Hokkaido Mathematical Journal Vol. 19 (1990) p. 481～493

On block-schematic Steiner systems $S(t, t+2, v)$ and $S(t, t+3, v)$

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(Received October 25, 1989)

1. Introduction

A Steiner system $S(t, k, v)$ is a pair consisting of a set Ω of v points and a family $\boldsymbol{B}$ of k-point subsets, called blocks, of Ω with the property that any t-point subset is contained in a unique block. We assume $1<t<k<v$, excluding trivial designs. A Steiner system is called block-schematic if the blocks form an association scheme with the relations determined by size of intersection. By [9], for each $n\geqq 1$ there exist finitely many block-schematic Steiner systems $S(t, k, v)$ with $k-t=n$ and $t\geqq 3$. Yoshizawa [10] proved that $S(t, t+1, v)$ is block-schematic if and only if $t=2$ or $(t, v)=(3,8)$, $(4,11)$ or $(5,12)$. The purpose of this paper is to prove the following theorems.

THEOREM 1. *A Steiner system $S(t, t+2, v)$ is block-schematic if and only if $t=2$.*

THEOREM 2. *A Steiner system $S(t, t+3, v)$ is block-schematic if and only if $t=2$ or $(t, v)=(3,22)$, $(4,23)$ or $(5,24)$.*

It is well known that $S(3,6,22)$, $S(4,7,23)$ and $S(5,8,24)$ are unique (cf. [8]), and $S(2, k, v)$, $S(3,6,22)$, $S(4,7,23)$ and $S(5,8,24)$ are block-schematic (cf. [2], [4], [5]).

2. Notation and preliminaries

For a Steiner system $S=S(t, k, v)$ we use $\lambda_i (0\leqq i\leqq t)$ to represent the number of blocks which contain given i points of S. Then we have

$$\lambda_i=\frac{(v-i)(v-i-1)\cdots\cdots(v-t+1)}{(k-i)(k-i-1)\cdots\cdots(k-t+1)}\ (0\leqq i\leqq t-1).$$

Let $x_i (0\leqq i\leqq k)$ denote the number of blocks each of which has exactly i points in common with a fixed block B. By [7] the number x_i depends on S, but not on the choice of a block B, and the following equality holds for $i=0, \ldots, t-1$.

$$x_i+\binom{i+1}{i}x_{i+1}+\cdots\cdots+\binom{t-1}{i}x_{t-1}=(\lambda_i-1)\binom{k}{i}. \tag{1}$$

We remark that $x_t=\cdots\cdots=x_{k-1}=0$ and $x_k=1$.

Let $B_1, \dots, B_{\lambda_0}$ be the blocks of S. Let $A_h(0\leqq h\leqq k)$ be the h- adjacency matrix of S of degree λ_0 defined by $A_h(i, j)=1$ if $|B_i\cap B_j|=h$, 0 otherwise. We remark that $A_t=\cdots\cdots=A_{k-1}=0$ (the zero matrix) and $A_k=I$ (the identity matrix). Let $\alpha_1, \dots, \alpha_v$ be the points of S, and $\Delta_{h1}, \dots, \Delta_{h\binom{v}{h}}$ be the h-point subsets of $\Omega(0\leqq h\leqq t-1)$. Let $M_h(0\leqq h\leqq t-1)$ be the $\binom{v}{h}$ by λ_0 incidence matrix of blocks on h-point subsets defined by $M_h(i, j)=1$ if $\Delta_{hi}\subseteq B_j$, 0 otherwise. Let $a_i(1\leqq i\leqq v)$ be the column vector of degree λ_0 defined by the j^{th} component of $a_i=1$ if $\alpha_i\in B_j$, 0 otherwise.

PROPOSITION 1. *The adjacency matrix $A_h(h=0, \dots, t-1)$ of a Steiner system $S(t, k, v)$ has an eigenvalue d_h belonging to a_i-a_j $(i\neq j)$ such that*

$$\sum_{h=0}^{t-1}\binom{h}{r}d_h+\binom{k}{r}=\binom{k-1}{r-1}(\lambda_r-\lambda_{r+1}) \quad (r=0, \dots, t-1).$$

PROOF. By the definitions we have

$$\sum_{h=0}^{t-1}\binom{h}{r}A_h+\binom{k}{r}I={}^tM_rM_r \quad (r=0, \dots, t-1), \tag{2}$$

and we have the following for $a_i\neq a_j$:

$${}^tM_rM_r(a_i-a_j)={}^t(f_{r1}, \dots, f_{r\lambda_0}),$$

where

$$f_{rq}=\sum_{\substack{m=1\\(\alpha_i\in B_m)}}^{\lambda_0}\binom{|B_m\cap B_q|}{r}-\sum_{\substack{m=1\\(\alpha_j\in B_m)}}^{\lambda_0}\binom{|B_m\cap B_q|}{r}.$$

