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A twisted version of the Frobenius-Schur indicator and multiplicity-free permutation representations

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1. Introduction

Let G be a finite group, and

$$\tau: g \longrightarrow g^{\tau},\ g \in G$$

an automorphism of G such that $\tau^2=1$. For a complex irreducible character χ of G, we define the *twisted Frobenius-Schur indicator* $c_\tau(\chi)$ by

$$c_\tau(\chi)=|G|^{-1}\sum_{g\in G}\chi(gg^{\tau}).$$

When the τ-action is trivial, this is nothing but the classical Frobenius-Schur indicator [2], which we denote by $c(\chi)$. The purpose of this paper is to show that some of the standard properties (found, e.g. in [1; § 12C, § 73A]) of $c(\cdot)$ can naturally be generalized to those of $c_\tau(\cdot)$. Partly this was also observed by R. Gow [3].

Let χ be an irreducible character of G. There are following three possibilities:

(1_τ) The character χ is afforded by a matrix representation R of G such that

$$R(g^{\tau})=\overline{R(g)},\ g \in G, \tag{1.1}$$

where the bar means the complex conjugation.

(2_τ) The character χ satisfies

$$\chi(g^{\tau})=\overline{\chi(g)},\ g\in G, \tag{1.2}$$

but it can not be afforded by a representation R with the property (1.1).

(3_τ) The character χ does not satisfy (1.2).

Our main result is the following genaralization of a theorem of

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Frobenius and Schur [2].

THEOREM 1. 3. *Let χ be an irreducible character of G. Then*

$$c_\tau(\chi)=\begin{cases} 1 & \text{if } \chi \text{ is of type } (1_\tau), \\ -1 & \text{if } \chi \text{ is of type } (2_\tau), \\ 0 & \text{if } \chi \text{ is of type } (3_\tau). \end{cases}$$

This is proved in Section 3. In Section 4, we remark that the twisted Frobenius-Schur indicators already appeared implicitly in a classical work [9] of G. W. Mackey. Once this is recognized, it is not difficult to formulate and prove the τ-version of the Mackey's result. In Section 5, as an application of Theorem 1. 3, we prove:

THEOREM 1. 4. *Let the pair (G, τ) be one of the following:*

(a) *G is of odd order and τ is any involutive automorphism of G,*

(b) *$G=\{g\in \boldsymbol{G} | g^{\sigma^2}=g\}$ and $g^\tau=g^\sigma$ for $g\in G$, where $\boldsymbol{G}$ is the group of invertible elements of an associative algebra with unity over an algebraically closed field, and σ is an algebraic group endomorphism of $\boldsymbol{G}$ such that the σ^2-fixed point set G is finite.*

Let G_τ be the fixed point set of τ in G. Then we have:

(i) *The induced character $1_{G_\tau}^G(=(1_{G_\tau})^G)$ is multiplicity-free, and an irreducible character χ of G is a component of $1_{G_\tau}^G$ if and only if $\chi^\tau=\bar{\chi}$.*

(ii) *Any irreducible character of G is either of type (1_τ) or of type (3_τ).*

When G is the general linear group $GL_n(\boldsymbol{F}_{q^2})$ over a finite field $\boldsymbol{F}_{q^2}$ of q^2 elements and the τ-action on G is given by

$$(x_{ij})^\tau=(x_{ij}^q), \quad (x_{ij})\in GL_n(\boldsymbol{F}_{q^2}),$$

or by

$$(x_{ij})^\tau=(x_{ji}^q)^{-1}, \quad (x_{ij})\in GL_n(\boldsymbol{F}_{q^2}).$$

(these two cases are covered by case (b) of Theorem 1. 4), Theorem 1. 4 (i) was proved by Gow [5] by a totally different method. (Motivated by this result of Gow, the first-named author proved Theorem 1. 4 (i) in an unpublished paper [7].) A further application to "almost" multiplicity-free permutation representations of a finite reductive group will be given in a forthcoming paper of the first-named author.

NOTATION: For a set X, $|X|$ denotes its cardinality. Let Y be a subset of X. For a map f from X to another set, $f|_Y$ denotes the restriction of f to Y. Let G be a finite group. Then $\hat{G}$ (or $G^\wedge$) means the set of complex irreducible characters of G. For a complex valued class function α on a subgroup H of G, α^G denotes the class function on G induced from α.

