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On some sublattices of the Leech lattice

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§ 0. Introduction

This paper is a continuation of Harada-Lang [5] and M. L. Lang [8] to investigate the behaviour of the conjugacy classes of the automorphism group of the Leech lattice. In [8], the second author showed that fifteen conjugacy classes behave differently from other classes in connection with Conway-Norton's "Monstrous Moonshine" [2]. In [5], the nonzero-genus property of these fifteen conjugacy classes was studied and five elliptic curves defined over integers were produced.

In this paper, the invariance sublattices, their automorphism groups and realization of them in $.O$ will be investigated. More precisely, let $g \in G = .O$, the automorphism group of the Leech lattice Λ, and $\Lambda_g = \{\lambda \in \Lambda | g(\lambda) = \lambda\}$. For each $g \in G$, the structure of Λ_g will be explicitly determined. The results, including the ranks, the Gram matrices, the automorphism groups will be listed in TABLE 1 at the end of this paper. The identification of some of the well known lattices, such as E_6, E_8, D_{12}^2, Coxeter-Todd, and Barnes-Wall lattices, is also given.

The Leech lattice Λ and its automorphism group $G = .O$ are well known for their unique properties, a dense packing of the 24 dimensional Euclidean space, for example. The question discussed in this paper is: how complete is the Leech lattice in connection with the conjugacy classes of $G = .O$? More precisely, is $Aut(\Lambda_g)$ induced from the normalizer $N_G(<g>)$ of $<g>$ in G for all $g \in G$? It has turned out that it is true for all but 9 conjugacy classes of G. The exceptional classes are -2_A, 2_C, 3_D, -4_A, 4_F, 6_I, 9_C, 12_J, and -20_C.

§ 1. Invariance sublattices

The Leech lattice Λ may be defined in many equivalent ways (see Conway-Sloane [3], Kondo-Tasaka [6]). The following is one of the typical ones. Let

(1) $\Omega = \{\infty, 0, 1, \dots, 22\}$,

(2) G = *the Golay code viewed as a subset of the power set* $P(\Omega)$ *and*

as a vector space over $GF(2)$,

(3) $\{e_\infty, e_0, e_1, \dots, e_{22}\}$ = *the canonical basis of the Euclidean space* $\mathbf{R}^{24}$ *with inner product* $<e_i, e_j> = 2\delta_{ij}$,

(4) $\Lambda_\delta = \{X = (x_i) \in \mathbf{Z}^{24} | \Sigma\, x_i \equiv \delta (mod\ 2)\}$ *for* $\delta = 0$ *or* 1; *and*

(5) $e_X = \sum_{i \in X} e_i$ *for* $X \in P(\Omega)$ (*i. e.* $X \subset \Omega$).

Under the notation above, the Leech lattice Λ is defined to be the union of

$$\{\frac{1}{2} e_X + \Lambda_0 | X \in G\} \text{ and}$$

$$\{\frac{1}{4} e_\Omega + \frac{1}{2} e_X + \Lambda_1 | X \in G\}.$$

Λ is a positive definite even integral unimodular lattice inherited from the inner product $<\ ,\ >$ of $\mathbf{R}^{24}$ defined above. The minimal vectors of Λ have square length 4.

Define $\Lambda_g = \{\lambda \in \Lambda | g(\lambda) = \lambda\}$ for $g \in G$. Λ_g is also a positive definite even integral lattice but not unimodular in general. It is not in general easy to determine Λ_g explicitly. The Mathieu group M_{24} and its extension $2^{12}M_{24}$ are naturally embedded in G. If $g \in M_{24}$ then

$$\Lambda_g = \bigcup_{X \in G_g} \{\{\frac{1}{2} e_X + (\Lambda_0)_g\},\ \{\frac{1}{4} e_\Omega + \frac{1}{2} e_X + (\Lambda_1)_g\}\}$$

where G_g, $(\Lambda_0)_g$, $(\Lambda_1)_g$ are the subsets consisting of the elements fixed by g. Using this result, Kondo-Tasaka [6, 7] treated the case $g \in 2^{12}M_{24}$ also. All necessary information for us can be deduced from these two papers [6] and [7] if $g \in 2^{12}M_{24}$. We will give below an example.

