\boldsymbol{Z}),\\
\psi := (\psi_i) : \tilde{\Omega}(\Gamma) &\longrightarrow \qquad\qquad \mathrm{Obs}(\Gamma)\\
: \quad \chi \quad &\longmapsto \Big(\sum_{\sigma\in\mathrm{Aut}\ i} \chi(i/\sigma) \quad \mathrm{mod}\ |\mathrm{Aut}\ i|\Big).
\end{aligned}$$

A. 12 THEOREM. *Under the hypotheses* (F) *and* (C), *the following sequence of abelian groups is exact*:

$$0 \longrightarrow \Omega(\Gamma) \xrightarrow{\varphi} \tilde{\Omega}(\Gamma) \xrightarrow{\psi} \mathrm{Obs}(\Gamma) \longrightarrow 0.$$

This theorem is proved by the Cauchy-Frobenius lemma and the fact that for any $\sigma\in\mathrm{Aut}\ i$,

$$|\Gamma(i, x)^{<\sigma>}| = |\Gamma(i/\sigma, x)|, \tag{3}$$

where σ acts on $\Gamma(i, x)$ by composition.

A. 13 THEOREM. *Under* (F) *and* (C), $\Omega(\Gamma)$ *is an abstract Burnside ring.*

This is a corollary of the above theorem and is proved by a similar way as in the case of generalized Burnside rings.

A. 14 EXAMPLE. Let $\mathfrak{X}$ be a family of subgroups of G which is closed under G-conjugation and satisfies the condition $(\mathrm{C})_\infty$ of (3.6). Let $\Gamma := \{G/S \mid S\in\mathfrak{X}\}$ be the full subcategory of transitive $(G, \mathfrak{X})$-sets and G-maps. In this case, the abstract Burnside ring $\Omega(\Gamma)$ and the generalized Burnside ring $\Omega(G, \mathfrak{X})$ is isomorphic.

As another example, there is a Möbius ring of a finite poset. See [Yo 84], [Ai 79], [St 86].

A. 15 In order to obtain an idempotent formula for the abstract Burnside ring, we have to introduce a concept corresponding to the subgroup lattice or a family $\mathfrak{X}$ of subgroups of G which is ordered by inclusion.

A *discrete cofibration* $f : \tilde{\Gamma} \longrightarrow \Gamma$ is a functor from a finite category $\tilde{\Gamma}$ such that the diagram

$$\begin{array}{ccc}
\mathrm{Mor}(\tilde{\Gamma}) & \xrightarrow{\mathrm{dom}} & \mathrm{Obj}(\tilde{\Gamma})\\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f}\\
\mathrm{Mor}(\Gamma) & \xrightarrow{\mathrm{dom}} & \mathrm{Obj}(\Gamma)
\end{array}$$

is a pullback diagram.

Note that the category $\tilde{\Gamma}$ in a discrete cofibration $f:\tilde{\Gamma}\longrightarrow\Gamma$ can not be replaced by a category equivalent to $\tilde{\Gamma}$. For example, the restriction of a discrete cofibration $f:\tilde{\Gamma}\longrightarrow\Gamma$ into a skeleton of $\tilde{\Gamma}$ is not a discrete cofibration in general.

A. 16 If $f:\tilde{\Gamma}\longrightarrow\Gamma$ is a discrete cofibration, then

$$F:\Gamma\longrightarrow \boldsymbol{Set_f}: i\longmapsto f^{-1}(i)$$

makes a functor. Conversely, if $F:\Gamma\longrightarrow \boldsymbol{Set_f}$ is a functor, then we have a discrete cofibration $f:\tilde{\Gamma}\longrightarrow\Gamma$ defined by

$$\begin{aligned}
&\mathrm{Obj}(\tilde{\Gamma}) &&:=\coprod_{a\in\Gamma}\{a\}\times F(a),\\
&\tilde{\Gamma}((a, s),\ (b, t)) &&:=\{\lambda: a\longrightarrow b \,|\, F(\lambda)(s)=t\}\\
&f:\tilde{\Gamma}\longrightarrow\Gamma &&:\ \ (a, s)\longmapsto a,\ \lambda\longmapsto\lambda.
\end{aligned}$$

Thus there exists a one-to-one correspondence between discrete cofibrations over Γ and functors $\Gamma\longrightarrow \boldsymbol{Set_f}$.

A. 17 In general, $\tilde{\Gamma}$ in a discrete cofibration $f:\tilde{\Gamma}\longrightarrow\Gamma$ is not equivalent to a poset. So in order to get a better idempotent formula for the abstract Burnside ring $\Omega(\Gamma)$, we have to choose a convenient discrete cofibration over Γ.