2. Twisted Frobenius-Schur indicators

Let $\tilde{G}$ be a finite group, and G a subgroup of $\tilde{G}$ of index 2. We choose an element τ in $\tilde{G}-G$. (This situation is slightly more general than that of Section 1.) For a complex irreducible character χ of G, we put

$$c_\tau(\chi)=|G|^{-1}\sum_{g\in G}\chi((\tau g)^2)=|G|^{-1}\sum_{x\in\tilde{G}-G}\chi(x^2).$$

When $\tilde{G}=<\tau>\times G$ (direct product), this reduces to the Frobenius-Schur indicator $c(\chi)$. For any complex valued class function χ on G, we define $c_\tau(\chi)$ (and $c(\chi)$) by the same formula as above. The following lemma is easy to see.

LEMMA 2.1. *Let H be a subgroup of G, and α a class function on H. Then*

$$c(\alpha^{\tilde{G}})=c(\alpha^G)+c_\tau(\alpha^G).$$

Let $\chi\in\hat{G}$. If $\chi=\chi^\tau$ (resp. $\chi\neq\chi^\tau$), where χ^τ is defined by $\chi^\tau(g)=\chi(g^\tau)$ for $g\in G$, then we denote by $\tilde{\chi}$ an (resp. the) element of $(\tilde{G})^\wedge$ such that

$$\tilde{\chi}|_G=\chi \quad (\textit{resp. } \tilde{\chi}|_G=\chi+\chi^\tau). \tag{2.2}$$

LEMMA 2.3. *In the above notations, we have*

$$c(\chi)+c_\tau(\chi)=\begin{cases}2c(\tilde{\chi}) & \text{if } \chi^\tau=\chi,\\ c(\tilde{\chi}) & \text{if } \chi^\tau\neq\chi.\end{cases}$$

PROOF. This follows from Lemma 2.1 by putting $H=G$ and $\alpha=\chi$.

PROPOSITION 2.4. For an element g of G, we have

$$\sum_{\chi\in\hat{G}}c_\tau(\chi)\chi(g)=|\{h\in G|(\tau h)^2=g\}|.$$

PROOF. This follows from Lemma 2.3 and the classical counterpart (see [2], [1; §73, Ex. 4]) of Proposition 2.4.

The next result, given implicitly in R. Gow [3; Lemma 2.1] (see also [4]), generalizes a part of the Frobenius-Schur theorem [2].

THEOREM 2.5. *Let $\chi\in\hat{G}$. Then*

$$c_\tau(\chi)=\begin{cases}\pm 1 & \textit{if } \chi^\tau=\bar{\chi},\\ 0 & \textit{if } \chi^\tau\neq\bar{\chi},\end{cases}$$

where the bar means the complex conjugation.

PROOF. We consider the following five cases seperately:

(Aa) $\chi^\tau=\bar{\chi}=\chi$, (Ab) $\chi^\tau=\bar{\chi}\neq\chi$,
(Ba) $\chi^\tau=\chi\neq\bar{\chi}$, (Bb) $\chi^\tau\neq\bar{\chi}=\chi$,
(Bc) $\chi^\tau\neq\bar{\chi}\neq\chi$, $\chi^\tau\neq\chi$.

In case (Aa), we have $c(\chi)=\pm 1$ by [2]. Hence, if $c(\tilde{\chi})=0$, we have $c^\tau(\chi)=\mp 1$ by Lemma 2.3. Next, if $c(\tilde{\chi})=1$, then $c(\chi)=1$ by [2]. Hence we have $c_\tau(\chi)=1$ by Lemma 2.3. Therefore we may assume $c(\tilde{\chi})=-1$. In this case, we cannot have $c(\chi)=1$. In fact, if $c(\chi)=1$, then the induced character $\chi^{\tilde{G}}$ is afforded by a real representation of $\tilde{G}$. Moreover, by (2.2), we have the irreducible decomposition (over the complex number field):

$$\chi^{\tilde{G}}=\tilde{\chi}+\tilde{\chi}', \ \tilde{\chi}\neq\tilde{\chi}'.$$

Hence, we have either

(1) both $\tilde{\chi}$ and $\tilde{\chi}'$ are afforded by real representations of $\tilde{G}$, or

(2) $\tilde{\chi}'$ is complex conjugate to $\tilde{\chi}$.

Accordingly, $c(\tilde{\chi})$ is equal to 1 or 0, which contradicts to our hypothesis. Hence $c(\chi)$ must be -1. This and Lemma 2.3 imply that $c_\tau(\chi)=-1$. This proves the theorem in case (Aa).