Let us assume that $\alpha_i\in B_q$, $\alpha_j\notin B_q$. Then we have

$$\sum_{\substack{m=1\\(\alpha_j\in B_m)}}^{\lambda_0}\binom{|B_m\cap B_q|}{r}=|(J, B_m): J\subseteq B_q\cap B_m,\ |J|=r,\ \alpha_j\in B_m\}|=\binom{k}{r}\lambda_{r+1},$$

and

$$\sum_{\substack{m=1\\(\alpha_i\in B_m)}}^{\lambda_0}\binom{|B_m\cap B_q|}{r}=\sum_{\substack{m=1\\(\alpha_i\in B_m)}}^{\lambda_0}\binom{|B_m\cap B_q|-1}{r-1}+\sum_{\substack{m=1\\(\alpha_i\in B_m)}}^{\lambda_0}\binom{|B_m\cap B_q|-1}{r}$$
$$=|\{(J, B_m): J\subseteq B_m\cap B_q-\{\alpha_i\},\ |J|=r-1,\ \alpha_i\in B_m\}|$$
$$+|\{(J, B_m): J\subseteq B_m\cap B_q-\{\alpha_i\},\ |J|=r,\ \alpha_i\in B_m\}|$$

$$=\binom{k-1}{r-1}\lambda_r+\binom{k-1}{r}\lambda_{r+1}.$$

Therefore, $f_{rq}=\binom{k-1}{r-1}(\lambda_r-\lambda_{r+1})$ when $a_i\in B_q$, $a_j\notin B_q$. Hence we have

$$^tM_rM_r(a_i-a_j)=\binom{k-1}{r-1}(\lambda_r-\lambda_{r+1})(a_i-a_j). \tag{3}$$

Setting $r=t-1$, $t-2, \ldots$ and 0 in (2) and (3) by turns, we find that a_i-a_j is an eigenvector of A_{t-1}, $A_{t-2}, \ldots$ and A_0 and get the result.

PROPOSITION 2. *For any point a and any block B of a Steiner system $S(t, k, v)$ with $a\in B$, the number of blocks which contain a and have exactly h points in common with B is $d_h+k(x_h-d_h)/v$ $(0\leqq h\leqq t-1)$.*

PROOF. We may assume that $a=a_1$, $B=B_1$ and that all the blocks containing a are $B_1, \ldots, B_{\lambda_1}$. Let us set $b_i=a_1-a_i$ $(i=1, \ldots, v)$. Then we have $A_h(b_1+\cdots\cdots+b_v)=d_h(b_1+\cdots\cdots+b_v)$ $(0\leqq h\leqq t-1)$, where $b_1+\cdots\cdots+b_v={}^t(v-k, \ldots, v-k, -k, \ldots, -k)$. Let f be the number of blocks each of which contains a_1 and has exactly h points in common with B_1. Calculating the 1st component of $A_h(b_1+\cdots\cdots+b_v)$, we get $f(v-k)+(x_h-f)(-k)=d_h(v-k)$. Hence, $f=d_h+k(x_h-d_h)/v$.

PROPOSITION 3. *Let S be a Steiner system $S(t, k, v)$. If there is an integer i $(0\leqq i\leqq t-1)$ with $x_i>0$ and $v>(k-t+1)(k-i)+k$, then for any integer j with $0\leqq j\leqq i$ and $i+t-1-j\leqq k$ there exist three blocks B_1, B_2 and B_3 of S such that $|B_1\cap B_2|=i$, $|B_1\cap B_3|=t-1$ and $|B_2\cap B_3|=j$.*

PROOF. By the assumption there exist two blocks B_1 and B_2 with $|B_1\cap B_2|=\text{i}$. Let $a_1, \ldots, a_j$ be j points of $B_1\cap B_2$ and $a_{j+1}, \ldots, a_{t-1}$ be $t-1-j$ points of B_1-B_2. Let $W_1(=B_1)$, $W_2, \ldots, W_{\lambda_{t-1}}$ be the blocks which contain $a_1, \ldots, a_{t-1}$. Let us suppose $W_h\cap B_2\supsetneqq\{a_1, \ldots, a_j\}$ for $h=2, \ldots, \lambda_{t-1}$. Then $W_h\cap(B_2-B_1)\neq\phi$ holds for $h=2, \ldots, \lambda_{t-1}$. Hence we have

$$\frac{v-t+1}{k-t+1}-1\leqq k-i,$$

that is, $v\leqq(k-t+1)(k-i)+k$, a contradiction.

