Hokkaido Mathematical Journal Vol. 19(1990) p. 431~434

A note on amalgams

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(Received March 13, 1989, Revised March 6, 1990)

To state our result, we account the situation along [1], [3] or [6]. We use the standard notation and one of [4] unless otherwise specified. Let P_1 and P_2 be distinct finite subgroups of a group G. We assume throughout this paper that

(A. 1) $G=\langle P_1, P_2\rangle$;
(A. 2) no non-trivial normal subgroup of G is contained in $P_1\cap P_2$;
(A. 3) $P_1\cap P_2\in Syl_2(P_1)\cap Syl_2(P_2)$; and
(A. 4) $C_{P_i}(O_2(P_i))\leqq O_2(P_i)$ for i=1, 2.

By a graph Γ, we mean a set Γ with a symmetric and irrefiexive relation which we call *adjacent*. For $0\in\Gamma$, we define $\Delta(0)$ the set of all vertices adjacent to 0. For an ordered $(n+1)$-tuple $\gamma=(\lambda_0, \lambda_1, \ldots, \lambda_n)$, γ is an *arc of lenath n* if $\lambda_i\in\Delta(\lambda_{i+1})$, $0\leqq i\leqq n-1$ (possibly, $\lambda_i=\lambda_j$ if $i\neq j$). Γ is *connected* if every pair of vertices is joined by an arc. For $\lambda\in\Gamma$, we denote by $d(0, \lambda)$ the minimal length of arcs connecting 0 and λ. Let $\Gamma=\Gamma(G, P_1, P_2)$ be the set of the right cosets of G with respect to P_1 and P_2. Let two cosets be adjacent if they are different and have non-empty intersection. Then we obtain a graph Γ, the *right coset graph of G with respect to P_1 and P_2* that is defined in [2], and G operates on Γ by right multiplication. The following fundamental properties of Γ can be also found in [2].

(a) Γ is connected.
(b) G is edge-transitive on Γ.
(c) Each vertex-stabilizer in G is conjugate to P_1 or P_2.
(d) Each edge-stabilizer in G is conjugate to $P_1\cap P_2$.

Throughout this note, we use the following notation. $X\leqq Y$ means X is a subgroup of Y. For a subset Λ of Γ, $G_\Lambda=\{g\in G;\ \lambda^g=\lambda$ for all $\lambda\in\Lambda\}$. For $\lambda\in\Gamma$,

$$Q_\lambda=O_2(G_\lambda),$$
$$Z_\lambda=\langle\Omega_1 Z(G_{\lambda\mu});\ \mu\in\Delta(\lambda)\rangle,$$
$$C_\lambda=\langle C_{Z_\mu}(O^2(G_\lambda));\ \mu\in\Delta(\lambda)\rangle \text{ and } V_\lambda=\langle z\in\bigcup_{\mu\in\Delta(\lambda)} Z_\mu;\ [z, Q_\lambda]\leqq C_\lambda\rangle$$

if $Z_\lambda\leqq Z(G_\lambda)$, and $C_\lambda=1$ and $V_\lambda=Z_\lambda$ otherwise.

$b_\lambda = \min\{d(\mu, \lambda)\ ;\ V_\mu \nleqq Q_\lambda,\ \mu \in \Gamma\}$.
ν_λ the number of non-central composition facors of G_λ within Q_λ.

Let $Q_\lambda = Q_0 > Q_1 > \cdots > Q_r = 1$ be a composition series of G_λ within Q_λ. For $x \in G_\lambda$, define $\|[Q_\lambda/-, x]\| = \prod_{i=0}^{r-1} \|[Q_i/Q_{i+1}, x]\|$. We note that $\|[Q_\lambda/-, x]\|$ is independent of the choice of $\{Q_i\ ;\ 0 \leqq i \leqq r\}$ by the Jordan-Hörder's theorem.

Let $0 \in \Gamma$ and $b = b_0$. To determine the structure of G_0, ν_0 plays an important role in pushig up problems using amalgam method. In many cases, it is shown that b is rather small. The purpose of this note is to give an estimation of ν_0 by using information about an arc $(0, 1, \ldots, b)$ with $V_b \nleqq Q_0$.