In case (Ab), we have $c(\chi)=0$ by [2], and $\tilde{\chi}=\chi^{\tilde{G}}$ and $\tilde{\chi}|_G=\chi+\bar{\chi}$ by (2.2). Hence $c(\tilde{\chi})=\pm 1$ by [2]. Hence $c_\tau(\chi)=\pm 1$ by Lemma 2.3.

In case (Ba), we have $c(\tilde{\chi})=c(\chi)=0$. Hence $c_\tau(\chi)=0$.

In case (Bb), we have $c(\chi)=\pm 1$. Moreover, we can show that $c(\chi)=1$ if and only if $c(\tilde{\chi})=1$. In fact, if $c(\tilde{\chi})=1$, the character $\tilde{\chi}|_G=\chi+\chi^\tau$ is afforded by a real representation, which implies that $c(\chi)=1$. Conversely, if $c(\chi)=1$, $c(\tilde{\chi})$ must be 1 because $\tilde{\chi}=\chi^{\tilde{G}}$ is afforded by a real representation. Thus we have shown that $c(\chi)=c(\tilde{\chi})$. Hence, by Lemma 2.3, we have $c_\tau(\chi)=0$.

In case (Bc), we have $c(\tilde{\chi})=c(\chi)=0$. Hence $c_\tau(\chi)=0$. This completes the proof of Theorem 2.5.

3. Proof of Theorem 1.3

In this section, we prove Theorem 1.3. Let G and τ be as in Section 1, and $\tilde{G}$ the semi-direct product of G with $<\tau>$:

$$\tilde{G}=<\tau>G.$$

Let χ be an irreducible character of G, and R a matrix representation of G affording χ:

$$R: G \longrightarrow GL_n(\boldsymbol{C}).$$

Assume that χ is either of type (1_τ) or of type (2_τ). This implies that there exists a matrix $X \in GL_n(\boldsymbol{C})$ such that

$$(3.1) \qquad XR(g^\tau)X^{-1}=\overline{R(g)}, \ g \in G.$$

LEMMA 3.2. *In the above situation, we have*

$$\bar{X}X=\alpha 1_n,$$

for a non-zero real number α, *where* 1_n *is the* n-*by*-n *identity matrix. Moreover,* χ *is of type* (1_τ) *(resp. type* (2_τ)*) if* α *is positive (resp. negative).*

PROOF. By (3.1), we have

$$\bar{X}XR(g^\tau)X^{-1}\bar{X}^{-1}=R(g^\tau), \ g \in G.$$

Hence, by Schur's lemma, we have

$$\bar{X}X=\alpha 1_n$$

for some $\alpha \in \boldsymbol{C}-\{0\}$. Taking the traces of both sides, we see that α is real. If χ is of type (1_τ), then there exists a matrix $Y \in GL_n(\boldsymbol{C})$ such that

$$(3.3) \qquad YR(g^\tau)Y^{-1}=\bar{Y}\,\overline{R(g)}\,\bar{Y}^{-1}, \ g \in G.$$

Comparing (3.1) with (3.3), and using Schur's lemma, we have

$$X=\beta\bar{Y}^{-1}Y$$

for some $\beta \in \boldsymbol{C}-\{0\}$. Hence

$$\bar{X}X=\bar{\beta}\beta 1_n.$$

This implies that $\alpha=\bar{\beta}\beta>0$. Conversely, if $\alpha>0$, then

$$(\sqrt{\alpha}^{-1}\bar{X})(\sqrt{\alpha}^{-1}X)=1_n.$$

Hence, by the triviality of the Galois cohomology $H^1(\boldsymbol{C}/\boldsymbol{R}, GL_n(\boldsymbol{C}))$ (see, e.g., [10; ch. X, Prop. 3]), we have

$$\sqrt{\alpha}^{-1}X=\bar{Y}^{-1}Y$$

for some $Y\in GL_n(\boldsymbol{C})$. This and (3.1) lead to (3.3), which means that χ is of type (1_τ). This proves the lemma.

PROOF OF THEOREM 1.3. By Theorem 2.5, we already know that $c_\tau(\chi)=0$ if and only if χ is of type (3_τ). Hence we may assume that $\chi^\tau=\bar{\chi}$. We consider the following four cases seperately:

(A) $\chi^\tau=\bar{\chi}\neq\chi$,
(Ba) $\chi^\tau=\bar{\chi}=\chi,\ c(\tilde{\chi})=1$,
(Bb) $\chi^\tau=\bar{\chi}=\chi,\ c(\tilde{\chi})=-1$,
(Bc) $\chi^\tau=\bar{\chi}=\chi,\ c(\tilde{\chi})=0$.