Hereafter we assume that S is block-schematic.
Then we have

$$A_iA_j=\sum_{h=0}^{k}\mu(i, j, h)A_h \quad (0\leqq i, j\leqq k), \tag{4}$$

where $\mu(i, j, h)$ is a non-negative integer defined by the following: When there exist blocks B_p and B_q with $|B_p \cap B_q| = h$, $\mu(i, j, h) = |\{B \in \boldsymbol{B} : |B_p \cap B| = i, |B_q \cap B| = j\}|$, and when there exist no blocks B_p and B_q with $|B_p \cap B_q| = h$, $\mu(i, j, h) = 0$. Now the following equalities are easily verified (cf. [3]):

$$\begin{aligned} &\sum_{i=0}^{k} \mu(i, j, h) = x_j \text{ if } x_h > 0, \\ &\mu(i, j, h) = \mu(j, i, h), \quad \mu(i, j, k) = \delta_{ij} x_i, \\ &\mu(i, j, h) x_h = \mu(h, j, i) x_i = \mu(h, i, j) x_j. \end{aligned} \tag{5}$$

The intersection matrix $P_j (j=0, \ldots, t-1, k)$ is the $(t+1) \times (t+1)$ matrix whose (i, h) entry is $\mu(i, j, h) (i, h=0, \ldots, t-1, k)$. We remark that the map $A_j \mapsto P_j$ $(j=0, \ldots, t-1, k)$ extends to an isomorphism from the algebra $\langle A_0, \ldots, A_{t-1}, A_k \rangle$ over $\boldsymbol{R}$ (the set of real numbers) to the algebra $\langle P_0, \ldots, P_{t-1}, P_k \rangle$ over $\boldsymbol{R}$ (cf. [3]). In particular, any eigenvalue of Aj is an eigenvalue of P_j, and vice versa. We say that a vector x is standard when the first entry of x is 1. Then by [1, Theorem 2.4.1] we have

PROPOSITION 4. *If the multiplicity of an eigenvalue θ of P_j $(0 \leqq j \leqq t-1)$ is one, then the multiplicity of θ as an eigenvalue of A_j is*

$$\lambda_0 / (u, v)$$

where $(\ ,\ )$ denotes the usual inner product of row vectors and u and ${}^t v$ are unique standardized left and right eigenvectors of P_j such that $uP_j = \theta u$ and $p_j\, {}^t v = \theta\, {}^t v$ respectively.

3. Proof of Theorem 1

Let S be a block-schematic Steiner system $S(t, t+2, v)$ $(t \geqq 3)$. By (1) we get the following lemma.

LEMMA 1. $x_i + \binom{i+1}{i} x_{i+1} + \cdots\cdots + \binom{t-1}{i} x_{t-1} = (\lambda_i - 1) \binom{t+2}{i}$ $(i=0, \ldots, t-1)$. *In particular*

$$x_{t-1} = \frac{(v-t-2)(t+2)(t+1)t}{18},$$

$$x_{t-2} = \frac{(v-t-11)(v-t-2)(t+2)(t+1)t(t-1)}{288},$$

$$x_{t-3} = \frac{(v-t-2)(t+2)(t+1)t(t-1)(t-2)\{(v-t-2)^2 - 13(v-t-2) + 72\}}{7200},$$

$$x_{t-4} = \frac{(v-t-2)(t+2)(t+1)t(t-1)(t-2)(t-3)\{(v-t-2)^3 - 18(v-t-2)^2 + 137(v-t-2) - 600\}}{259200}.$$

By [6] we have

$$x_i>0(i=0, \dots, t-1). \tag{6}$$

Then by Lemma 1 we have

$$v\geqq t+13(t\geqq 4),\ v\geqq 15(t=3). \tag{7}$$

LEMMA 2. *$(v-t)(v-t-1)$ is not divisible by 4 nor by any prime between 3 and $t+2$.*

PROOF. Let p be 4 or a prime between 3 and $t+2$. Then we have

$$\lambda_{t+2-p}=\frac{(v-t-2+p)(v-t-3+p)\cdots\cdots(v-t+1)}{p(p-1)\cdots\cdots 3}.$$

Since λ_{t+2-p} is an integer, we get the result.

By Proposition 1 we have

LEMMA 3. *The adjacency matrix $A_h(h=0, \cdots, t-1)$ of S has an eigenvalue d_h belonging to $a_i-a_j(i\neq j)$ such that*

$$\sum_{h=0}^{t-1}\binom{h}{r}d_h+\binom{t+2}{r}=\binom{t+1}{r-1}(\lambda_r-\lambda_{r+1})\ (r=0, \dots, t-1).$$

The following lemma is easily verified.