EXAMPLE 1. *Let* $g = 11_A = 1^2 11^2 = (\infty)(0)(1, 2, 4, 8, 16, 9, 18, 13, 3, 6, 12)$ $(5, 10, 20, 17, 11, 22, 21, 19, 15, 7, 14) \in M_{24} \subset G = Aut(\Lambda)$. *Then*

$$\theta_{11_A}(z) = \theta(z, A)$$

Where

$$A = \begin{bmatrix} 4 & 0 & 2 & -1 \\ 0 & 4 & -1 & 2 \\ 2 & -1 & 4 & -1 \\ -1 & 2 & -1 & 4 \end{bmatrix}$$

PROOF. It suffices to find a suitable basis for Λ_{11_A}. Let λ be an element in Λ_{11_A}, then λ is of the form

$$ae_\infty + be_0 + ce_{X_1} + de_{X_2}$$

where

$$X_1 = (1, 2, 4, 8, 16, 9, 18, 13, 3, 6, 12)$$
$$X_2 = (5, 10, 20, 17, 11, 22, 21, 19, 15, 7, 14)$$
$$a,\ b,\ c,\ d \in \mathbf{Z} + \frac{1}{4}\mathbf{Z}$$

It is immediate that λ can be generated by f_1, f_2, f_3, and f_4, where

$$f_1 = \frac{1}{2}e_\infty + \frac{1}{2}e_{X_2}$$
$$f_2 = -e_\infty + \frac{1}{4}e_\Omega$$
$$f_3 = e_\infty + \frac{1}{4}e_\Omega$$
$$f_4 = -e_0 + \frac{1}{4}e_\Omega$$

Let $v_1 = f_1 - f_4$, $v_2 = f_2$, $v_3 = f_3 - f_4$, and $v_4 = f_4$, we have Gram matrix of $\{v_1, v_2, v_3, v_4\} = A$.

The harder cases are when $g \notin 2^{12}M_{24}$. In [8], the second auther found Λ_g explicitly and determined its theta series for all such g.

§ 2. Isometries of Λ_g

In this section, we will discuss the group $Aut(\Lambda_g)$ of the invariance sublattice Λ_g, $g \in G = .O = Aut(\Lambda)$. The normalizer $N_G(<g>)$ acts naturally on Λ_g and so induces a subgroup of $Aut(\Lambda_g)$. It is not hard to determine the kernel of the action of $N_G(<g>)$ on Λ_g. This will give us a lower bound of $|Aut(\Lambda_g)|$. As expected, $N_G(<g>)$ does induce the full $Aut(\Lambda_g)$ for the majority of the conjugacy classes of G.

If the rank of Λ_g is small, say 2 or 4, then a direct computation by hand is in general sufficient. An example is given below.

EXAMPLE 2. *Let* $g = 11_A \in M_{24} \subset G$. *Then* $Aut(\Lambda_g) \cong D_{24}$, *a dihedral group of order* 24.

PROOF. Let $\{v_1, v_2, v_3, v_4\}$ be basis with Gram matrix

$$A_g = \begin{bmatrix} 4 & 0 & 2 & -1 \\ 0 & 4 & -1 & 2 \\ 2 & -1 & 4 & -1 \\ -1 & 2 & -1 & 4 \end{bmatrix}$$

The minimal vectors of Λ_g are of square length 4 and if $v_5=v_1-v_3$, $v_6=v_2-v_4$, then

$$X_g=\{\pm v_1, \pm v_2, \pm v_3, \pm v_4, \pm v_5, \pm v_6\}$$

is the set of all minimal vectors of Λ_g. The Gram matrix of the set $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ is

$$B=\begin{bmatrix} 4 & 0 & 2 & -1 & 2 & 1 \\ 0 & 4 & -1 & 2 & 1 & 2 \\ 2 & -1 & 4 & -1 & -2 & 0 \\ -1 & 2 & -1 & 4 & 0 & -2 \\ 2 & 1 & -2 & 0 & 4 & 1 \\ 1 & 2 & 0 & -2 & 1 & 4 \end{bmatrix}$$

Next we will show that $Aut(\Lambda_g)$ contains a subgroup $<\sigma\rho, \tau>\cong D_{24}$ which is transitive on X_g. Define

$$\sigma: v_1\to v_2,\ v_2\to -v_1,\ v_3\to v_6,\ v_4\to -v_5$$
$$\rho: v_1\to -v_3,\ v_2\to -v_6,\ v_3\to v_5,\ v_4\to -v_2$$
$$\tau: v_1\to v_2,\ v_2\to v_1,\ v_3\to v_4,\ v_4\to v_3$$

Then $\sigma, \rho, \tau \in Aut(\Lambda_g)$, $o(\sigma\rho)=12$, $o(\tau)=2$ and $(\rho\sigma)^\tau=(\sigma\rho)^{-1}$. Hence $<\sigma\rho, \tau>\cong D_{24}$. It can readily be checked that $<\sigma\rho, \tau>$ is transitive on X_g.

To complete the proof, it suffices to show $|Aut(\Lambda_g)_{v_1}|=2$. Let $\gamma\in Aut(\Lambda_g)$ such that $\gamma(v_1)=v_1$. The matrix B implies $\gamma(v_2)=\varepsilon v_2$ with $\varepsilon=\pm 1$. Suppose $\varepsilon=1$: i.e. $\gamma(v_1)=v_1$, $\gamma(v_2)=v_2$. Since $2=<v_1, v_3>=<v_1, \gamma(v_3)>$, we get $\gamma(v_3)=v_3$ or v_5 by inspecting the first row of B. But an inspection of the second row forces $\gamma(v_3)=v_3$. Likewise, $\gamma(v_4)=v_4$. Thus $\gamma=1$.

