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The character table of the commutative association scheme coming from the action of $GL(n, q)$ on non-incident point-hyperplane pairs

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Abstract. The theme discussed in this paper is a continuation of that of our previous papers [1, 2, 3, 5, 6], i.e., the character tables of certain commutative association schemes are controlled by the character tables of certain smaller association schemes. In this paper, we will see this for the commutative association scheme, denoted by $\mathscr{X}(GL(n, q), \Delta)$, of class $q+2$ (for $n\geq 3$) coming from the action of the group $GL(n, q)$ on the set of non-incident point-hyperplane pairs. Namely, it is shown that this character table is controlled by the character table of the association scheme of class $q+1$, denoted by $\mathscr{X}(GL(2, q), \Delta)$ or $\mathscr{X}(PGL(2, q)/Z_{q-1})$, which controls the character table of the association scheme coming from the action of $PSp(2n, q)$ on the set of non-isotropic projective lines.

0. Introduction

The theme discussed in this paper is a continuation of that of our previous papers [1, 2, 3, 5, 6], i.e., the character tables of certain commutative association schemes are controlled by the character tables of certain smaller association schemes. We will see this for the commutative association schemes, denoted by $\mathscr{X}(GL(n, q), \Delta)$, of class $q+2$ coming from the action of the group $GL(n, q)$ on the set of non-incident point-hyperplane pairs.

Let $V=V_{n}(q)$ be a vector space of dimension n over $GF(q)$. Let Δ be the set of non-incident point-hyperplane pairs of the projective space attached to V. Then $|\Delta|=q^{n-1}(q^{n}-1)/(q-1)$. The group $GL(n, q)$(or $PGL(n, q)$) acts transitively on the set Δ. Darafsheh [8] determined the decomposition of this permutation character into irreducible characters of $GL(n, q)$ for $n\geq 3$, and showed in particular that the permutation character is with inner product $q+5$ and is not multiplicity-free. However, if we consider the action of the involutive graph automorphism of $GL(n, q)$ on the set Δ, we can see that two pairs of relations (of the original noncom-

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mutative association scheme obtained from the action) are fused, and so we get a symmetric (hence commutative) association scheme of class $q+2$ (i.e., of rank $q+3$ as a permutation group). The purpose of this paper is to study the character table of this association scheme $\mathscr{X}(GL(n, q), \Delta)$ in what follows. For $n=2$, it is also shown that $GL(2, q)$ acts transitively on the set Δ, which is identified with the set of ordered pairs of distinct projective points. This permutation character is with inner product $q+4$, and is not multiplicity-free. Let Z_2 be the group action on Δ interchanging the order of projective points in each element of Δ. Then the group $Z_2\times GL(2, q)$ acts on Δ, and two pairs of relations are fused, and so we get a symmetric (hence a commutative) association scheme of class $q+1$, which will be denoted by $\mathscr{X}(GL(2, q), \Delta)$ in what follows. This is exactly the same as the association scheme obtained from the action of $PGL(2, q)$ on the cosets by Z_{q-1}, denoted here by $\mathscr{X}(PGL(2, q)/Z_{q-1})$, which was described in [3]. In [3], *it was proved that the character table of the association scheme of* $PSp(2n, q)$, with $n\geq 2$, acting on the set of non-isotropic projective lines is controlled by the character table of the association scheme $\mathscr{X}(PGL(2, q)/Z_{q-1})$. In this paper we will show that the character table of the same association scheme $\mathscr{X}(GL(2, q), \Delta)$ controls the character table of the association scheme $\mathscr{X}(GL(n, q), \Delta)$ with $n\geq 3$.

REMARK. It seems that the rank of the multiplicity-free permutation group corresponding to the commutative association scheme $\mathscr{X}(GL(n, q), \Delta)$ was given incorrectly in [9]. The correct rank is $q+3$ as it is easily seen from [8] (also, see [7]). We understood that the commutative association scheme $\mathscr{X}(GL(n, q), \Delta)$ was also studied by A. Cohen recently very extensively. We heard from him at Pingree Park Conference that he observed that $PSL(2, q)$ also controlled (geometrically) the behavior of the association scheme $\mathscr{X}(GL(n, q), \Delta)$. His observation can be regarded as a counterpart of the algebraic control (i.e., controls among the character tables) that is discussed in this paper. We also heard from him that he calculated the parameters p_{ij}^h of this association scheme. The calculation given in this paper has been done independently for the purpose of determination of the character table of the association scheme which is our main object here.

1. Association scheme $\mathscr{X}(GL(n, q), \Delta)$, $n\geq 3$

Let $V_n(q)$ be the n-dimensional vector space over the finite field $F=GF(q)$. Let $\mathscr{U}$ be the set of 1-dimensional subspace U and $\mathscr{H}$ be the set of $(n-1)$-dimensional subspaces H of $V_n(q)$. Let Δ be the set $\{(U, H)\in \mathscr{U}\times\mathscr{H} | U\not\subseteq H\}$. Then by the straightforward counting, we have $|\mathscr{U}|=$

$|\mathscr{H}|=(q^n-1)/(q-1)$, and $|\Delta|=q^{n-1}(q^n-1)/(q-1)$. In the vector space $V_n(q)$, we denote the usual inner product by $\langle\, ,\rangle$, i. e., $\langle u_s, u_t\rangle=\sigma_1\tau_1+\sigma_2\tau_2+\cdots+\sigma_n\tau_n$ for the vectors $u_s=(\sigma_1, \sigma_2,\cdots, \sigma_n)$ and $u_t=(\tau_1, \tau_2,\cdots, \tau_n)$. With this inner product the orthogonal complement relation between the projective points and the hyperplanes yields a one-to-one correspondence between $\mathscr{U}=\{U_1, U_2, \cdots, U_r\}$ and $\mathscr{H}=\{H_1, \cdots, H_r\}$, where $r=(q^n-1)/(q-1)$. In what follows, we will always assume that the hyperplane H_l, indexed by l, represents the orthogonal complement of U_l, the projective point indexed by l, and vice versa for each $l\in\{1, 2,\cdots, r\}$. Also, we will denote the representing vector for the suspace U_l by $u_l=(\mu_1, \mu_2,\cdots, \mu_n)$ with the convention that the first nonzero component of u_l is 1. Finally, the ordered pair $(U_l, H_m)\in\Delta$ will also be denoted by f_{lm} in short.