LEMMA 4. *If $i-j\geqq 4$, then $\mu(i, t-1, j)=0$ $(0\leqq j<i\leqq t-1)$.*

LEMMA 5.

(i) $x_{t-1}^2=\sum_{i=t-4}^{t-1}\mu(t-1, t-1, i)x_i+x_{t-1}$,

$d_{t-1}^2=\sum_{i=t-4}^{t-1}\mu(t-1, t-1, i)d_i+x_{t-1}$.

(ii) $x_{t-2}x_{t-1}=\sum_{i=t-5}^{t-1}\mu(t-2, t-1, i)x_i$,

$d_{t-2}d_{t-1}=\sum_{i=t-5}^{t-1}\mu(t-2, t-1, i)d_i$, *where*

$\mu(t-2, t-1, t-1)=\mu(t-1, t-1, t-2)x_{t-2}/x_{t-1}$.

(iii) $x_{t-3}x_{t-1}=\sum_{i=t-6}^{t-1}\mu(t-3, t-1, i)x_i$,

$d_{t-3}d_{t-1}=\sum_{i=t-6}^{t-1}\mu(t-3, t-1, i)d_i$, *where*

$\mu(t-3, t-1, t-1)=\mu(t-1, t-1, t-3)x_{t-3}/x_{t-1}$ *and*

$\mu(t-3, t-1, t-2)=\mu(t-2, t-1, t-3)x_{t-3}/x_{t-2}$.

PROOF. Multiplying both sides of (4) by the all-1 vector of degree λ_0, we have

$$x_i x_j = \sum_{h=0}^{t-1} \mu(i, j, h) x_h + \mu(i, j, k).$$

On the other hand, multiplying both sides of (4) by $a_q - a_r$ $(q \neq r)$, we have the following by Lemma 3:

$$d_i d_j = \sum_{h=0}^{t-1} \mu(i, j, h) d_h + \mu(i, j, k).$$

Hence by (5) and Lemma 4, we get (i), (ii) and (iii).

LEMMA 6.

(i) *$\mu(t-1, t-1, i) \leqq x_{t-1}$ holds and $\mu(t-1, t-1, i)$ is divisible by $x_{t-1}/(x_{t-1}, x_i)$, where (x_{t-1}, x_i) is the greatest common divisor of x_{t-1} and x_i. Moreover if $t-4 \leqq i \leqq t-1$, then $1 \leqq \mu(t-1, t-1, i)$.*

(ii) *$\mu(t-2, t-1, i) \leqq x_{t-1}$ holds and $\mu(t-2, t-1, i)$ is divisible by $x_{t-2}/(x_{t-2}, x_i)$. Moreover if $v \geqq t+15$ and $t-5 \leqq i \leqq t-1$, then $1 \leqq \mu(t-2, t-1, i)$.*

(iii) *$\mu(t-3, t-1, i) \leqq x_{t-1}$ holds and $\mu(t-3, t-1, i)$ is divisible by $x_{t-3}/(x_{t-3}, x_i)$. Moreover if $v \geqq t+18$ and $t-6 \leqq i \leqq t-1$, then $1 \leqq \mu(t-3, t-1, i)$.*

PROOF. By (5), (6), (7) and Proposition 3, we get (i), (ii) and (iii).

LEMMA 7. $3 \leqq t \leqq 40$.

PROOF. Let us suppose $t \geqq 41$. Let us set $z = v-t-2$. Then we have

$$z^3 - 18z^2 + 137z - 600 \geqq z^3 - 18z^2 + 122z,$$

because $z = v-t-2 \geqq (2t+3)-t-2 \geqq 42$. Now by Lemmas 5(i) and 6(i), we have $x_{t-1}^2 > x_{t-4}$. Hence,

$$\frac{800(t+2)(t+1)t}{(t-1)(t-2)(t-3)} > z^2 - 18z + 122.$$

Since the left-hand side of the above inequality is less than 1000, we get $z < 40$, a contradiction.

LEMMA 8. *If $t=3$, then $15 \leqq v \leqq 32$. If $t \geqq 4$, then $\max(2t+3, t+13) \leqq v$ and the following inequality holds:*

$$800(t+2)(t+1)t > (t-1)(t-2)(t-3)(v-t-2)(v-t-20).$$

PROOF. By (7) and $v \geqq 2t+3$, we get the lower bounds on v. If $t=3$, we have $v \leqq 32$ by [5]. If $t \geqq 4$, then by $x_{t-1}^2 > x_{t-4}$ we get the following (cf. Proof of Lemma 7):

$$\frac{800(v-t-2)(t+2)(t+1)t}{(t-1)(t-2)(t-3)} >$$
$$(v-t-2)^3 - 18(v-t-2)^2 + 137(v-t-2) - 600.$$

Since the right-hand side of the above inequality is greater than $(v-t-2)^3 - 18(v-t-2)^2$, we complete the proof.