Suppose $\varepsilon=-1$. Again $\gamma(v_3)=v_3$ or v_5. But by inspecting the second row of B, we must conclude $\gamma(v_3)=v_5$ and $\gamma(v_5)=v_3$. Likewise, $\gamma(v_4)=-v_6$ and $\gamma(v_6)=-v_4$. Thus γ is uniquely determined and $\gamma^2=1$, as desired.

If the rank of Λ_g is 6 or more, the method described in EXAMPLE 2 is sometimes tedious. In general, we do the following procedure:

STEP 1. *Find a subset $X_g\subset\Lambda_g$ on which $Aut(\Lambda_g)$ acts faithfully.* A typical choice of such an X_g. is

$$X_g=\{\lambda\in\Lambda_g | <\lambda, \lambda>=<v, v> \text{ for some } v\in B_g\}$$

where

B_g is a basis for Λ_g.

3_C $3^9/1^3$	$3E_6^{-1}$	$W(E_6)$	$W(E_6)$
3_D 3^8	$3E_8$	$W(E_8)$	*
-4_A $1^8 4^8/2^8$	$2E_8$	$W(E_8)$	*
4_B $4^8/2^4$	$2D_4$	$W(D_4)$	$W(D_4)$
4_C $1^4 2^2 4^4$	$\begin{bmatrix} 4 & 2 & 1 & 1 & 1 & -2 & 0 & 1 & -1 & -1 \\ 2 & 4 & 1 & 1 & 1 & 0 & 2 & 1 & -1 & -1 \\ 1 & 1 & 4 & 1 & 1 & 1 & 1 & 0 & -1 & -1 \\ 1 & 1 & 1 & 4 & 2 & 0 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 2 & 4 & 0 & 2 & 2 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 & 4 & 2 & 1 & 1 & 1 \\ 0 & 2 & 1 & 2 & 2 & 2 & 4 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2 & 2 & 1 & 1 & 4 & 1 & 1 \\ -1 & -1 & -1 & 2 & 0 & 1 & 1 & 1 & 4 & 2 \\ -1 & -1 & -1 & 2 & 0 & 1 & 1 & 1 & 2 & 4 \end{bmatrix}$	$[2^{14}3^2 5]$	$[2^{14}3^2 5]$
-4_C $2^6 4^4/1^4$	$\begin{bmatrix} 6 & 2 & 2 & 2 & 2 & 2 \\ 2 & 4 & 0 & 0 & 0 & 0 \\ 2 & 0 & 4 & 0 & 0 & 0 \\ 2 & 0 & 0 & 4 & 0 & 0 \\ 2 & 0 & 0 & 0 & 4 & 0 \\ 2 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}$	$2^6 S_6$	$2^6 S_6$
4_D $2^4 4^4$	$\begin{bmatrix} 2D_4 & 0 \\ 0 & 2D_4 \end{bmatrix}$	$W(D_4)\wr Z_2$	$W(D_4)\wr Z_2$
4_F 4^6	41_6	$2^6 S_6$	*
5_B $1^4 5^4$	$\begin{bmatrix} 4 & -2 & -2 & -2 & 1 & 1 & 2 & 2 \\ -2 & 4 & 2 & 2 & 0 & 0 & 1 & -1 \\ -2 & 2 & 4 & 2 & 0 & 1 & 1 & -1 \\ -2 & 2 & 2 & 4 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 4 & 1 & 3 & 1 \\ 1 & 0 & 1 & 0 & 1 & 4 & 1 & 3 \\ 2 & 1 & 1 & 0 & 3 & 1 & 6 & 0 \\ 2 & -1 & -1 & 0 & 1 & 3 & 0 & 6 \end{bmatrix}$	$[2^7 3^2 5^2]$	$[2^7 3^2 5^2]$
5_C $5^5/1$	A_4^{-1}	$W(A_4)$	$W(A_4)$
6_C $1^4 2.6^5/3^4$	$2E_6$	$W(E_6)$	$W(E_6)$
-6_C $2^5 3^4 6/1^4$	$6E_6^{-1}$	$W(E_6)$	$W(E_6)$
-6_D $1^5 3.6^4/2^4$	$3E_6^{-1}$	$W(E_6)$	$W(E_6)$
6_E $1^2 2^2 3^2 6^2$	$\begin{bmatrix} 4 & 0 & 1 & 1 & 2 & 0 & 2 & 0 \\ 0 & 4 & 1 & 1 & 2 & 0 & 2 & 2 \\ 1 & 1 & 4 & 2 & 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 4 & 2 & -1 & 2 & 2 \\ 2 & 2 & 1 & 2 & 4 & -2 & 1 & 1 \\ 0 & 0 & 1 & -1 & -2 & 4 & 0 & 0 \\ 2 & 0 & 2 & 2 & 1 & 0 & 4 & 0 \\ 0 & 2 & 2 & 2 & 1 & 0 & 0 & 4 \end{bmatrix}$	$[2^8 3^3]$	$[2^8 3^3]$
-6_E $2^4 6^4/1^2 3^2$	$\begin{bmatrix} 4 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix}$	$D_{12}\wr Z_2$	$D_{12}\wr Z_2$