$GL(n, q)$ acts transitively on the set Δ of the ordered nonincident projective point and hyperplane pairs. This action carries $q+5$ orbits of $GL(n, q)$ on $\Delta\times\Delta$. As usual, by taking these $q+5$ orbits as association relations, we have an association scheme of class $q+4$. It is known that this association scheme is not commutative because the transitive permutation representation on Δ is not multiplicity free. However, we get a symmetric association scheme, denoted by $\mathscr{X}(GL(n, q), \Delta)$, of class $q+2$ from the noncommutative association scheme of class $q+4$ by combining two pairs of relations pairwise (to eliminate non-symmetric association relations). This can be explained if we consider the action of the involutive graph automorphism, the automorphism which sends a pair of nonincident point and hyperplane to its dual pair (the orthogonal complement of hyperplane and that of projective point pair), of $GL(n, q)$ on the set Δ, which fuses two pairs of relations.

We now describe the association relations $\{R_i\}$ of the symmetric association scheme $\mathscr{X}(GL(n, q), \Delta)$ explicitly in the following.

$$R_0=\{(f_{lm}, f_{lm})|f_{lm}\in\Delta\}$$
$$R_1=\{(f_{lm}, f_{xy})\in\Delta\times\Delta|\,U_x\subseteq H_m,\ U_l\subseteq H_y\}$$
$$R_{q+1}=\{(f_{lm}, f_{xy})\in\Delta\times\Delta|\,\textit{either } U_l\subseteq H_y \textit{ and } U_x\not\subseteq H_m,\ \textit{or } U_l\not\subseteq H_y \textit{ and } U_x\subseteq H_m\}$$
$$R_{q+2}=\{(f_{lm}, f_{xy})\in\Delta\times\Delta|\,\textit{either } U_l=U_x \textit{ and } H_m\neq H_y,\ \textit{or } U_l\neq U_x \textit{ and } H_m=H_y\}.$$

For $j=2, 3,\cdots, q$,

$$R_j=\left\{(f_{lm}, f_{xy})\in\Delta\times\Delta\,\middle|\,\frac{\langle u_l, u_m\rangle\langle u_x, u_y\rangle}{\langle u_l, u_y\rangle\langle u_x, u_m\rangle}=\eta^{j-1},\ f_{lm}\neq f_{xy}\right\}$$

where η is a primitive element of F.

LEMMA 1. 1. *The valencies of the association scheme* $\mathscr{X}(GL(n, q), \Delta)$ *are given as follows*:

$$\begin{aligned} k_0 &= 1 \\ k_1 &= q^{n-2}(q^{n-1}-1)/(q-1) \\ k_2 &= k_3 = \cdots = k_{q-1} = q^{n-2}(q^{n-1}-1) \\ k_q &= (q^{n-2}-1)(q^{n-1}-1) \\ k_{q+1} &= 2q^{n-2}(q^{n-1}-1) \\ k_{q+2} &= 2(q^{n-1}-1). \end{aligned}$$

PROOF: Let $\Gamma_i(f_{lm}) = \{f_{st} \in \Delta | (f_{lm}, f_{st}) \in R_i\}$, and u_l and u_m be the representing vectors for U_l and $U_m = H_m^{\perp}$, the orthogonal complement of H_m, etc. Set $u_l = u_m = (1, 0, 0, \cdots, 0)$, $u_s = (\sigma_1, \sigma_2, \cdots, \sigma_n)$, and $u_t = (\tau_1, \tau_2, \cdots, \tau_n)$. Then

$$\begin{aligned} k_1 &= |\Gamma_1(f_{lm})| = |\{(U_s, H_t) \in \Delta | U_s \subseteq H_m \text{ and } U_l \subseteq H_t\}| \\ &= |\{(U_s, U_t) \in \mathscr{U} \times \mathscr{U} | \langle u_s, u_t \rangle \neq 0,\ \langle u_l, u_t \rangle = \langle u_s, u_m \rangle = 0\}| \\ &= \frac{1}{(q-1)^2} |\{(\sigma_1, \sigma_2, \cdots, \sigma_n ; \tau_1, \tau_2, \cdots, \tau_n) \in F^{2n} | \sigma_1 = \tau_1 = 0, \\ &\qquad \sigma_2\tau_2 + \cdots + \sigma_n\tau_n \neq 0\}| \\ &= \frac{1}{q-1} q^{n-2}(q^{n-1}-1). \end{aligned}$$