LEMMA 9. $\mu(t-1, t-1, t-4) \leqq 40$, $\mu(t-1, t-1, t-3) \leqq \min\{10t-10, 120\}$, $16 \leqq \mu(t-1, t-1, t-2) \leqq 16 + \min\{12(t-2), 36\} + \min\{2(t-2)(t-3), 16\}$, $\frac{v-t-5}{3} + 9(t-1) \leqq \mu(t-1, t-1, t-1) \leqq \frac{v-t-5}{3} + 9(t-1) + \frac{3(t-1)(t-2)}{2} + \frac{(t-1)(t-2)(t-3)}{6}$.

PROOF. Let B_1 and B_2 be blocks of S with $|B_1 \cap B_2| = i \,(t-4 \leqq i \leqq t-1)$. Let us set

$$C_{i,j,h}(B_1, B_2) = |\{B \in \boldsymbol{B} : |B \cap B_1 \cap B_2| = h, \ |B \cap (B_1 - B_2)| = |B \cap (B_2 - B_1)| = j\}|$$

$(0 \leqq j \leqq 3,\ t-4 \leqq h \leqq t-1)$. Then we have

$$\mu(t-1, t-1, i) = \sum_{j=t-1-i}^{3} C_{i,j,t-1-j}(B_1, B_2),$$

where

$$C_{i,j,t-1-j}(B_1, B_2) \leqq \binom{t+2-i}{j}\binom{i}{t-1-j}[(t+2-i)/j] \quad (j \neq 0), \text{ and}$$
$$C_{i,j,t-1-j}(B_1, B_2) \leqq \binom{t+2-i}{j}^2 \text{ if } i \leqq t-2.$$

($[(t+2-i)/j]$ is the greatest integer not exeeding $(t+2-i)/j$.) Moreover we find

$$C_{t-2,1,t-2}(B_1, B_2) = 4 \times 4 = 16,$$
$$C_{t-1,1,t-2}(B_1, B_2) = 3 \times (t-1) \times 3 = 9(t-1) \text{ and}$$
$$C_{t-1,0,t-1}(B_1, B_2) = (v-t-5)/3.$$

Hence we get the result.

By [8] we have

LEMMA 10. *There is no Steiner system $S(4, 6, 18)$.*

Now by using computer we search the elements of the set

$W_1=\{$all $(t, v, \mu(t-1, t-1, t-4), \mu(t-1, t-1, t-3), \mu(t-1, t-1, t-2), \mu(t-1, t-1, t-1))$ satisfying Lemmas 1, 2, 3, 5(i), 6(i), 7, 8, 9, 10$\}$.

We note that every element of W_1 satisfies $(t, v)=(3, 17)$ or $(t, t+23)$ $(3\leqq t\leqq 8)$.

Next by using computer we search the elements of the set

$W_2=\{$all $(t, v, \mu(t-1, t-1, t-4), \mu(t-1, t-1, t-3), \mu(t-1, t-1, t-2), \mu(t-1, t-1, t-1), \mu(t-2, t-1, t-5), \mu(t-2, t-1, t-4), \ldots, \mu(t-2, t-1, t-1))$ satisfying Lemmas 5(ii) and 6(ii) each of which satisfies that $(t, v, \mu(t-1, t-1, t-4), \mu(t-1, t-1, t-3), \mu(t-1, t-1, t-2), \mu(t-1, t-1, t-1))$ is an element of $W_1\}$.

We note that every element of W_2 satisfies $(t, v)=(3, 17)$ or $(t, t+23)$ $(3\leqq t\leqq 5)$.

Similarly by using computer we search the elements of the set

$W_3=\{$all $(t, v, \mu(t-1, t-1, t-4), \mu(t-1, t-1, t-3), \mu(t-1, t-1, t-2), \mu(t-1, t-1, t-1), \mu(t-2, t-1, t-5), \mu(t-2, t-1, t-4), \ldots, \mu(t-2, t-1, t-1), \mu(t-3, t-1, t-6), \mu(t-3, t-1, t-5), \ldots, \mu(t-3, t-1, t-1))$ satisfying Lemmas 5(iii) and 6(iii) each of which satisfies that $(t, v, \mu(t-1, t-1, t-4), \mu(t-1, t-1, t-3), \mu(t-1, t-1, t-2), \mu(t-1, t-1, t-1), \mu(t-2, t-1, t-5), \mu(t-2, t-1, t-4), \ldots, \mu(t-2, t-1, t-1))$ is an element of $W_2\}$.