6_F	$3^3 6^3/1.2$	$3D_4$	$(W(D_4)$	$W(D_4)$
-6_F	$1.6^6/2^2 3^3$	$\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$	D_{12}	D_{12}
6_G	$2^3 6^3$	$\begin{bmatrix} 6 & 2 & 2 & 2 & 2 & 0 \\ 2 & 6 & 4 & 4 & -2 & 2 \\ 2 & 4 & 6 & 4 & -2 & 2 \\ 2 & 4 & 4 & 6 & 0 & 2 \\ 2 & -2 & -2 & 0 & 6 & 2 \\ 0 & 2 & 2 & 2 & 2 & 6 \end{bmatrix}$	$[2^7 3^2]$	$[2^7 3^2]$
6_I	6^4	$6I_4$	$2^4 S_4$	*
7_B	$1^3 7^3$	$\begin{bmatrix} 4 & 0 & 2 & -1 & 2 & 0 \\ 0 & 4 & 2 & -2 & 1 & 0 \\ 2 & 2 & 4 & -1 & 2 & 1 \\ -1 & -2 & -1 & 4 & 0 & 2 \\ 2 & 1 & 2 & 0 & 4 & 2 \\ 0 & 0 & 1 & 2 & 2 & 4 \end{bmatrix}$	$[2^5 3.7]$	$[2^5 3.7]$
8_B	$2^4 8^4/4^4$	$4I_4$	$2^4 S_4$	$2^4 S_4$
-8_C	$1^4 8^4/2^2 4^2$	$2D_4$	$W(D_4)$	$W(D_4)$
8_D	$8^4/4^2$	$4I_2$	D_8	D_8
8_E	$1^2 2.4.8^2$	$\begin{bmatrix} 4 & 0 & 1 & 1 & 2 & 1 \\ 0 & 4 & -1 & -1 & 2 & -1 \\ 1 & -1 & 4 & 0 & -1 & 1 \\ 1 & -1 & 0 & 4 & -1 & 1 \\ 2 & 2 & -1 & -1 & 4 & 1 \\ 1 & -1 & 1 & 1 & 1 & 4 \end{bmatrix}$	$[2^7 3]$	$[2^7 3]$
-8_E	$2^3 4.8^2/1^2$	$\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 6 & 4 & 4 \\ 0 & 4 & 8 & 0 \\ 0 & 4 & 0 & 8 \end{bmatrix}$	$[2^5 3]$	$[2^5 3]$
8_F	$4^2 8^2$	$4D_4$	$W(D_4)$	$W(D_4)$
9_B	$9^3/3$	$\begin{bmatrix} 6 & 3 \\ 3 & 6 \end{bmatrix}$	D_{12}	D_{12}
9_C	$1^3 9^3/3^3$	$\begin{bmatrix} 4 & 1 & 1 & 2 \\ 1 & 4 & 1 & 2 \\ 1 & 1 & 4 & -1 \\ 2 & 2 & -1 & 4 \end{bmatrix}$	$[2^4 3^2]$	*
10_D	$1^2 2.10^3/5^2$	$2A_4$	$W(A_4)$	$W(A_4)$
-10_D	$2^3 5^2 10/1^2$	$10A_4^{-1}$	$W(A_4)$	$W(A_4)$
-10_E	$1^3 5.10^2/2^2$	$5A_4^{-1}$	$W(A_4)$	$W(A_4)$
10_F	$2^2 10^2$	$\begin{bmatrix} 6 & 4 & 0 & 0 \\ 4 & 6 & 0 & 0 \\ 0 & 0 & 6 & 4 \\ 0 & 0 & 4 & 6 \end{bmatrix}$	$2^4.2$	$2^4.2$
11_A	$1^2 11^2$	$\begin{bmatrix} 4 & 0 & 2 & -1 \\ 0 & 4 & -1 & 2 \\ 2 & -1 & 4 & -1 \\ -1 & 2 & -1 & 4 \end{bmatrix}$	D_{24}	D_{24}

If B_g consists of minimal vectors of Λ_g only, then X_g is the set of all minimal vectors of Λ_g, which was the case for $g=11_A$. We will get a (crude) upper bound $|Aut(\Lambda_g)|\leqq|X_g|!$.