For $j = 2, 3, \cdots, q-1$,

$$\begin{aligned} k_j &= \left| \left\{ (U_s, H_t) \in \Delta \left| \frac{\langle u_l, u_m \rangle \langle u_s, u_t \rangle}{\langle u_l, u_t \rangle \langle u_s, u_m \rangle} = \eta^{j-1} \right. \right\} \right| \\ &= |\{(\sigma_1, \sigma_2, \cdots, \sigma_n ; \tau_1, \tau_2, \cdots, \tau_n) \in F^{2n} | \sigma_1 = \tau_1 = 1, \\ &\qquad \sigma_2\tau_2 + \sigma_3\tau_3 + \cdots + \sigma_n\tau_n = \eta^{j-1} - 1\}| \\ &= q^{n-2}(q^{n-1}-1) \\ k_q &= |\{(\sigma_1, \sigma_2, \cdots, \sigma_n ; \tau_1, \tau_2, \cdots, \tau_n) \in F^{2n} | \sigma_1 = \tau_1 = 1, \\ &\qquad \sigma_2\tau_2 + \cdots + \sigma_n\tau_n = 0,\ u_s \neq u_l,\ u_t \neq u_m\}| \\ &= (q^{n-2}-1)(q^{n-1}-1). \end{aligned}$$

By the similar counting argument, we complete the proof.

LEMMA 1. 2. *Let* $\{p_{ij}^h\}$ *be the set of intersection numbers of the association scheme* $\mathscr{X}(GL(n, q), \Delta)$.

$$(1.2.1)\quad p_{1j}^{1}=\begin{cases} q^{n-3}(q^{n-2}-1)/(q-1) & \text{if } j=1,\\ q^{n-3}(q^{n-2}-1) & \text{if } j=2,3,\cdots,q-1,\\ (q^{n-3}-1)(q^{n-2}-1) & \text{if } j=q,\\ 2q^{n-3}(q^{n-2}-1) & \text{if } j=q+1,\\ 2(q^{n-2}-1) & \text{if } j=q+2. \end{cases}$$

(1.2.2) *For* $h,\ j\in\{2,3,\cdots,q-1\}$,

$$p_{1j}^{h}=\begin{cases} q^{n-3}(q^{n-2}-1)+q^{n-2} & \text{if } \eta^{h+j-2}=\eta^{h-1}+\eta^{j-1},\\ q^{n-3}(q^{n-2}-1) & \text{if } \eta^{h+j-2}\neq\eta^{h-1}+\eta^{j-1}. \end{cases}$$

$$(1.2.3)\quad p_{1q}^{h}=\begin{cases} q^{n-3}(q^{n-2}-1) & \text{if } h=2,3,\cdots,q-1\\ q^{2n-5} & \text{if } h=p \end{cases}$$

$$p_{1q+1}^{h}=\begin{cases} 2q^{n-3}(q^{n-2}-1) & \text{if } h=2,3,\cdots,q-1,\ q+1,\\ 2q^{2n-5} & \text{if } h=q. \end{cases}$$

(1.2.4) *For* $h,\ i,\ j\in\{2,3,\cdots,q-1\}$,

$$p_{ij}^{h}=\begin{cases} q^{n-3}(q^{n-1}-q^{n-2}+1) & \text{if } (h,i,j)\in T_0\\ q^{n-3}(q^{n-1}-q^{n-2}+q+1) & \text{if } (h,i,j)\in T_1\\ q^{n-3}(q^{n-1}-q^{n-2}-q+1) & \text{if } (h,i,j)\in T_2, \end{cases}$$

where T_0, T_1, *and* T_2 *are defined as follows.*

(i) *If* q *is odd prime power, then set*

$$D=\eta^{-2(j-1)}\{(\eta^{h+i-2}+\eta^{i+j-2}+\eta^{j+h-2}-\eta^{h+i+j-3})^2-4\eta^{h+i+j-3}\},$$

and define

$$T_0=\{(h,i,j)|D=0\},$$
$$T_1=\{(h,i,j)|D \text{ is a non-zero square element in } F\}$$
$$T_2=\{(h,i,j)|D \text{ is a non-square element in } F\}.$$

(ii) *If* q *is a power of* 2, *then set*

$$D=\eta^{h-1}+\eta^{i-1}+\eta^{h+i-j-1}+\eta^{h+i-2},$$

and define

$$T_0=\{(h,i,j)|D=0\}$$
$$T_1=\{(h,i,j)|D\neq 0,\ (1+\eta^{h-1})(1+\eta^{i-1})D^{-2}\in\{\tau^2+\tau|\tau\in F\}\}$$
$$T_2=\{(h,i,j)|D\neq 0,\ (1+\eta^{h-1})(1+\eta^{i-1})D^{-2}\in\{\tau^2+\tau|\tau\notin F\}\}.$$

(1.2.5) *For* $h,\ i\in\{2,3,\cdots,q-1\}$,

$$p_{iq}^{h}=\begin{cases} q^{n-3}(q-1)(q^{n-2}-1) & \text{if } h\neq i\\ (q^{n-2}-q^{n-3}-1)(q^{n-2}-1) & \text{if } h=i, \end{cases}$$

$$p_{iq}^{q}=\begin{cases} q^{n-2}(q^{n-2}-q^{n-3}-2) & \textit{if } i=2,3,\cdots,q-1 \\ q^{n-2}(q^{n-2}-q^{n-3}+q-3)+4 & \textit{if } i=q. \end{cases}$$