THEOREM. *Let $(0, 1, 2, \ldots, b)$ be an arc of Γ such that $b = b_0$ and V_b is not contained in Q_0. Set $n_i = |G_{i-1,1} : G_{i-1,i} \cap G_{i,i+1}|$ for $1 \leqq i \leqq b-1$. Then $\|[Q_0/-, x]\| \leqq \prod_{i=1}^{b-1} n_i \times |G_{b-1,b}/Q_b| \times |C_b|$ for all $x \in V_b$.*

COROLLARY. *Continue with the assumption and the notation of the theorem. Let $m = min\{\|[V, x]\|\ ;\ x \in G_{0,1} - N\}$, where N ranges over all the proper normal subgroups of G_0, ann V does over all the finite dimensional faithful $GF(2)G_0/N$-madules. Then $m^{\nu_0} \leqq \prod_{i=1}^{b-1} n_i \times |G_{b-1,b}/Q_b| \times |C_b|$.*

For the proof of the theorem, we require two elementary lemmas.

LEMMA 1. *Let H be a finite group, and $Q = O_2(H)$. Then $\|[Q/-, x]\| \leqq |Q : D| \times \|[D, x]\|$ for all $x \in H$ with $x^2 \in Q$.*

PROOF. Fix $x \in H$ with $x^2 \in Q$. Let $Y = [D, x]$. Let $Q = Q_0 \geqq Q_1 \geqq \cdots \geqq Q_r = 1$ be a composition series of H within Q. We proceed using induction on r. Let $B = Q_{r-1}$ and $A = B \cap D$. Take elements b_i of B, $1 \leqq i \leqq s$, so that $\{b_i A\ ;\ 1 \leqq i \leqq s\}$ is a basis of B/A as a vector space over $GF(2)$. Since $B \leqq \Omega_1 Z(Q)$ and $[A, x] \leqq [B, x] \cap [D, x] \leqq B \cap Y$, it follows that $\|[B, x]\| \leqq |\langle [Ab_i, x]\ ;\ 1 \leqq i \leqq s \rangle| \leqq \|[A, x]\| \times |\langle [b_i, x]\ ;\ 1 \leqq i \leqq s \rangle| \leqq |B \cap Y| \times 2^s \leqq |B \cap Y\|B|/|A|$. Using induction, we have that $\|[Q/-, x]\| = \|[B, x]\| \times \prod_{i=1}^{r-1} \|[Q_i/Q_{i+1}, x]\| \leqq |B \cap Y\|B|/|A| \times |QB : DB| \times |YB/B| = |B \cap Y| \times |B|/|A| \times |Q\|B|/|Q \cap B| \times |D \cap B|/|D\|B| \times |Y|/|Y \cap B| = |Q : D| \times |Y|$, as desired.

LEMMA 2. *Let b be a positive integer, and $(0, 1, \ldots, b)$ be an arc of Γ. Set $n_i = |G_{i-1,i} : G_{i-1,i} \cap G_{i,i+1}|$ for $1 \leqq i \leqq b-1$. Then*

(a) $|G_{0,1} : G_{0,1} \cap G_{b-1,b}| \leqq \prod_{i=1}^{b-1} n_i$.

(b) $|Q_0: Q_0 \cap Q_b| \leqq \prod_{i=1}^{b-1} n_i \times |G_{b-1,b}/Q_b|$.

PROOF. By induction on b, we have that $|G_{0,1}:G_{0,1} \cap G_{b-1,b}| = |G_{0,1}: G_{0,1} \cap G_{b-2,b-1}| \times |G_{0,1} \cap G_{b-2,b-1}: G_{0,1} \cap G_{b-2,b-1} \cap G_{b-1,b}| \leqq \prod_{i=1}^{b-2} n_i \times |G_{b-2,b-1}: G_{b-1,b}| = \prod_{i=1}^{b-1} n_i$, proving (a). It is easy to see that $|Q_0: Q_0 \cap Q_b| \leqq |Q_0: Q_0 \cap G_{b-1,b}| \times |G_{b,b-1}/Q_b| \leqq |G_{0,1}: G_{0,1} \cap G_{b-1,b}| \times |G_{b-1,b}/Q_b|$. Then (b) follows from (a).