STEP 2. *Express X_g as a union of $Aut(\Lambda_g)$ invariant subsets.* A typical expression is

$$X_g=\cup S_g(\mu_1, \mu_2)$$

where $S_g(\mu_1, \mu_2)=\{x\in X_g|$ the number of $y\in X_g$ such that $<x, y>=\mu_1$ is $\mu_2\}$. We will obtain an upper bound

$$|Aut(\Lambda_g)|\leqq\prod(|S_g(\mu_1, \mu_2)|!)$$

If $Aut(\Lambda_g)$ acts transitively on X_g then $X_g=S_g(\mu_1, \mu_2)$ whenever $S_g(\mu_1, \mu_2)\neq\phi$ and so this step will yield no information. Knowing that $Aut(\Lambda_g)$ is transitive, however, is a useful information.

STEP 3. Suppose $Aut(\Lambda_g)$ is transitive on X_g. Pick $x\in X_g$ and investigate the action of the stabilizer $Aut(\Lambda_g)_x$ on $X_g\backslash\{x\}$ expressing $X_g\backslash\{x\}$ as a union of $Aut(\Lambda_g)$ invariant subsets of some kind.

The following is one of the harder cases.

EXAMPLE 3. *Let $g=6_E=1^2 2^2 3^2 6^2\in M_{24}\subset G$. Then $|Aut(\Lambda_{6_E})|=6912=2^8 3^3$.*

PROOF. Let $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$ be basis with Gram matrix

$$A_g=\begin{bmatrix} 4 & 0 & 1 & 1 & 2 & 0 & 2 & 0 \\ 0 & 4 & 1 & 1 & 2 & 0 & 0 & 2 \\ 1 & 1 & 4 & 2 & 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 4 & 2 & -1 & 2 & 2 \\ 2 & 2 & 1 & 2 & 4 & -2 & 1 & 1 \\ 0 & 0 & 1 & -1 & -2 & 4 & 0 & 0 \\ 2 & 0 & 2 & 2 & 1 & 0 & 4 & 0 \\ 0 & 2 & 2 & 2 & 1 & 0 & 0 & 4 \end{bmatrix}$$

The minimal vectors of Λ_g are of square length 4 and the set of all minimal vectors X_g is of size 72. Let $X_g=\{\pm v_1, \pm v_2, \dots, \pm v_{36})$. The Gram matrix of $\{v_1, v_2, \dots, v_{36}\}=\mathrm{B}$ is listed in TABLE 2. By the method we described above we conclude that

$$|Aut(\Lambda_{6_E})|\leq 6912$$

Next, we will show that the lower bound of $|Aut(\Lambda_{6_E})|$ is also 6912. By Wilson [9], the normalizer $N_G(<6_E>)$ *has order 82944. Let* $\sigma\in$

$N_G(<6_E>)$ act trivally on Λ_{6E}. Inspecting the matrix A_g, σ fixes two vectors w_1 and w_2 with $<w_1, w_1>=<w_2, w_2>=6$, and $<w_1, w_2>=0$. This implies that $\sigma \in \cdot 633 \cong M_{12}$ (see Conway [1] for notation). Since $N_{M_{12}}(<6_E>)$ has order 12, we conclude that $\overline{N_G(6_E)}$ has order$\geq 82944/12=6912$.

We summarize our results in the table below.

(0). *The last column is the group of isometries induced by* $N_G(<g>)$.
(1). * *means that* $N_G(<g>)$ *does not induces* $Aut(\Lambda_g)$.
(2). $W(\ )$ *denotes the Weyl group.*
(3). $[n]$ *denotes an arbitrary group of order* n.
(4). Λ_{2A} *is the Barnes-Wall lattice (see Conway-Sloane* [3]).
(5). Λ_{2C} *is the* D_{12}^2 *lattice (see Coxeter-Todd* [4]).
(6). Λ_{3B} *is the Coxeter-Todd lattice (see Conway-Sloane* [3]).

TABLE 1.