Here $\tilde{\chi}$ is an irreducible character of $\tilde{G}$ with the property (2.2).

We begin with case (A). Let $R: G\longrightarrow GL_n(\boldsymbol{C})$ be a representation of G affording χ. Then there exists a representation $\tilde{R}$ of $\tilde{G}$ affording $\tilde{\chi}$ with the following form:

$$(3.4)\qquad \tilde{R}(g)=\begin{pmatrix} R(g) & 0 \\ 0 & \overline{R(g)} \end{pmatrix},\ g\in G,$$

$$(3.5)\qquad \tilde{R}(\tau)=\begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix},\ P,\ Q\in GL_n(\boldsymbol{C}),\ PQ=1_n.$$

We put

$$(3.6)\qquad \tilde{R}^A(x)=A\tilde{R}(x)A^{-1},\ x\in\tilde{G},$$

where

$$A=\begin{pmatrix} 1_n & 1_n \\ -i1_n & i1_n \end{pmatrix},\ i=\sqrt{-1}.$$

Then $\tilde{R}^A|_G$ is a real representation of G affording $\chi+\bar{\chi}$. By the proof of Theorem 2.5, $c_\tau(\chi)=1$ if and only if $c(\tilde{\chi})=1$. Assume that $c(\tilde{\chi})=1$. Then there exists a real representation $T: \tilde{G}\longrightarrow GL_{2n}(\boldsymbol{R})$ which is equivalent to $\tilde{R}$ as complex representations. We have

$$(3.7)\qquad B\tilde{R}^A(x)B^{-1}=T(x),\ x\in\tilde{G},$$

for some $B\in GL_{2n}(\boldsymbol{C})$. Moreover, since $\tilde{R}^A|_G$ and $T|_G$ are equivalent as real representations, we have

$$C\tilde{R}^A(g)C^{-1}=T(g),\ g\in G$$

for some $C\in GL_{2n}(\boldsymbol{R})$. Hence,

$$B\tilde{R}^A(g)B^{-1}=C\tilde{R}^A(g)C^{-1},\ g\in G.$$

Hence, by Schur's lemma,

$$B=CA\begin{pmatrix}\lambda 1_n & 0\\ 0 & \mu 1_n\end{pmatrix}A^{-1}$$

for some $\lambda,\ \mu\in \boldsymbol{C}-\{0\}$. Using this, (3.6) and (3.7), we have

$$T(\tau)=CA\begin{pmatrix}0 & P'\\ Q' & 0\end{pmatrix}A^{-1}C^{-1}=\frac{1}{2}C\begin{pmatrix}P'+Q' & -i(P'-Q')\\ i(P'-Q') & -(P'+Q')\end{pmatrix}C^{-1},$$

where $P'=\lambda\mu^{-1}P,\ Q'=\lambda^{-1}\mu Q$. Since $T(\tau)$ and C are real matrices, we see from this that $P'+Q'$ and $i(P'-Q')$ are real matrices. This implies that

$$Q'=\overline{P'}.$$

By (3.4) and (3.5), we have

$$\begin{pmatrix}0 & P\\ Q & 0\end{pmatrix}\begin{pmatrix}R(g) & 0\\ 0 & \overline{R(g)}\end{pmatrix}\begin{pmatrix}0 & P\\ Q & 0\end{pmatrix}=\begin{pmatrix}R(g^\tau) & 0\\ 0 & \overline{R(g^\tau)}\end{pmatrix}$$

Hence

$$Q'R(g^\tau)(Q')^{-1}=QR(g^\tau)Q^{-1}=\overline{R(g)},\ g\in G.$$

Moreover

$$\overline{Q'}Q'=P'Q'=PQ=1_n.$$

Hence, by Lemma 3.2, we see that χ is of type (1_τ). Conversely, assume that χ is of type (1_τ). Then the representation $R: G\longrightarrow GL_n(\boldsymbol{C})$ can be taken so that

$$\overline{R(g)}=R(g^\tau).$$

Then we can take $P=Q=1_n$ in (3.5). Then the representation $\tilde{R}^A$ of $\tilde{G}$ defined by (3.6) in a real representation. Hence $c(\tilde{\chi})=1$, which implies $c_\tau(\chi)=1$. This proves the theorem in case (A).