(1.2.6) *For* $h,\ i\in\{2,3,\cdots,q-1\}$,

$$p_{iq+1}^{h}=\begin{cases} 2q^{n-3}(q^{n-1}-q^{n-2}-q+1) & \textit{if } \eta^{h+i-2}=\eta^{h-1}+\eta^{i-1} \\ 2q^{n-3}(q^{n-1}-q^{n-2}+1) & \textit{if } \eta^{h+i-2}\neq\eta^{h-1}+\eta^{i-1}. \end{cases}$$

$$p_{iq+1}^{q}=\begin{cases} 2q^{n-3}(q^{n-1}-q^{n-2}) & \textit{if } i=2,3,\cdots,q-1 \\ 2q^{n-3}(q^{n-1}-q^{n-2}-q) & \textit{if } i=q. \end{cases}$$

$$p_{iq}^{q+1}=\begin{cases} 2q^{n-3}(q-1)(q^{n-2}-1)+q^{n-2} & \textit{if } i=2,3,\cdots,q-1 \\ 2q^{n-2}(q-1)(q^{n-2}-1)-q^{n-2}+1 & \textit{if } i=q \\ 4q^{n-3}(q-1)(q^{n-2}-1)+q^{n-2} & \textit{if } i=q+1. \end{cases}$$

The intersection numbers which are not listed above are obtained from the above by the following basic equalities

(i) $p_{ij}^{h}=p_{ji}^{h}$

(ii) $\sum_{j=1}^{q+2}p_{ij}^{h}=\begin{cases} k_i-1 & \textit{if } h=i \\ k_i & \textit{if } h\neq i \end{cases}$

(iii) $k_ip_{hj}^{i}=k_hp_{ij}^{h}=k_jp_{ih}^{j}$.

PROOF: For (1.2.1), set $u_l=u_m=(1,0,0,\cdots,0)$, $u_x=u_y=(0,1,0,\cdots,0)$, $u_s=(\sigma_1,\sigma_2,\cdots,\sigma_n)$, and $u_t=(\tau_1,\tau_2,\cdots,\tau_n)$.

$$\begin{aligned} p_{11}^{1}&=|\Gamma_1(f_{lm})\cap\Gamma_1(f_{xy})| \\ &=|\{(U_s,H_t)\in\Delta\,|\,U_s\subseteq H_m,\ U_l\subseteq H_t,\ U_s\subseteq H_y,\ U_x\subseteq H_t\}| \\ &=|\{(u_s,u_t)\,|\,\langle u_l,u_t\rangle=\langle u_s,u_m\rangle=\langle u_x,u_t\rangle=\langle u_s,u_y\rangle=0, \\ &\qquad \langle u_s,u_t\rangle\neq 0\}| \\ &=\frac{1}{(q-1)^2}|\{(\sigma_1,\sigma_2,\cdots,\sigma_n\,;\tau_1,\tau_2,\cdots,\tau_n)\in F^{2n}\,|\,\sigma_1=\tau_1=\sigma_2=\tau_2=0, \\ &\qquad \sigma_3\tau_3+\sigma_4\tau_4+\cdots+\sigma_n\tau_n\neq 0\}| \\ &=q^{n-3}(q^{n-2}-1)/(q-1). \end{aligned}$$

$$\begin{aligned} p_{1j}^{1}&=|\Gamma_1(f_{lm})\cap\Gamma_j(f_{xy})| \\ &=|\{(u_s,u_t)\,|\,\langle u_l,u_t\rangle=\langle u_s,u_m\rangle=0,\ \langle u_s,u_t\rangle=\eta^{j-1}\langle u_x,u_t\rangle\langle u_s,u_y\rangle\}| \\ &=\frac{1}{(q-1)^2}|\{(\sigma_1,\sigma_2,\cdots,\sigma_n\,;\tau_1,\tau_2,\cdots,\tau_n)\in F^{2n}\,|\,\sigma_1=\tau_1=0, \\ &\qquad \sigma_2\tau_2+\sigma_3\tau_3+\cdots+\sigma_n\tau_n=\eta^{j-1}\sigma_2\tau_2\}| \\ &=|\{(0,1,\sigma_3,\cdots,\sigma_n\,;0,1,\tau_3,\cdots,\tau_n)\in F^{2n}\,|\,\sigma_3\tau_3+\cdots+\sigma_n\tau_n=\eta^{j-1}-1, \\ &\qquad (\sigma_3,\cdots,\sigma_n)\neq(0,\cdots,0),\ (\tau_3,\cdots,\tau_n)\neq(0,\cdots,0)\}| \\ &=\begin{cases} q^{n-3}(q^{n-2}-1) & \textit{if } j=2,3,\cdots,q-1 \\ (q^{n-3}-1)(q^{n-2}-1) & \textit{if } j=q. \end{cases} \end{aligned}$$

$$\begin{aligned} p_{1q+1}^{1}&=|\{(u_s,u_t)\,|\,\langle u_s,u_t\rangle\neq 0,\ \langle u_l,u_t\rangle=\langle u_s,u_m\rangle=0, \\ &\qquad \langle u_x,u_t\rangle=0,\ \langle u_s,u_y\rangle\neq 0\}| \end{aligned}$$

$$+|\{(u_s, u_t)|\langle u_s, u_t\rangle\neq 0,\ \langle u_l, u_t\rangle=\langle u_s, u_m\rangle=0,$$
$$\langle u_x, u_t\rangle\neq 0,\ \langle u_s, u_y\rangle=0\}|$$
$$=\frac{2}{q-1}|\{(0, 1, \sigma_3,\cdots, \sigma_n ; 0, 0, \tau_3,\cdots, \tau_n)\in F^{2n}|\sigma_3\tau_3+\sigma_4\tau_4$$
$$+\cdots+\sigma_n\tau_n\neq 0\}|$$
$$=2q^{n-3}(q^{n-2}-1).$$
$$p^1_{1q+2}=k_1-\sum_{j=0}^{q+1}p^1_{1j}=2(q^{n-2}-1).$$

This completes the proof of (1.2.1).