Frame shape	*Gram matrix*	*Isometry*	$\overline{N_G(g)}$
$1_A\ 1^{24}$	Leech lattice	$.O$	$.O$
$2_A\ 1^8 2^8$	Barnes-Wall [3]	$2^{1+8}O_8^+(2)$	$2^{1+8}O_8^+(2)$
$-2_A\ 2^{16}/1^8$	$2E_8$	$W(E_8)$	*
$2_C\ 2^{12}$	$\begin{bmatrix} 4 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 4 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 4 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 4 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 4 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 4 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 4 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 4 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 4 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 4 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 6 \end{bmatrix}$	$2^{11}S_{12}$	*
$3_B\ 1^6 3^6$	$\begin{bmatrix} 4 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 2 & 2 & 2 & 2 & 1 & 2 & 2 & 1 & 1 \\ 2 & 2 & 4 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 1 \\ 2 & 2 & 2 & 4 & 2 & 1 & 2 & 2 & 2 & 2 & 1 & 2 \\ 2 & 2 & 2 & 2 & 4 & 1 & 1 & 1 & 1 & 1 & 2 & 1 \\ 2 & 2 & 2 & 1 & 1 & 4 & 2 & 2 & 2 & 2 & -1 & -1 \\ 2 & 2 & 1 & 2 & 1 & 2 & 4 & 2 & 2 & 2 & -1 & 1 \\ 2 & 1 & 2 & 2 & 1 & 2 & 2 & 4 & 2 & 2 & -1 & 1 \\ 1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 4 & 2 & -1 & 1 \\ 2 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 4 & -1 & 1 \\ 1 & 1 & 1 & 1 & 2 & -1 & -1 & -1 & -1 & -1 & 4 & 2 \\ 1 & 1 & 1 & 2 & 1 & -1 & 1 & 1 & 1 & 1 & 2 & 4 \end{bmatrix}$	$6U_4(3)2$	$6U_4(3)2$

-12_D $2.3^3 12^3/1.4.6^3$	$\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$	D_{12}	D_{12}
-12_E $1^2 3^2 4^2 12^2/2^2 6^2$	$\begin{bmatrix} 4 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix}$	$D_{12}\wr Z_2$	$D_{12}\wr Z_2$
12_G $4^2 12^2/2.6$	$\begin{bmatrix} 8 & 4 \\ 4 & 8 \end{bmatrix}$	D_{12}	D_{12}
12_H $2^3 6.12^2/1.3.4^2$	$6I_2$	D_8	D_8
-12_H $1.2^2 3.12^2/4^2$	$\begin{bmatrix} 4 & 2 & 2 & 2 \\ 2 & 4 & 1 & 1 \\ 2 & 1 & 4 & 1 \\ 2 & 1 & 1 & 4 \end{bmatrix}$	$[2^5 3]$	$[2^5 3]$
12_I $1^2 4.6^2 12/3^2$	$\begin{bmatrix} 6 & 2 & 2 & 2 \\ 2 & 4 & 0 & 0 \\ 2 & 0 & 4 & 0 \\ 2 & 0 & 0 & 4 \end{bmatrix}$	$[2^5 3]$	$[2^5 3]$
-12_I $2^2 3^2 4.12/1^2$	$\begin{bmatrix} 10 & 6 & 6 & 6 \\ 6 & 12 & 0 & 0 \\ 6 & 0 & 12 & 0 \\ 6 & 0 & 0 & 12 \end{bmatrix}$	$[2^5 3]$	$[2^5 3]$
12_J $2.4.6.12$	$\begin{bmatrix} 4 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 8 & 4 \\ 0 & 0 & 4 & 8 \end{bmatrix}$	$D_{12}\times D_{12}$	*
-12_K $1^3 12^3/2.3.4.6$	$\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$	D_{12}	D_{12}
12_M 12^2	$12I_2$	D_8	D_8
14_B $1.2.7.14$	$\begin{bmatrix} 4 & 1 & 3 & 1 \\ 1 & 6 & 0 & 3 \\ 3 & 0 & 6 & 1 \\ 1 & 3 & 1 & 4 \end{bmatrix}$	$D_8\times Z_2$	$D_8\times Z_2$
-14_B $2^2 14^2/1.7$	$\begin{bmatrix} 4 & 2 \\ 2 & 8 \end{bmatrix}$	$Z_2\times Z_2$	$Z_2\times Z_2$
15_D $1.3.5.15$	$\begin{bmatrix} 4 & 1 & -2 & -2 \\ 1 & 6 & 1 & 2 \\ -2 & 1 & 4 & 1 \\ -2 & 2 & 1 & 6 \end{bmatrix}$	$D_{12}\times Z_2$	$D_{12}\times Z_2$
15_E $1^2 15^2/3.5$	$\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$	$Z_2\times Z_2$	$Z_2\times Z_2$
16_A $2^2 16^2/4.8$	$\begin{bmatrix} 4 & 0 \\ 0 & 8 \end{bmatrix}$	$Z_2\times Z_2$	$Z_2\times Z_2$
-16_B $1^2 16^2/2.8$	$4I_2$	D_8	D_8
-18_B $1^2 9.18/2.3$	$\begin{bmatrix} 6 & 3 \\ 3 & 6 \end{bmatrix}$	D_{12}	D_{12}
18_C $1.2.18^2/6.9$	$\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$	D_{12}	D_{12}