We note that $|W_3|$ is thirty-one and there exist one element with $(t, v)=(3, 17)$, one element with $(t, v)=(3, 26)$ and twenty-nine elements with $(t, v)=(4, 27)$ in W_3. Really the elements of W_3 with $(t, v)=(4, 27)$ satisfy the following:

$$(\mu(3,3,0),\ \mu(3,3,1),\ \mu(3,3,2),\ \mu(3,3,3),\ \mu(2,3,0),\ \mu(2,3,1),\ \mu(2,3,2), \mu(1,3,0),\ \mu(1,3,1))$$
$$=(4,10,28,34,6,50,28,96,40),\ (4,10,28,34,12,45,32,84,50),$$
$$=(4,10,28,34,18,40,36,72,60),\ (4,10,28,34,24,35,40,60,70),$$
$$=(4,10,28,34,30,30,44,48,80),\ (4,10,28,34,36,25,48,36,90),$$
$$=(4,10,28,34,42,20,52,24,100),\ (4,10,28,34,48,15,56,12,110),$$
$$=(6,10,24,40,3,50,34,96,40),\ (6,10,24,40,9,45,38,84,50),$$

$=(6, 10, 24, 40, 15, 40, 42, 72, 60)$, $(6, 10, 24, 40, 21, 35, 46, 60, 70)$,
$=(6, 10, 24, 40, 27, 30, 50, 48, 80)$, $(6, 10, 24, 40, 33, 25, 54, 36, 90)$,
$=(6, 10, 24, 40, 39, 20, 58, 24, 100)$, $(6, 10, 24, 40, 45, 15, 62, 12, 110)$,
$=(10, 5, 32, 34, 3, 55, 18, 84, 45)$, $(10, 5, 32, 34, 9, 50, 22, 72, 55)$,
$=(10, 5, 32, 34, 15, 45, 26, 60, 65)$, $(10, 5, 32, 34, 21, 40, 30, 48, 75)$,
$=(10, 5, 32, 34, 27, 35, 34, 36, 85)$, $(10, 5, 32, 34, 33, 30, 38, 24, 95)$,
$=(10, 5, 32, 34, 39, 25, 42, 12, 105)$, $(12, 5, 28, 40, 6, 50, 28, 72, 55)$,
$=(12, 5, 28, 40, 12, 45, 32, 60, 65)$, $(12, 5, 28, 40, 18, 40, 36, 48, 75)$,
$=(12, 5, 28, 40, 24, 35, 40, 36, 85)$, $(12, 5, 28, 40, 30, 30, 44, 24, 95)$ or
$=(12, 5, 28, 40, 36, 25, 48, 12, 105)$.

Since $t=3$ or 4, each element of W_3 determines just one intersection matrix P_{t-1} (cf. (5)).

Let us suppose $(t, v)=(3, 17)$. So we note

$$P_2=\begin{pmatrix} 10 & 8 & 6 & 0 \\ 10 & 8 & 9 & 0 \\ 20 & 24 & 24 & 40 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Since the multiplicity of the eigenvalue $d_2=6$ of P_2 is one, we can calculate that of the eigenvalue 6 of A_2 by Proposition 4. Really we find easily that it is 33.333……, a contradiction.

Next let us suppose $(t, v)=(3, 26)$. So we note

$$P_2=\begin{pmatrix} 30 & 24 & 12 & 0 \\ 30 & 24 & 33 & 0 \\ 10 & 22 & 24 & 70 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

By the similar argument to that in the case $(t, v)=(3, 17)$, we find easily that the multiplicity of the eigenvalue 18 of A_2 is 119.047……, a contradiction.

Last let us suppose $(t, v)=(4, 27)$. So we note that P_3 is one of the twenty-nine matrices, but we find the following by using computer and the similar argument to that in the case $(t, v)=(3, 17)$: If P_3 is anyone of the twenty-nine matrices, the multiplicity of the eigenvalue 50 of A_3 is 445.714 ……, a contradiction.

Thus we complete the proof of Theorem 1.

4. Proof of Theorem 2

Let S be a block-schematic Steiner system $S(t, t+3, v)$ $(t\geqq 3)$. By

(1), we get the following lemma.