For (1.2.4), set $u_l=u_m=(1, 0,\cdots, 0)$, $u_x=(1, 1, 0,\cdots, 0)$, $u_y=(1, \eta^{h-1}-1, 0, 0,\cdots, 0)$, $u_s=(\sigma_1, \sigma_2,\cdots, \sigma_n)$ and $u_t=(\tau_1, \tau_2,\cdots, \tau_n)$.

$$p^h_{ij}=|\Gamma_i(f_{lm})\cap\Gamma_j(f_{xy})|$$
$$=\left|\{(u_s, u_t)|\frac{\langle u_s, u_t\rangle}{\langle u_s, u_m\rangle\langle u_l, u_t\rangle}=\eta^{j-1},\ \frac{\eta^{h-1}\langle u_s, u_t\rangle}{\langle u_s, u_y\rangle\langle u_x, u_t\rangle}=\eta^{j-1}\}\right|$$
$$=\frac{1}{(q-1)^2}|\{(\sigma_1, \sigma_2,\cdots, \sigma_n ; \tau_1, \tau_2,\cdots, \tau_n)\in F^{2n}|\sigma_1\neq 0,\ \tau_1\neq 0,$$
$$\sigma_1+(\eta^{h-1}-1)\sigma_2\neq 0,\ \tau_1+\tau_2\neq 0,\ \sigma_1\tau_1+\cdots+\sigma_n\tau_n=\eta^{i-1}\sigma_1\tau_1,$$
$$\sigma_1\tau_1+\cdots+\sigma_n\tau_n=\eta^{j-h}(\tau_1+\tau_2)\{\sigma_1+(\eta^{h-1}-1)\sigma_2\}\}|$$
$$=|\{(1, \sigma_2,\cdots, \sigma_n ; 1, \tau_2,\cdots, \tau_n)\in F^{2n}|1+(\eta^{h-1}-1)\sigma_2\neq 0,$$
$$1+\tau_2\neq 0,\ \sigma_2\tau_1+\cdots+\sigma_n\tau_n=\eta^{i-1}-1,$$
$$\sigma_2\tau_2+\cdots+\sigma_n\tau_n=\eta^{j-h}(1+\tau_2)\{1+(\eta^{h-1}-1)\sigma_2\}-1\}|$$
$$=\left|\left\{(\sigma_2,\cdots, \sigma_n ; \tau_2,\cdots, \tau_n)\in F^{2(n-1)}|\sigma_3\tau_3+\cdots+\sigma_n\tau_n\right.\right.$$
$$\left.\left.=\eta^{i-1}-1-\frac{\sigma_2\{\eta^{h+i-j-1}-(\eta^{h-1}-1)\sigma_2-1\}}{1+(\eta^{h-1}-1)\sigma_2}\right\}\right|.$$

The last equality is obtained by replacing τ_2 by $\dfrac{\eta^{h+i-j-1}-(\eta^{h-1}-1)\sigma_2-1}{1+(\eta^{h-1}-1)\sigma_2}$ for each σ_2 except $\sigma_2=-(\eta^{h-1}-1)^{-1}$. Notice that there are $(q-1)$ pairs (σ_2, τ_2) which satisfy the equality $\eta^{i-1}=\eta^{j-h}(1+\tau_2)\{1+(\eta^{h-1}-1)\sigma_2\}$. Since the number of choices in $(\sigma_3, \sigma_4,\cdots, \sigma_n ; \tau_3, \tau_4,\cdots, \tau_n)\in F^{2(n-2)}$ satisfying $\sigma_3\tau_3+\cdots+\sigma_n\tau_n=\theta$ is given by $q^{n-2}+q^{n-3}(q^{n-2}-1)$ if $\theta=0$, and $q^{n-3}(q^{n-2}-1)$ if $\theta\neq 0$, p^h_{ij} does depend on the number of solutions of the equation

$$\eta^{i-1}-\frac{\sigma_2\{\eta^{h+i-j-1}-(\eta^{h-1}-1)\sigma_2-1\}}{1+(\eta^{h-1}-1)\sigma_2}=0,$$

or equivalently, the equation

$$(*)\qquad (\eta^{h-1}-1)\sigma_2^2+\{(\eta^{h-1}-1)(\eta^{i-1}-1)+1-\eta^{h+i-j-1}\}\sigma_2+\eta^{i-1}-1=0.$$

This quadratic equation (*) for σ_2 has either two, one, or no solution from the set $F-\{-(\eta^{h-1}-1)^{-1}\}$ depending on the sign of its discriminant D which is also depending on the choice of (h, i, j). The discriminant of the quadratic equation $(*)$ is described as follows.

(i) If the characteristic of the field F is odd, then

$$\begin{aligned} D&=\{(\eta^{h-1}-1)(\eta^{i-1}-1)+1-\eta^{h+i-j-1}\}^2-4(\eta^{h-1}-1)(\eta^{i-1}-1)\\ &=\eta^{-2(j-1)}\{(\eta^{h+i-2}+\eta^{i+j-2}+\eta^{j+h-2}-\eta^{h+i+j-3})^2-4\eta^{h+i+j-3}\}. \end{aligned}$$