Let $g_1, g_2, \cdots$ be a set of generators of Γ, and let $f:\tilde{\Gamma}\longrightarrow\Gamma$ be a discrete cofibration over Γ corresponding to the functor

$$\prod_i \mathrm{Epi}(g_1, -):\Gamma\longrightarrow \boldsymbol{Set_f}.$$

Then $\tilde{\Gamma}$ is a quasi-ordered set (that is, $|\tilde{\Gamma}(a, b)|\leq 1$ for any $a,\ b\in\tilde{\Gamma}$).

Furthermore, when Γ is a full subcategory of a finite category Γ' which has an object $g'\in\Gamma'$ such that $\mathrm{Epi}_{\Gamma'}(g', a)\neq\emptyset$ for every $a\in\Gamma$, the category $\tilde{\Gamma}$ in the discrete cofibration $f:\tilde{\Gamma}\longrightarrow\Gamma$ corresponding to the functor $\mathrm{Epi}_{\Gamma'}(g', -)$ is a quasi-ordered set, too.

A. 18 EXAMPLE. Assume that $\Gamma:=\{G/S|S\in\mathfrak{X}\}$, where $\mathfrak{X}$ is a family of subgroups of G closed under G-conjugation as before. Define a finite category $\tilde{\Gamma}:=\{gS\subseteq G|g\in G,\ S\in\mathfrak{X}\}$ by

$$\tilde{\Gamma}(xS, yT):=\begin{cases}\{x^{-1}y\} & \text{if } {}^xS\subseteq{}^yT\\ \emptyset & \text{otherwise.}\end{cases}$$

Then

$$f:\tilde{\Gamma}\longrightarrow\Gamma: gS\longmapsto G/S$$

is a discrete cofibration with $\tilde{\Gamma}$ a quasi-ordered set. The isomorphism classes of $\tilde{\Gamma}$ makes a poset and it is isomorphic to the subposet $\mathfrak{X}$ of the subgroup lattice by the correspondence $gS \longleftrightarrow {}^gS$.

A. 19 Since $\boldsymbol{Q}\otimes\Omega(\Gamma)$ is isomorphic to $\boldsymbol{Q}\otimes\tilde{\Omega}(\Gamma)$ as rings via $1\otimes\varphi$ under the condition (F), there is a primitive idempotent e_t corresponding to each $t\in\Gamma$ such that

$$\varphi_s(e_t)=\begin{cases}1 & \text{if } s\cong t\\ 0 & \text{otherwise.}\end{cases}$$

We wish to obtain an explicit formula for e_t. By the above definition of e_t, we have that

$$e_t=\sum_{s\in\Gamma/\cong} H_{s,t}^{-1}s,$$

where $H_{s,t}^{-1}$ is the (s, t)-entry of the inverse matrix of $H=(\langle s, t\rangle)_{s,t}$. So we have to calculate the inverse matrix of the hom-set matrix H.

A. 20 We give an idempotent formula under the following stronger assumption than (F).

(F′) *All morphisms in* Γ *are epimorphisms.*

An idempotent formula without the assumption (F′) is found in [Yo 87a].

A. 21 THEOREM. *Assume that all morphisms in* Γ *are epimorphisms. Let* $f:\tilde{\Gamma}\longrightarrow\Gamma$ *be a discrete cofibration such that* $\tilde{\Gamma}$ *is a quasi-ordered set and* f *is surjective on objects. Let* μ *be the Möbius function of the poset* $\bar{\Gamma}:=\tilde{\Gamma}/\cong$. *Let* $\bar{f}:\bar{\Gamma}\longrightarrow\Gamma$ *be the functor induced by* f, *so that* $\bar{f}$ *maps an isomorphism class* $\bar{a}$ *to* $f(a)$. *Let* e_t *be the primitive idempotent corresponding to* $t\in\Gamma$. *Then*

$$e_t=\sum_{a\in\bar{\Gamma}}\sum_{t'\in\bar{f}^{-1}(t)}\frac{\mu(a, t')}{|\mathrm{Aut}\,\bar{f}(a)|\cdot|\bar{f}^{-1}\bar{f}(a)|}\bar{f}(a).$$

The theorem follows from the decomposition of the hom-set matrix H. See [Yo 87a], and it coincides with Theorem 4.2 when Γ is the category of transitive $(G, \mathfrak{X})$-sets.

A. 22 THEOREM. *Assume that the hypothesis* (C) *and that all morphisms in* Γ *are epimorphisms. Let* p *be a prime or* ∞. *Let* $\sim_p$ *be the equivalence relation on* Γ *generated by the relation*

$$i/\sigma\sim_p i \quad \textit{for } i\in\Gamma,\ \sigma\in(\mathrm{Aut}\ i)_p,$$

where $(\mathrm{Aut}\, i)_p$ *is a Sylow p-subgroup of* $\mathrm{Aut}\, i$. *Then any primitive idempotent of* $\Omega(\Gamma)_{(p)}$ *has the form*

$$e_s^{(p)} := \sum_{t \sim_p s} e_t$$

for some $s \in \Gamma$, *where the summation is taken over a set of complete representatives of isomorphism classes of objects of* Γ *equivalent to s.*

Of course, this theorem gives theorem 4.12 about a p-local generalized Burnside ring when Γ is the category of transitive $(G, \mathfrak{X})$-sets.