-18_C $2^2 9.18/1.6$	$\begin{bmatrix} 12 & 6 \\ 6 & 12 \end{bmatrix}$	D_{12}	D_{12}
20_B 4.20	$\begin{bmatrix} 4 & 0 \\ 0 & 20 \end{bmatrix}$	$Z_2 \times Z_2$	$Z_2 \times Z_2$
20_C 1.2.10.20/4.5	$\begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix}$	$Z_2 \times Z_2$	$Z_2 \times Z_2$
-20_C $2^2 5.20/1.4$	$10I_2$	D_8	*
21_C 3.21	$\begin{bmatrix} 6 & 3 \\ 3 & 12 \end{bmatrix}$	$Z_2 \times Z_2$	$Z_2 \times Z_2$
22_A 2.22	$\begin{bmatrix} 6 & 4 \\ 4 & 10 \end{bmatrix}$	Z_2	Z_2
23_A 1.23	$\begin{bmatrix} 4 & 1 \\ 1 & 6 \end{bmatrix}$	Z_2	Z_2
24_E 2.6.8.24//4.12	$\begin{bmatrix} 4 & 0 \\ 0 & 12 \end{bmatrix}$	$Z_2 \times Z_2$	$Z_2 \times Z_2$
24_F 1.4.6.24/3.8	$\begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}$	$Z_2 \times Z_2$	$Z_2 \times Z_2$
-24_F 2.3.4.24/1.8	$\begin{bmatrix} 6 & 0 \\ 0 & 12 \end{bmatrix}$	$Z_2 \times Z_2$	$Z_2 \times Z_2$
-28_A 1.4.7.28/2.14	$\begin{bmatrix} 4 & 2 \\ 2 & 8 \end{bmatrix}$	$Z_2 \times Z_2$	$Z_2 \times Z_2$
-30_D 2.3.5.30/1.15	$\begin{bmatrix} 4 & 2 \\ 2 & 16 \end{bmatrix}$	$Z_2 \times Z_2$	$Z_2 \times Z_2$
30_D 1.6.10.15/3.5	$\begin{bmatrix} 8 & 2 \\ 2 & 8 \end{bmatrix}$	$Z_2 \times Z_2$	$Z_2 \times Z_2$
-30_E 2.3.5.30/6.10	$\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$	$Z_2 \times Z_2$	$Z_2 \times Z_2$