(ii) If the characteristic of F is 2, then the equation $(*)$ becomes

$$(**)\qquad (\eta^{h-1}+1)\sigma_2^2+(\eta^{h-1}+\eta^{i-1}+\eta^{h+i-2}+\eta^{h+i-j-1})\sigma_2+\eta^{i-1}+1=0,$$

and has single solution if $D=\eta^{h-1}+\eta^{i-1}+\eta^{h+i-2}+\eta^{h+i-j-1}$ is equal to zero. If $D\neq 0$, then $(**)$ becomes

$$\frac{D^2}{1+\eta^{h-1}}\{t^2+t+(\eta^{h-1}+1)(\eta^{i-1}+1)D^{-2}\}=0$$

by setting $\sigma^2=\dfrac{Dt}{1+\eta^{h-1}}$. This equation has two or no solution according as either $(\eta^{h-1}+1)(\eta^{i-1}+1)D^{-2}$ belongs or does not belong to the set $\{\tau^2+\tau \mid \tau\in F\}$, respectively. Therefore, afterall we have

$$p_{ij}^h=\begin{cases} 2\{q^{n-2}+q^{n-3}(q^{n-2}-1)\}+(q-3)q^{n-3}(q^{n-2}-1) & \text{if } (h, i, j)\in T_1\\ 1\{q^{n-2}+q^{n-3}(q^{n-2}-1)\}+(q-2)q^{n-3}(q^{n-2}-1) & \text{if } (h, i, j)\in T_0\\ 0\{q^{n-2}+q^{n-3}(q^{n-2}-2)\}+(q-1)q^{n-3}(q^{n-2}-1) & \text{if } (h, i, j)\in T_2 \end{cases}$$

which gives (1.2.4).
This completes (1.2.4).
The rest of the parameters are computed in the same counting argument.

2. Association scheme $\mathscr{X}(GL(2, q), \Delta)$

When $n=2$, the association scheme $\mathscr{X}(GL(2, q), \Delta)$ is defined exactly the same way as for $n\geq 3$ cases except the fact that it has one less class than what $\mathscr{X}(GL(n, q), \Delta)$ does for $n\geq 3$. However, we treat this association scheme $\mathscr{X}(GL(2, q), \Delta)$ separately because of the following two reasons. Firstly, its character table controls that of $\mathscr{X}(GL(n, q), \Delta)$ for every $n\geq 3$. Secondly, it has exactly the same parameters, and thus the same character table, as those of the association scheme $\mathscr{X}(PGL(2, q)/Z_{q-1})$ which is coming from the action of $PGL(2, q)$ on the cosets by the cyclic subgroup Z_{q-1} [3].

Let $\mathscr{U}$ be the set of 1-dimensional subspaces of the 2-dimensional vector space $V_2(q)$ over $F=GF(q)$. Denote the elements of $\mathscr{U}$ by $U_1, U_2, \cdots, U_{q+1}$, while their representing vectors by $u_1, u_2, \cdots, u_{q+1}$, respectively. We will use the convention that for every m, the representing vector u_m is one of the vectors in the set $\{(1,0), (0,1), (1, \eta), (1, \eta^2), \cdots, (1, \eta^{q-1})\}$ where η is a primitive element of F. Let Δ be the set of ordered pairs $(U_l, U_m) \in \mathscr{U} \times \mathscr{U}$ such that $U_l \neq U_m$. i.e.

$$\Delta = \{f_{lm} | f_{lm} = (U_l, U_m) \in \mathscr{U} \times \mathscr{U},\ U_l \neq U_m\}.$$

For each $m \in \{1, 2, \cdots, q+1\}$, there is $l \in \{1, 2, \cdots, q+1\}$ such that $U_m^{\perp} = U_l$. In this case we denote l by m' so that the orthogonal complement of U_m is to be $U_{m'}$ (with respect to the usual inner product of the vectors). The corresponding respresenting vector for $U_{m'}$ will be denoted by $u_{m'}$ as well.

The association classes for $\mathscr{X}(GL(2, q), \Delta)$ are defined by

$$\begin{aligned} R_0 &= \{(f_{lm}, f_{lm}) | f_{lm} \in \Delta\}, \\ R_1 &= \{(f_{lm}, f_{ml}) | f_{lm} \in \Delta\}, \\ R_{q+1} &= \{(f_{lm}, f_{st}) \in \Delta \times \Delta |\ \text{either } U_s = U_m \text{ and } U_l \neq U_t, \\ &\qquad \text{or } U_s \neq U_m \text{ and } U_l = U_t\} \\ R_{q+2} &= \{(f_{lm}, f_{st}) \in \Delta \times \Delta |\ \text{either } U_l = U_s \text{ and } U_m \neq U_t, \\ &\qquad \text{or } U_l \neq U_s \text{ and } U_m = U_t\}, \end{aligned}$$

for $j = 2, 3, \cdots, q-1$,

$$R_j = \left\{(f_{lm}, f_{st}) | \frac{\langle u_l, u_{m'} \rangle \langle u_s, u_{t'} \rangle}{\langle u_l, u_{t'} \rangle \langle u_s, u_{m'} \rangle} = \eta^{j-1}\right\}.$$

Notice that there is no class indexed by q. That is, this is a symmetric association scheme of class $q+1$ with $|\Delta| = q(q+1)$.

LEMMA 2. 1. *The valencies of $\mathscr{X}(GL(2, q), \Delta)$ are given by*

$$\begin{aligned} k_0 &= k_1 = 1 \\ k_2 &= k_3 = k_4 = \cdots = k_{q-1} = q-1 \\ k_{q+1} &= k_{q+2} = 2(q-1). \end{aligned}$$

PROOF: For $j = 2, 3, \cdots, q-1$, set $u_l = u_{m'} = (1, 0)$, $u_s = (\sigma_1, \sigma_2)$, $u_{t'} = (\tau'_1, \tau'_2)$.

$$\begin{aligned} k_j &= |\Gamma_j(f_{lm})| \\ &= \left|\left\{(u_s, u_t) | \frac{\langle u_s, u_{t'} \rangle}{\langle u_l, u_{t'} \rangle \langle u_s, u_{m'} \rangle} = \eta^{j-1}\right\}\right| \\ &= |\{(\sigma_1, \sigma_2; \tau'_1, \tau'_2) \in F^4 | \sigma_1 = \tau'_1 = 1,\ \sigma_2 \tau'_2 = \eta^{j-1} - 1\}| \\ &= q-1. \end{aligned}$$

All remaining valencies are computed by direct counting.