Next we consider cases (Ba)—(Bc). Let $\tilde{R}: \tilde{G}\longrightarrow GL_n(\boldsymbol{C})$ be a representation of $\tilde{G}$ affording $\tilde{\chi}$. Then $R=\tilde{R}|_G$ is a representation of G affording χ. We put $A=\tilde{R}(\tau)$. Then

(3.8) $\quad AR(g)A^{-1}=R(g^\tau),\ g\in G$

and

(3.9) $\quad A^2=1_n.$

If we are in case (Ba), then, by the proof of Theorem 2.5, we always have $c_\tau(\chi)=1$. Hence we have to show that χ is always of type (1_τ). But, in this case, $\tilde{R}$ can be taken as a real representation. Then, by

(3.8) and (3.9), we have

$$AR(g)A^{-1}=\overline{R(g^{\tau})},\ g\in G,$$

and

$$\bar{A}A=A^{2}=1_{n}.$$

Hence, by Lemma 3.2, χ is of type (1_{τ}).

If we are in case (Bb), then by the proof of Theorem 2.5, we have $c_{\tau}(\chi)=c(\chi)=-1$. Hence we have to show that χ is always of type (2_{τ}) in this case. Since the representation $\tilde{R}$ is equivalent to $\bar{\tilde{R}}$, there exists a matrix $B\in GL_{n}(\boldsymbol{C})$ such that

$$(3.10)\quad B\tilde{R}(x)B^{-1}=\overline{\tilde{R}(x)},\ x\in\tilde{G}.$$

Since $c(\chi)=-1$, we have

$$(3.11)\quad \bar{B}B=\alpha 1_{n},\ \alpha<0,$$

by Lemma 3.2. By (3.8) and (3.10), we have

$$(3.12)\quad BAR(g)A^{-1}B^{-1}=\overline{R(g^{\tau})},\ g\in G,$$

and

$$(3.13)\quad BAB^{-1}=\bar{A}.$$

Now

$$BA\overline{(BA)}=\bar{A}B\overline{BA}=\alpha\bar{A}^{2}=\alpha 1_{n}$$

by (3.13), (3.11) and (3.9). Hence, by (3.12) and Lemma 3.2, We see that χ is of type (2_{τ}).

If we are in case (Bc), then by the proof of Theorem 2.5, we have $c_{\tau}(\chi)=-c(\chi)=\pm 1$. Let $\varepsilon:\tilde{G}\longrightarrow\{\pm 1\}$ be the 1-dimensional representation of $\tilde{G}$ defined by

$$\varepsilon|_{G}=1,\ \varepsilon(\tau)=-1.$$

Since χ is real valued, $\bar{\tilde{\chi}}|_{G}=\tilde{\chi}|_{G}=\chi$. This and the assumption $c(\tilde{\chi})=0$ imply that $\bar{\tilde{\chi}}=\varepsilon\otimes\tilde{\chi}$. Hence the representation $\bar{\tilde{R}}$ is equivalent to $\varepsilon\otimes\tilde{R}$. Hence there exists a matrix $B\in GL_{n}(\boldsymbol{C})$ such that

$$(3.14)\quad B(\varepsilon\otimes\tilde{R})(x)B^{-1}=\overline{\tilde{R}(x)},\ x\in\tilde{G}.$$

Hence

$$(3.15)\quad BR(g)B^{-1}=\overline{R(g)},\ g\in G.$$

By Lemma 3.2, we have

(3.16) $\bar{B}B=\alpha 1_n,\ \alpha c(\chi)>0.$

Br (3.8) and (3.15), we have

(3.17) $BAR(g)A^{-1}B^{-1}=\overline{R(g^\tau)},\ g\in G.$

By (3.14) with $x=\tau$, we have

(3.18) $-BAB^{-1}=\bar{A}.$

Now

$$BA\overline{(BA)}=-\bar{A}B\overline{BA}=-\alpha 1_n$$

by (3.18), (3.16) and (3.9). Since

$$\mathrm{sign}(-\alpha)=-\mathrm{sign}\ c(\chi)=\mathrm{sign}\ c_\tau(\chi),$$

we see, from (3.17) and Lemma 3.2, that χ is of type (1_τ) (resp. (2_τ)) if $c_\tau(\chi)$ is equal to 1 (resp. -1). This proves the theorem in cases (Ba)—(Bc). The proof of Theorem 1.3 is now complete.