LEMMA 11. $x_i+\binom{i+1}{i}x_{i+1}+\cdots\cdots+\binom{t-1}{i}x_{t-1}=(\lambda_i-1)\binom{t+3}{i}$ $(i=0, \ldots, t-1)$. *In particular*

$$x_{t-1}=\frac{(t+3)(t+2)(t+1)t(v-t-3)}{96},$$
$$x_{t-2}=\frac{(t+3)(t+2)(t+1)t(t-1)(v-t-3)(v-t-19)}{2400},$$
$$x_{t-3}=\frac{(t+3)(t+2)(t+1)t(t-1)(t-2)(v-t-3)\{(v-t-3)^2-21(v-t-3)+200\}}{86400},$$
$$x_{t-4}=(t+3)(t+2)(t+1)t(t-1)(t-2)(t-3)(v-t-3)(v-t-19)\times \{(v-t-3)^2-11(v-t-3)+150\}/4233600,$$
$$x_{t-5}=(t+3)(t+2)(t+1)t(t-1)(t-2)(t-3)(t-4)(v-t-3)\times \{(v-t-3)^4-34(v-t-3)^3+515(v-t-3)^2-4682(v-t-3)+29400\}/270950400.$$

If $x_i=0$ holds for some i $(0\leqq i\leqq t-1)$, then S is a Steiner system $S(3, 6, 22)$, $S(4, 7, 23)$ or $S(5, 8, 24)$ by [6]. Hence from now on we consider the case

$$x_i>0(i=0, \ldots, t-1). \tag{8}$$

Then by Lemma 11 we have

$$v>t+20. \tag{9}$$

LEMMA 12. *$v-t$, $v-t-1$ and $v-t-2$ are divisible by no prime between 5 and $t+3$, and $v-t+1$ is divisible by 4.*

PROOF. Let p be a prime between 5 and $t+3$. Then we have

$$\lambda_{t+3-p}=\frac{(v-t-3+p)(v-t-4+p)\cdots\cdots(v-t+1)}{p(p-1)\cdots\cdots 4} \text{ and}$$
$$\lambda_{t-1}=\frac{v-t+1}{4}.$$

Since λ_{t+3-p} and λ_{t-1} are integers, we get the result.

By Proposition 1 we have

LEMMA 13. *The adjacency matrix $A_h(h=0, \ldots, t-1)$ of S has an eigenvalue d_h belonging to $a_i-a_j(i\neq j)$ such that*

$$\sum_{h=0}^{t-1}\binom{h}{r}d_h+\binom{t+3}{r}=\binom{t+2}{r-1}(\lambda_r-\lambda_{r+1}) \quad (r=0, \ldots, t-1).$$

The following lemma is easily verified.

LEMMA 14. *If* $i-j\geqq 5$, *then* $\mu(i, t-1, j)=0$ $(0\leqq j<i\leqq t-1)$.

By the similar proof to that of Lemma 5 we have

LEMMA 15.

(i) $x_{t-1}^2=\sum_{i=t-5}^{t-1}\mu(t-1, t-1, i)x_i+x_{t-1}$,

$d_{t-1}^2=\sum_{i=t-5}^{t-1}\mu(t-1, t-1, i)d_i+x_{t-1}$.

(ii) $x_{t-2}x_{t-1}=\sum_{i=t-6}^{t-1}\mu(t-2, t-1, i)x_i$,

$d_{t-2}d_{t-1}=\sum_{i=t-6}^{t-1}\mu(t-2, t-1, i)d_i$, *where*

$\mu(t-2, t-1, t-1)=\mu(t-1, t-1, t-2)x_{t-2}/x_{t-1}$.

LEMMA 16.

(i) $\mu(t-1,t-1, i)\leqq x_{t-1}$ *holds and* $\mu(t-1, t-1, i)$ *is divisible by* $x_{t-1}/(x_{t-1}, x_i)$. *Moreover if* $t-5\leqq i\leqq t-1$, *then* $1\leqq\mu(t-1, t-1, i)$.

(ii) $\mu(t-2, t-1, i)\leqq x_{t-1}$ *holds and* $\mu(t-2, t-1, i)$ *is divisible by* $x_{t-2}/(x_{t-2}, x_i)$. *Moreover if* $v\geqq t+24$ *and* $t-6\leqq i\leqq t-1$, *then* $1\leqq\mu(t-2, t-1, i)$.

PROOF. By (5), (8), (9) and Proposition 3, we get (i) and (ii).

LEMMA 17. $3\leqq t\leqq 49$.

PROOF. Let us suppose $t\geqq 50$. Let us set $z=v-t-3$. Then we have $z^4-34z^3+515z^2-4682z+29400\geqq z^2(z-17)^2$.
Now by Lemmas 15(i) and 16(i), we nave $x_{t-1}^2>x_{t-5}$. Hence,

$$\frac{29400(t+3)(t+2)(t+1)t}{(t-1)(t-2)(t-3)(t-4)}>z(z-17)^2.$$

Since the left-hand side of the above inequality is less than 41000, we get $z<50$. Hence we have $2t+4\leqq v<53+t$, and so $t<49$, a contradiction.