LEMMA 2. 2. *Let $\{b_{ij}^h\}$ be the set of intersection numbers for the* *association scheme* $\mathscr{X}(GL(2, q), \Delta)$.

(2. 2. 1)

$$b_{1j}^{1}=\begin{cases}1 & \text{if } j=0\\ 0 & \text{otherwise.}\end{cases}$$

(2. 2. 2)

$$b_{1j}^{h}=\begin{cases}1 & \text{if } \eta^{h+j-2}=\eta^{h-1}+\eta^{j-1}\\ 0 & \text{otherwise}\end{cases} \qquad \text{for } h, j\in\{2, 3,\cdots, q-1\}$$

(2. 2. 3)

$$b_{1q+1}^{h}=\begin{cases}0 & \text{if } h=0, 1, 2,\cdots, q-1, q+1\\ 1 & \text{if } h=q+2.\end{cases}$$

(2. 2. 4) *For* $h, i, j\in\{2, 3,\cdots, q-1\}$

$$b_{ij}^{h}=\begin{cases}1 & \text{if } (h, i, j)\in T_0\\ 2 & \text{if } (h, i, j)\in T_1\\ 0 & \text{if } (h, i, j)\in T_2,\end{cases}$$

where T_0, T_1, *and* T_2 *are defined as in Lemma* 1.2.

(2. 2. 5) *For* $h, i\in\{2, 3,\cdots, q-1\}$

$$b_{iq+1}^{h}=\begin{cases}0 & \text{if } \eta^{h+i-2}=\eta^{i-1}+\eta^{h-1}\\ 2 & \text{if } \eta^{h+i-2}\neq\eta^{i-1}+\eta^{h-1}\end{cases}$$

$$b_{iq+1}^{q+1}=\begin{cases}0 & \text{if } i=1\\ q-2 & \text{if } i=q+2\\ 1 & \text{otherwise.}\end{cases}$$

The rest of them are easily computed from the above by using the basic equalities given in Lemma 1. 2.

PROOF: For (2.2.4), set $u_l=(1, 0)$, $u_{m'}=(1, 0)$, $u_x=(1, 1)$, $u_{y'}=(1, \eta^{h-1}-1)$, $u_s=(\sigma_1, \sigma_2)$, and $u_{t'}=(\tau_1', \tau_2')$. Then

$$\begin{aligned}b_{ij}^{h}&=|\Gamma_i(f_{lm})\cap\Gamma_i(f_{xy})|\\ &=\frac{1}{(q-1)^2}\left|\left\{(\sigma_1, \sigma_2; \tau_1', \tau_2')\in F^4\,\middle|\,\frac{\sigma_1\tau_1'+\sigma_2\tau_2'}{\sigma_1\tau_1'}=\eta^{i-1},\right.\right.\\ &\qquad\left.\left.\frac{\eta^{h-1}(\sigma_1\tau_1'+\sigma_2\tau_2')}{(\tau_1'+\tau_2')\{\sigma_1+(\eta^{h-1}-1)\sigma_2\}}=\eta^{j-1}\right\}\right|\\ &=\left|\left\{\sigma_2\in F\,|\,1+(\eta^{h-1}-1)\sigma_2\neq0,\ (\eta^{h-1}-1)\sigma_2^2\right.\right.\\ &\qquad\left.\left.+\{(\eta^{h-1}-1)(\eta^{i-1}-1)+1-\eta^{h+i-j-1}\}\sigma_2+\eta^{i-1}-1=0\right\}\right|.\end{aligned}$$

Hence from the proof of (1.2.4), we have (2.2.4) as we desired. Everything else will be analogue of the proof of Lemma 1.2.

By comparing with the results given in § 2.5 of [3], we observe that all parameters of this association scheme *coincide* with those of the association scheme $\mathscr{X}(PGL(2, q)/Z_{q-1})$ and $\mathscr{X}(SO_3(q), \Theta)$ in [3]. Therefore, we have the following lemma.

LEMMA 2. 3. *The character table P of $\mathscr{X}(GL(2, q), \Delta)$ is given as follows.*

$$P=\begin{bmatrix} 1 & 1 & q-1 & q-1 & \cdots & q-1 & 2(q-1) & 2(q-1) \\ 1 & 1 & & & & & -2 & -2 \\ \vdots & \vdots & & & & & \vdots & \vdots \\ \vdots & 1 & & p_j(i) & & & -2 & \vdots \\ \vdots & -1 & & & 1\leq i\leq q-1 & & 2 & \vdots \\ \vdots & & & & 2\leq j\leq q-1 & & & \vdots \\ \vdots & \vdots & & & & & \vdots & \vdots \\ 1 & -1 & & & & & 2 & -2 \\ 1 & 1 & -2 & -2 & \cdots & -2 & q-3 & q-3 \\ 1 & -1 & 0 & 0 & \cdots & 0 & -q+1 & q-1 \end{bmatrix}$$

REMARK 2. 4. As a consequence of above Lemmas 2.2 and 2.3, the character table of $\mathscr{X}(GL(2, q), \Delta)$ controls that of $\mathscr{X}(Sp_{2n}(q), \Omega)$ for every $n\geq 2$, the association scheme obtained from the action of $Sp_{2n}(q)$ on the set of non-isotropic lines in symplectic geometry on $V_{2n}(q)$. [cf. 3].