REMARK 3.18. (i) By Theorem 1.3, we have the following interpretation of the twisted Frobenius-Schur indicator $c_\tau(\cdot)$ (in the case $\tau^2=1$). Let M_χ be a G-module over $\boldsymbol{C}$ affording $\chi\in\hat{G}$. Let $\mathrm{Bil}^+_{G,\tau}(M_\chi)$ (resp. $\mathrm{Bil}^-_{G,\tau}(M_\chi)$) be the space of symmetric (resp. skew symmetric) bilinear forms $B(\cdot,\cdot)$ on M_χ which are G-invariant in the following sense:

$$B(g\cdot m_1, g^\tau\cdot m_2)=B(m_1, m_2),\ g\in G_1,\ m_1,\ m_2\in M_\chi.$$

Then

$$c_\tau(\chi)=\dim \mathrm{Bil}^+_{G,\tau}(M_\chi)-\dim \mathrm{Bil}^-_{G,\tau}(M_\chi).$$

Compare with [1; §73A].

(ii) A result of A. A. Klyachko [8; Th. 4.1] and R. Gow [4; Th. 3] is equivalent to the following statement:

If G is a general linear group over a finite field, and τ is the transpose-inverse automorphism of G, then any $\chi\in\hat{G}$ is of type (1_τ).

(iii) Theorem 1.3 (and Theorem 2.5) can be generalized in the obvious manner to the case when G is a compact topological group.

4. Induced characters

We recall a result of G. W. Mackey [9] on the Frobenius-Schur indicators of induced characters following an exposition by C. W. Curtis and I. Reiner [1; §12C]. Let H be a subgroup of a finite group G. Let D_{-1}

be a set of representatives of the self-inverse (H, H)-double cosets, i. e., the double cosets HxH $(x\in G)$ such that $(HxH)^{-1}=HxH$. For $x\in D_{-1}-H$, choose $z=z_x\in xH\cap Hx^{-1}$. Then $H(x, z)=<z, {}^xH\cap H>$ contains ${}^xH\cap H=xHx^{-1}\cap H$ as a normal subgroup of index 2. Let L be a (possibly reducible) H-module over $\boldsymbol{C}$. Then, on the vector space $L\otimes L$, we can define an $({}^xH\cap H)$-module structure by

$$(4.1)\qquad h(l\otimes l')=(x^{-1}hx)l\otimes hl',\ \ l,\ l'\in L,\ \ h\in{}^xH\cap H.$$

We denote this $({}^xH\cap H)$-module by ${}^xL\otimes L$. We also define a linear transformation Z on $L\otimes L$ by

$$(4.2)\qquad Z(l\otimes l')=(x^{-1}z)l'\otimes(zx)l,\qquad l,\ l'\in L.$$

Then, by letting z acts as Z (resp. $-Z$), the $({}^xH\cap H)$-module ${}^xL\otimes L$ extends to an $H(x, z)$-module, which we denote by $L^+_{x,z}$ (resp. $L^-_{x,z}$). If α denotes the character of L, the one of ${}^xL\otimes L$ is given by

$${}^x\alpha\cdot\alpha:\ h\longrightarrow\alpha(x^{-1}hx)\cdot\alpha(h),\ \ h\in{}^xH\cap H.$$

We denote by $({}^x\alpha\cdot\alpha)^{\pm}$ the characters of $L^{\pm}_{x,z}$. The values of $({}^x\alpha\cdot\alpha)^{\pm}$ are given by

$$(4.3)\qquad ({}^x\alpha\cdot\alpha)^{\pm}|_{{}^xH\cap H}={}^x\alpha\cdot\alpha,$$

and

$$(4.4)\qquad ({}^x\alpha\cdot\alpha)^{\pm}(y)=\pm\alpha(y^2),\qquad y\ni z({}^xH\cap H).$$

In fact, by (4. 1) and (4. 2), we have

$$Zh(l_i\otimes l_j)=(x^{-1}zh)l_j\otimes(zhx)l_i$$

for $h\in{}^xH\cap H$ and $l_i,\ l_j\in L$. Hence

$$\begin{aligned}(\alpha^x\cdot\alpha)^{\pm}(zh)&=\pm\sum_{i,j}<(x^{-1}zh)l_j,\ l_i><(zhx)l_i,\ l_j>\\&=\pm\sum_j<(x^{-1}(zh)^2x)l_j,\ l_j>\\&=\pm\alpha(x^{-1}(zh)^2x)\\&=\pm\alpha((zh)^2),\end{aligned}$$

where $\{l_i\}$ is a basis of L, and, for $l\in L$, $<l,\ l_i>\in\boldsymbol{C}$ is defined by:

$$l=\sum_i<l,\ l_i>l_i.$$

This proves (4. 4). By [9 ; Th. 1] (or [1 ; Th. (12. 13)]), we have

$$(4.5)\qquad c(\alpha^G)=c(\alpha)+$$

$$\sum_{x\in D_{-1}-H}|H(x, z_x)|^{-1}\{\sum_{y\in H(x,z_x)}(({}^x\alpha\cdot\alpha)^+-({}^x\alpha\cdot\alpha)^-)(y)\}.$$