LEMMA 18. *If* $t=3$, *then* $23\leqq v\leqq 62$. *If* $t=4$, *then* $24\leqq v\leqq 280$. *If* $t\geqq 5$, *then* $\max\{2t+4, t+20\}\leqq v$ *and the following inequality holds*:

$$29400(t+3)(t+2)(t+1)t>(t-1)(t-2)(t-3)(t-4)(v-t-3)(v-t-20)^2.$$

PROOF. Since $v\geqq 2t+4$ and $v\geqq t+20$, we have lower bounds on v. If $t=3$, we have $v\leqq 62$ by [5]. If $t\geqq 5$, then by $x_{t-1}^2>x_{t-5}$ we get the inequality which yields an upper bound on v (cf. Proof of Lemma 17). If $t=4$, then by Lemmas 15(i) and 16(i), we have $x_3^2 > x_0$. Then,

$$\frac{7^2 6^2 5^2 4^2 (v-7)^2}{9216} > \frac{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1(v-7)(v-23)\{(v-7)^2-11(v-7)+150\}}{4233600}$$

Hence $64400 > (v-23)(v-18)$ holds, and so we have $v \leqq 280$.

LEMMA 19. $\mu(t-1, t-1, t-5) \leqq 140$, $\mu(t-1, t-1, t-4) \leqq \min\{35(t-2), 1295\}$, $\mu(t-1, t-1, t-3) \leqq 45 + \min\{40(t-3), 400\} + \min\{15(t-3)(t-4)/2, 225\}$, $25 \leqq \mu(t-1, t-1, t-2) \leqq 25 + \min\{20(t-2), 100\} + \min\{5(t-2)(t-3), 100\} + \min\{5(t-2)(t-3)(t-4)/6, 25\}$. $\frac{v-t-7}{4} + 16(t-1) \leqq \mu(t-1, t-1, t-1) \leqq \frac{v-t-7}{4} + 16(t-1) + 12\binom{t-1}{2} + 4\binom{t-1}{3} + \binom{t-1}{4}$.

PROOF. Let B_1 and B_2 be blocks of S with $|B_1 \cap B_2| = i \, (t-5 \leqq i \leqq t-1)$. Let us set

$C_{i,j,h}(B_1, B_2) = |\{B \in \boldsymbol{B} : |B \cap B_1 \cap B_2| = h,\ |B \cap (B_1 - B_2)| = |B \cap (B_2 - B_1)| = j\}|$ $(0 \leqq j \leqq 4,\ t-5 \leqq h \leqq t-1)$. Then we have

$$\mu(t-1, t-1, i) = \sum_{j=t-1-i}^{4} C_{i,j,t-1-j}(B_1, B_2),$$

where

$$C_{i,j,t-1-j}(B_1, B_2) \leqq \binom{t+3-i}{j}\binom{i}{t-1-j}[(t+3-i)/j] \quad (j \neq 0), \text{ and}$$
$$C_{i,j,t-1-j}(B_1, B_2) \leqq \binom{t+3-i}{j}^2 \text{ if } i \leqq t-2.$$

Moreover we find

$$C_{t-2,1,t-2}(B_1, B_2) = 5 \times 5 = 25,$$

$$C_{t-1,1,t-2}(B_1, B_2) = 4 \times (t-1) \times 4 = 16(t-1) \text{ and}$$
$$C_{t-1,0,t-1}(B_1, B_2) = (v-t-7)/4.$$

Hence we get the result.

By [5, Proposition 1.3] we have

LEMMA 20. *If* $t=3$, *then* $\mu(2, 2, 1) \neq 25$.

Now by using computer we search the elements of the set

$W_1 = \{$all $(t, v, \mu(t-1, t-1, t-5), \mu(t-1, t-1, t-4), \ldots,$

$\mu(t-1, t-1, t-1))$ satisfying Lemmas 11, 12, 13, 15(i), 17, 18, 19 and 20$\}$.

We note that every element of W_1 satisfies $(t, v)=(6, 45)$ or $(8, 47)$.

Next by using computer we search the elements of the set

$W_2=\{$all $(t, v, \mu(t-1, t-1, t-5), \mu(t-1, t-1, t-4), \ldots, \mu(t-1, t-1, t-1), \mu(t-2, t-1, t-6), \mu(t-2, t-1, t-5), \ldots, \mu(t-2, t-1, t-1))$ satisfying Lemmas 15(ii) and 16(ii) each of which satisfies that $(t, v, \mu(t-1, t-1, t-5), \mu(t-1, t-1, t-4), \ldots, \mu(t-1, t-1, t-1))$ is an element of $W_1\}$.

We note that there is no element in W_2.

Thus we complete the proof of Theorem 2.

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