References

[Ai 79] M. AIGNER, "Combinatorial Theory," Springer-Verlag, Berlin, 1979.

[Ar 82] S. ARAKI, Equivariant stable homotopy theory and idempotents of Burnside rings, Publ. RIMS, Kyoto Univ., 18 (1982), 1193-1212.

[Bo 89] R. BOLTJE, Canonical and explicit Brauer induction in the character ring of a finite group and a generalization for Mackey functors, Inaugural-Dissertation zur Erlangung des Doktorgrades der Naturwissenshaftlichen Fakultät der Universität Augsburg, 1989.

[Br 75] K. S. BROWN, Euler characteristics of groups: The p-fractional part, Invent. Math. 29 (1975), 1-5.

[BT 88] K. B. BROWN-J. THÉVENAZ, A generalization of Sylow's third theorem, J. Algebra, 115 (1988), 414-430.

[Di 79] T. TOM DIECK, "Transformation Groups and Representation Theory," Lecture Notes in Math. 766, Springer-Verlag, Berlin, 1979.

[Dr 71a] A. DRESS, "Notes on the Theory of Finite Grups I: The Burnside Rings of Finite Group and Some AGN Applications," multicopied notes, Bielefeld University, 1971.

[Dr 71b] A. DRESS, The ring of monomial representations I: Structure theory, J. Algebra, 18 (1971), 137-157.

[Dr 86] A. DRESS, Congruence relations characterizing the representation ring of the symmetric group, J. Algebra, 101 (1986), 350-364.

[DSY 90] A. DRESS-C. SIEBENEICHER-T. YOSHIDA, An application of Burnside rings in elementary finite group theory, Adv. Math., to appear.

[DY 90] A. DRESS-T. YOSHIDA, Private communication about the fundamental theorem of Burnside rings and congruences of Sylow-Frobenius-Quillen type.

[Gl 81] D. GLUCK, Idempotent formula for the Burnside algebra with applications to the p-subgroup simplicial complex, Illinois J. Math., 25 (1981), 63-67.

[GO 68] D. GORENSTEIN, "Finite Groups", Harper & Rew, New York, 1968.

[Gr 71] J. A. GREEN, Axiomatic representation theory of finite groups, J. Pure Appl. Algebra, 1 (1971), 41-77.

[Hu 67] B. HUPPERT, "Endliche Gruppen I", Harper & Row, New York, 1968.

[JK 81] G. JAMES-A. KERBER, "The Representation Theory of the Symmetric Group", Encyclopedia of Mathematics and its Applications, Vol. 16, 1981.

[Qu 78] D. QUILLEN, Homotopy properties of the poset of non-trivial p-subgroups of a group, Adv. in Math. 28 (1978), 101-128.

[Sn 88] V. SNAITH, Explicit Brauer induction, Invent. Math. 94 (1988), 455-478.

[St 86] R. P. STANLEY, " Enumerative Combinatorics I ", Wadsworth, Monterey, 1986.

[Th 87] J. THÉVENAZ, Permutation representation arising from simplicial complexes, J. of Combi. Theory (A) 46 (1987), 121-155.

[Wa 80] B. WAGNER, A permutation representation theoretical version of a theorem of Frobenius, Bayreuther Math., 6 (1980), 23-32.

[Yo 80] T. YOSHIDA, On G-functors I: Transfer theorems for cohomological G-functors, Hokkaido Math. J., 9 (1980), 222-257.

[Yo 83a] T. YOSHIDA, Idempotents of Burnside rings and Dress induction theorem, J. Algebra, 80 (1983), 90-105.

[Yo 83b] T. YOSHIDA, On G-functors II: Hecke operators and G-functors, J. Math. Soc. Japan 35 (1983), 179-190.

[Yo 84] T. YOSHIDA, The Möbius algebra as a Burnside ring, Hokkaido Math. J., 13 (1984), 362-376.

[Yo 85] T. YOSHIDA, Idempotents and transfer theorems of Burnside rings, character rings and span rings, in " Algebraic and Topological Theories," Kinokuniya, Tokyo. 1985, pp. 589-615.

[Yo 87a] T. YOSHIDA, On the Burnside rings of finite groups and finite categories, Advanced Studies in Pure Math. 11 (1987), " Commutative Algebra and Combinatorics ", 337-353.

[Yo 87b] T. YOSHIDA, Fisher's inequality for block designs with finite group action, J. Fac. Sci., Univ. Tokyo, 34 (1987), 513-544.

[Yo 90] T. YOSHIDA, On the unit groups of Burnside rings, J. Math. Soc. Japan, 42 (1990), 31-64.

[Yo 91] T. YOSHIDA, MacWilliams identity for linear codes with group action, in preparation.

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