Hence, by (4.3)—(4.5), we have

$$(4.6)\qquad c(\alpha^G)=\sum_{x\in D_{-1}}c_{z_x}(\alpha|_{{}^xH\cap H}).$$

This last formula, which is not stated explicitly in [9], shows that the twisted Frobenius-Schur indicator appears quite naturally in the study of its classical counterpart.

We now formulate the τ-version of (4.5) and (4.6).

THEOREM 4.7. *Let $\tilde{G}$, G and τ be as in Section 2. Let H be a subgroup of G such that $\tau^2\in H$, and $D_{-\tau}$ a set of representatives of the double cosets $H^\tau xH$, $x\in G$, such that $((H^\tau xH)^{-1})^\tau=H^\tau xH$.*

(i) *Let α be a (possibly reducible) character of H. For $x\in D_{-\tau}$, let $\alpha^{\tau x}\cdot\alpha$ be the character $h\longrightarrow\alpha(\tau xhx^{-1}\tau^{-1})\alpha(h)$ of $H^{\tau x}\cap H$. Choose $z=z_x\in x^{-\tau}H\cap H^\tau x$. Then $H(\tau x, \tau z)=<\tau z, H^{\tau x}\cap H>$ contains $H^{\tau x}\cap H$ as a normal subgroup of index 2. Moreover, there exist characters $(\alpha^{\tau x}\cdot\alpha)^\pm$ of $H(\tau x, \tau z)$ such that*

$$(\alpha^{\tau x}\cdot\alpha)^\pm|_{H^{\tau x}\cap H}=\alpha^{\tau x}\cdot\alpha$$

and that

$$(\alpha^{\tau x}\cdot\alpha)^\pm(y)=\pm\alpha(y^2),\ y\in\tau z(H^{\tau x}\cap H).$$

We also have

$$c_\tau(\alpha^G)=\sum_{x\in D_{-\tau}}(2|H^{\tau x}\cap H|)^{-1}\sum_{y\in H(\tau x,\tau z)}\{(\alpha^{\tau x}\cdot\alpha)^+-(\alpha^{\tau x}\cdot\alpha)^-\}(y)$$
$$=\sum_{x\in D_{-\tau}}c_{\tau z_x}(\alpha|_{H^{\tau x}\cap H}).$$

(ii) *Let α be a linear character of H. Then*

$$c_\tau(\alpha^G)=\sum_{x\in D_{-\tau}}j_\tau(x),$$

where, for $x\in D_{-\tau}$, we define $j_\tau(x)$ to be 0 or $\alpha((\tau z_x)^2)=\pm1$, $z_x\in x^{-\tau}H\cap H^\tau x$, according to whether $\alpha^{\tau x}\cdot\alpha\neq1$ or 1 on $H^{\tau x}\cap H$. In particular, we have

$$c_\tau(1_H^G)=|D_{-\tau}|.$$

PROOF. A self-inverse (H, H)-double coset in $\tilde{G}$ is either of the form HxH, $x\in D_{-1}$, or of the form $H\tau xH$, $x\in D_{-\tau}$. Hence, applying (4.5) and (4.6) (resp. [9; Cor. 1, 2] or [1; Cor. (12.19), (12.20)]) to $\alpha^{\tilde{G}}$, and using Lemma 2.1, we get part (i) (resp. (ii)).

5. Multiplicity-free permutation representations

Let $\boldsymbol{G}$ be a (not necessarily connected) linear algebraic group over an algebraically closed field. Let σ be an endomorphism of $\boldsymbol{G}$ such that the group G of σ^2-fixed points of $\boldsymbol{G}$ is finite. Let τ be an automorphism of the finite group G defined by

$$x^{\tau}=x^{\sigma},\ x\in G.$$

Then $\tau^2=1$. We put

$$G_{\tau}=\{x\in G\ ;\ x^{\tau}=x\}.$$

By [11 ; III, 3.22], for a proof of Theorem 1.4, it is enough to prove the following.