Proof of the theorem and corollary. The preceeding lemma shows that $|Q_0: Q_0 \cap Q_b| \leqq \prod_{i=1}^{b-1} n_i \times |G_{b-1,b}/Q_b|$. Note that $[V_b, Q_0 \cap Q_b] \leqq C_b$. On the other hand, it follows from (A. 4), the defnition of V_b and minimality of b that $C_b \leqq Z_{b-1} \leqq Q_0$. Since V_b/C_b is elementary abelian, so is V_bQ_0/Q_0. Now applying Lemma 1 (with $H = G_0$, $S = G_{0,1}$ and $D = Q_0 \cap Q_b$), we have that $|[Q_0/-, x]| \leqq |Q_0: Q_0 \cap Q_b| \times |C_b|$ for all $x \in V_b$. Then the theorem follows from the above two inequalities, and the corollary follows immediately from the theorem.

Now we show two examples:

EXAMPLE 1. Let G be the Tits's simple group ${}^2F_4(2)'$. Let G_0 and G_1 be subgroups of G with a common Sylow 2-subgroup such that $|Q_0| = 2^9$, $|Q_1| = 2^{10}$, G_0/Q_0 is a Frobenius group of order 20 and G_1/Q_1 is one of order 6. Then $b_0 = 2$, $b_1 = 3$, and we can take an arc $(0, 1, 2, 3, 4)$ of Γ with $V_2 \not\leqq Q_0$ and $V_4 \not\leqq Q_1$. Since $|C_0| = 2$ and $C_1 = 1$, according to our results, we have that $|[Q_0/-, x]| \leqq |G_{0,1}/Q_0| \times |G_{0,1}/Q_1| \times |C_0| = 2^4$ for all $x \in V_2$, $|[Q_1/-, x]| \leqq |G_{1,2}/Q_1|^2 \times |G_{0,1}/Q_0| = 2^4$ for all $x \in V_2 - Q_0$, $|[Q_1/-, x]| = 2^3$ for all $x \in V_4 - Q_1$, $\nu_0 = 2$ and $\nu_1 = 3$. For precise, see [1] or [3].

EXAMPLE 2. Let $G = PSL_3(2^n)$. Let G_0 and G_1 be distinct minimal parabolic subgroups of G with a common Sylow 2-subgroup. Then we have that $C_0 = C_1 = 1$ and $b_0 = b_1 = 2$. Let $(0, 1, 2, 3)$ be an arc of Γ with $V_2 \not\leqq Q_0$ and $V_3 \not\leqq Q_1$. According to our results, for $i = 0$, 1, we have that $|[Q_i/-, x]| \leqq |G_{i,i+1}/Q_i|^2 = 2^{2n}$ for all $x \in V_{i+2}$, and $\nu_i \leqq 2$. Actually, for $i = 0$, 1, $|[Q_i/-, x]| = 2^n$ for all $x \in V_{3-i} - Q_i$, and $\nu_i = 1$.

Acknowledgment. The author would like to thank the refree for his pointing out many mistakes in the original version and giving useful comments.

References

[1] P. S. FAN, Amalgams of prime index, j. Alg. 98 (1986), 375-421.

[2] D. GOLDSCHMIDT, Automorphisms of trivalent graphs, ann. of Math. 111 (1980), 377-406.
[3] K. GOMI, A pair of groups having a common 2-subgroup of prime indeces, j. Alg. 97 (1985), 407-437.
[4] D. GORENSTEIN, " Finite Groups ", Harper & Row, New York, 1968.
[5] H. JIANHUA, B. Stellmacher and G. Stroth, Some Parabolic systems of rank 2 related to sporadic groups, J. Alg. 102 (1986), 78-118.
[6] B. STELLMACHER, Pushing up, Arch. Math. 46 (1986), 8-17.

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