3. Character table of $\mathscr{X}(GL(n, q), \Delta)$, $n\geq 3$

In this chapter, we will construct the character table of $\mathscr{X}(GL(n, q), \Delta)$ for $n\geq 3$, by using the relationship between the two sets of parameters $\{p_{ij}^h\}$ and $\{b_{ij}^h\}$. We have the following relation between the two sets.

For $h=2, 3, \cdots, q-1$,

$$p_{1j}^h=\begin{cases} q^{n-2}b_{11}^h+q^{n-3}(q^{n-2}-1)/(q-1) & \text{if } j=1 \\ q^{n-2}b_{1j}^h+q^{n-3}(q^{n-2}-1) & \text{if } j=2, 3, \cdots, q-1, \\ q^{n-2}b_{1q+1}^h+2q^{n-3}(q^{n-2}-1) & \text{if } j=q-1. \end{cases}$$

For $h,\ i\in\{2, 3, \cdots, q-1\}$,

$$p_{ij}^h=\begin{cases} q^{n-2}b_{i1}^h+q^{n-3}(q^{n-2}-1) & \text{if } j=1 \\ q^{n-2}b_{ij}^h+q^{n-3}(q-1)(q^{n-2}-1) & \text{if } j=2, 3, \cdots, q-1, \\ q^{n-2}b_{iq+1}^h+2q^{n-3}(q-1)(q^{n-2}-1) & \text{if } j=q+1. \end{cases}$$

$$p_{q+1\,q+1}^h=q^{n-2}b_{q+1\,q+1}^h+4q^{n-3}(q-1)(q^{n-2}-1),$$

and so on.

For the association scheme $\mathscr{X}(GL(2, q), \Delta)$, if we denote the i-th intersection matrix, whose (j, h)-entry is given by b_{ij}^h, by B_i, and the diagonal matrix with its diagonal entries $[p_i(0), p_i(1), \cdots, p_i(q+2)]$, the i-th column of the character table P, by P_i, then we have $B_i \cdot {}^tP = {}^tP \cdot P_i$ for all $i \in \{1, 2, \cdots, q-1, q+1, q+2\}$.

THEOREM 3. 1. *The character table $\tilde{P}$ of the association scheme $\mathscr{X}(GL(n, q), \Delta)$ is given as in Figure* 1.

PROOF: Suppose we denote the intersection matrix, the diagonal matrix with its diagonal entries $[\tilde{p}_i(0), \tilde{p}(1), \cdots, \tilde{p}_i(q+2)]$, the i-th column of the matrix $\tilde{P}$, for $\mathscr{X}(GL(n, q), \Delta)$ by $\tilde{B}_i$, $\tilde{P}_i$, then from the above relation between the two parameters and the equality $B_i \cdot {}^tP = {}^tP \cdot P_i$, we can show that $\tilde{B}_i \cdot {}^t\tilde{P} = {}^t\tilde{P} \cdot \tilde{P}_i$, which is enough to assert the statement, for all i.

$$\tilde{P}=\begin{bmatrix}
1 & \frac{q^{n-2}(q^{n-1}-1)}{q-1} & q^{n-2}(q^{n-1}-1) & \cdots & q^{n-2}(q^{n-1}-1) & (q^{n-2}-1)(q^{n-1}-1) & 2q^{n-2}(q^{n-1}-1) & 2(q^{n-1}-1) \\
1 & q^{n-2} & & & & -q^{n-2}+1 & -2q^{n-2} & -2 \\
\vdots & \vdots & & & & \vdots & \vdots & \vdots \\
\vdots & q^{n-2} & \tilde{p}_j(i)=q^{n-2}p_j(i) & & & \vdots & -2p^{n-2} & \vdots \\
\vdots & -q^{n-2} & 1\leq i\leq q-1 & & & \vdots & 2q^{n-2} & \vdots \\
\vdots & \vdots & 2\leq j\leq q-1 & & & \vdots & \vdots & \vdots \\
\vdots & \vdots & & & & \vdots & \vdots & \vdots \\
1 & -q^{n-2} & & & & -q^{n-2}+1 & 2q^{n-2} & -2 \\
1 & q^{n-3} & -q^{n-3}(q+1) & \cdots & -q^{n-3}(q+1) & q^{n-1}-q^{n-2}-q^{n-3}+1 & -2q^{n-3} & -2 \\
1 & q^{\frac{3}{2}(n-2)} & -q^{n-2}(q^{\frac{n-2}{2}}+1) & \cdots & -q^{n-2}(q^{\frac{n-2}{2}}+1) & (q^{n-2}-1)(-q^{\frac{n-2}{2}}-1) & q^{n-2}(q^{\frac{n}{2}}-2q^{\frac{n-2}{2}}-1) & q^{n-1}-q^{\frac{n-2}{2}}-2 \\
1 & -q^{\frac{3}{2}(n-2)} & q^{n-2}(q^{\frac{n-2}{2}}-1) & \cdots & q^{n-2}(q^{\frac{n-2}{2}}-1) & (q^{n-2}-1)(q^{\frac{n-2}{2}}-1) & -q^{n-2}(q^{\frac{n}{2}}-2q^{\frac{n-2}{2}}+1) & q^{n-1}+q^{\frac{n-2}{2}}-2
\end{bmatrix}$$

Figure 1

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