THEOREM 5.1. *Let $\boldsymbol{G}$, G and G_{τ} be as above. We denote by $Z_{\boldsymbol{G}}(x)$ and $Z_{\boldsymbol{G}}(x)^0$ the centralizer of x in $\boldsymbol{G}$, and its identity component, respectively. We assume that $|Z_{\boldsymbol{G}}(x)/Z_{\boldsymbol{G}}(x)^0|$ is odd for any $x\in G_{\tau}$. Then we have the following.*

(i) *The induced character $1_{G_{\tau}}^{G}$ is multiplicity-free.*

(ii) *Any $\chi\in\hat{G}$ is of type (1_{τ}) or (3_{τ}). Moreover, $\chi\in\hat{G}$ is a component of $1_{G_{\tau}}^{G}$* if and only if it is of type (1_{τ}).

LEMMA 5.2. *Let G be a finite group, and τ an automorphism of G such that $\tau^2=1$. For any $g\in G$, we put*

$$g^{G,\tau}=\{(h^{-1})^{\tau}gh\ ;\ h\in G\}$$

and

$$(g^{\tau}g)^{G}=\{h^{-1}(g^{\tau}g)h\ ;\ h\in G\}.$$

We assume:

(a) *For any $g\in G$*

$$|G|^{-1}|g^{G,\tau}|=|G_{\tau}|^{-1}|(g^{\tau}g)^{G}\cap G_{\tau}|.$$

(b) *Let g_1, $g_2\in G$. If $g_1^{G,\tau}\cap g_2^{G,\tau}=\phi$, then*

$$(g_1^{\tau}g_1)^{G}\cap(g_2^{\tau}g_2)^{G}\cap G_{\tau}=\phi.$$

Then conclusions (i) (ii) *of Theorem* 5.1 *hold.*

PROOF. We choose a set $\{g_i\}_{i=1}^{N}$ of elements of G such that

$$G=\bigcup_{i=1}^{N}g_i^{G,\tau}\ \ (disjoint).$$

Then, by conditions (a) (b), we have

$$G_\tau=\bigcup_{i=1}^{N}((g_i^\tau g_i)^G\cap G_\tau)\ (disjoint).$$

Hence, for any class function χ on G,

$$\begin{aligned}c_\tau(\chi)&=|G|^{-1}\sum_{g\in G}\chi(g^\tau g)\\&=|G|^{-1}\sum_{i=1}^{N}|g_i^{G,\tau}|\chi(g_i^\tau g_i)\\&=|G_\tau|^{-1}\sum_{i=1}^{N}|(g_i^\tau g_i)^G\cap G_\tau|\chi(g_i^\tau g_i)\\&=|G_\tau|^{-1}\sum_{i=1}^{N}\sum_{h\in(g_i^\tau g_i)^G\cap G_\tau}\chi(h)\\&=|G_\tau|^{-1}\sum_{h\in G_\tau}\chi(h).\end{aligned}$$

Hence, for $\chi\in\widehat{G}$, $c_\tau(\chi)$ is equal to the multiplicity $<1_{G_\tau}^G,\chi>$ of χ in the permutation character $1_{G_\tau}^G$ In particular it must be non-negative. Hence, by Theorem 1.3, we see that

$$<1_{G_\tau}^G,\chi>=c_\tau(\chi)=1\ or\ 0$$

according to whether χ is of type (1_τ) or of type (3_τ), and that χ cannot be of type (2_τ). This proves Lemma 5.2.

PROOF OF THEOREM 5.1. It is enough to show that conditions (a) (b) in Lemma 5.2 are satisfied for our (G,τ). But this is already known [6; Lemma 2.4.8, Lemma 2.4.5 (i)].

Let $\boldsymbol{G}$ be a connected reductive group defined over a finite field, and σ the Frobenius endomorphism of $\boldsymbol{G}$. Define G, τ and G_τ as in Theorem 5.1. Then the assumptions in Theorem 5.1 are not satisfied in general. But we can still modify the argument given above, and can show, e.g., that $1_{G_\tau}^G$ is "almost" multiplicity-free (in some rigorous sense). This and other topics on $1_{G_\tau}^G$ will be discussed in a forthcoming paper of the first-named author.

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Added in Proof. The authors have learned that Professor Michio Suzuki proved Theorem 1.4(i) in the case (a) more than thirty years ago (unpublished). His proof uses an anti-involution of G and is different from the one given in the present paper.

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