TABLE 2

4	0	1	1	2	0	2	0	0	2	−2	−1	2	2	−1	−1	0	1	1	0	2	−2	2	−1	1	1	1	−1	0	−2	0	1	−1	1	−1	0
0	4	1	1	2	0	0	2	0	−2	2	−1	2	0	1	1	2	−1	−1	0	2	2	−2	−1	−1	1	1	1	0	−2	0	−1	1	1	−1	0
1	1	4	2	1	1	2	2	2	0	0	1	2	−1	2	0	−1	2	0	−1	1	1	1	2	0	0	2	0	2	0	1	1	1	−1	0	1
1	1	2	4	2	−1	2	2	−2	−1	−1	2	1	−1	0	2	−1	0	2	1	0	0	0	1	1	2	0	1	2	−1	1	0	0	0	1	1
2	2	1	2	4	−2	1	1	−1	−2	−2	−2	2	1	0	1	1	0	1	−1	0	0	0	−1	−1	1	0	−1	1	−2	2	0	0	0	−1	−1
0	0	1	−1	−2	4	0	0	2	2	2	1	2	0	1	−1	0	1	−1	2	2	2	2	−1	1	−1	1	1	0	2	0	−1	−1	1	1	0
2	0	2	2	1	0	4	0	0	1	−1	1	1	−2	−2	−2	0	2	2	0	1	−1	1	1	−1	2	2	1	0	−1	0	−1	1	−1	1	0
0	2	2	2	1	0	0	4	0	−1	1	1	1	0	2	2	−2	−2	−2	0	1	1	−1	1	1	2	2	−1	0	−1	0	1	−1	−1	1	0
0	0	2	−2	−1	2	0	0	4	1	1	−1	1	0	2	−2	0	2	−2	−2	1	1	1	1	−1	−2	2	−1	0	1	0	1	1	−1	−1	0
2	−2	0	−1	−2	2	1	−1	1	4	0	1	0	1	−1	−2	−1	1	0	1	2	−2	2	0	2	0	1	0	−1	0	−2	1	−1	1	0	1
−2	2	0	−1	−2	2	−1	1	1	0	4	1	0	−1	1	0	1	−1	−2	1	2	2	−2	0	0	0	1	2	−1	0	−2	−1	1	1	0	1
−1	−1	1	2	−2	1	1	1	−1	1	1	4	−1	−2	0	1	−2	0	1	2	0	0	0	2	2	1	0	2	1	1	−1	0	0	0	2	2
2	2	2	1	2	2	1	1	1	0	0	−1	4	1	1	0	1	1	0	1	2	2	2	−2	0	0	1	0	1	0	2	−1	−1	1	0	−1
2	0	−1	−1	1	0	−2	0	0	1	−1	−2	1	4	1	1	0	−1	−1	0	1	−1	1	−2	2	−1	−1	−2	0	−1	0	2	−2	2	−2	0
−1	1	2	0	0	1	−2	2	2	−1	1	0	1	1	4	2	−1	0	−2	−1	0	2	0	1	1	−2	0	−1	2	1	1	2	0	0	−1	1
−2	1	0	2	1	−1	−2	2	−2	−2	0	1	0	1	2	4	−1	−2	0	1	−1	1	−1	0	2	0	−2	0	2	0	1	1	−1	1	0	1
0	2	−1	−1	1	0	0	−2	0	−1	1	−2	1	0	−1	−1	4	1	1	0	1	1	−1	−2	−2	−1	−1	2	0	−1	0	−2	2	2	−2	0
1	−1	2	0	0	1	2	−2	2	1	−1	0	1	−1	0	−2	1	4	2	−1	0	0	2	1	−1	−2	0	1	2	1	1	0	2	0	−1	1
1	−1	0	2	1	−1	2	−2	−2	0	−2	1	0	−1	−2	0	1	2	4	1	−1	−1	1	0	0	0	−2	2	2	0	1	−1	1	1	0	1
0	0	−1	1	−1	2	0	0	−2	1	1	2	1	0	−1	1	0	−1	1	4	1	1	1	−2	2	1	−1	2	0	1	0	−2	−2	2	2	0
2	2	1	0	0	2	1	1	1	2	2	0	2	1	0	−1	1	0	−1	1	4	0	0	−1	1	1	2	1	−1	−2	−2	0	0	2	−1	1
−2	2	1	0	0	2	−1	1	1	−2	2	0	2	−1	2	1	1	0	−1	1	0	4	0	−1	−1	−1	0	1	1	2	2	−2	0	0	1	−1
2	−2	1	0	0	2	1	−1	1	2	−2	0	2	1	0	−1	−1	2	1	1	0	0	4	−1	1	−1	0	−1	1	2	2	0	−2	0	1	−1
−1	−1	2	1	−1	−1	1	1	1	0	0	2	−2	−2	1	0	−2	1	0	−2	−1	−1	−1	4	0	0	1	0	1	0	−1	2	2	−2	0	2
1	−1	0	1	−1	1	−1	1	−1	2	0	2	0	2	1	2	−2	−1	0	2	1	−1	1	0	4	0	−1	0	1	0	−1	2	−2	2	0	2
1	1	0	2	1	−1	2	2	−2	0	0	1	0	−1	−2	0	−1	−2	0	1	1	−1	−1	0	0	4	2	0	−2	−2	−1	−1	−1	−1	2	−1
1	1	2	0	0	1	2	2	2	1	1	0	1	−1	0	−2	−1	0	−2	−1	2	0	0	1	−1	2	4	−1	−2	−1	−1	0	0	−2	1	−1
−1	1	0	1	−1	1	1	−1	−1	0	2	2	0	−2	−1	0	2	1	2	2	1	1	−1	0	0	0	−1	4	1	0	−1	−2	2	2	0	2
0	0	2	2	1	0	0	0	0	−1	−1	1	1	0	2	2	0	2	2	0	−1	1	1	1	1	−2	−2	1	4	1	2	1	1	1	−1	2
−2	−2	0	−1	−2	2	−1	−1	1	0	0	1	0	−1	1	0	−1	1	0	1	−2	2	2	0	0	−2	+1	0	1	4	2	−1	−1	−1	2	−1
0	0	1	1	2	0	0	0	0	−2	−2	−1	2	0	1	1	0	1	1	0	−2	2	2	−1	−1	−1	−1	−1	2	2	4	−1	−1	−1	1	−2
1	−1	1	0	0	−1	−1	1	1	1	−1	0	−1	2	2	1	−2	0	−1	−2	0	−2	0	2	2	−1	0	−2	1	−1	−1	4	0	0	−2	2
−1	1	1	0	0	−1	1	−1	1	−1	1	0	−1	−2	0	−1	2	2	1	−2	0	0	−2	2	−2	−1	0	2	1	−1	−1	0	4	0	−2	2
1	1	−1	0	0	1	−1	−1	−1	1	1	0	1	2	0	1	2	0	1	2	2	0	0	−2	2	−1	−2	2	1	−1	−1	0	0	4	−2	2
−1	−1	0	1	−1	1	1	1	−1	0	0	2	0	−2	−1	0	−2	−1	0	2	−1	1	1	0	0	2	1	0	−1	2	1	−2	−2	−2	4	−2
0	0	1	1	−1	0	0	0	0	1	1	2	−1	0	1	1	0	1	1	0	1	−1	−1	2	2	−1	−1	2	2	−1	−2	2	2	2	−2	4

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