' t-designs ' in $H(d, q)$

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Abatract

We define two kinds of ' t-designs ' in $H(d, q)$, which is a semilattice of all partial mappings from a d-element set to a q-element set, and prove Fisher type inequalities for those ' t-designs '. They are generalizations of the Ray-Chaudhuri and Wilson inequality for (combinatorial) t-designs and the Rao bound for orthogonal arrays of strength t. We give examples of ' t-designs ' which attain those bounds.

1. Introduction

Interesting similarities between (combinatorial) t-designs and orthogonal arrays have been pointed out by several authors. For example, P. Delsarte defined a concept of regular semilattices and t-designs in them ([2]) and now those two types of designs, namely, (combinatorial) t-designs and orthogonal arrays are understood as examples of t-designs in regular semilattices or those in Q-polynomial association schemes. We consider Hamming type (or hypercubic-type) regular semilattices and define two types of ' t-designs ', namely, $[t]$-designs and $\{t\}$-designs, both of which are generalizations of those two classical-type designs. See Definition 2.2. The concept of $[t]$-designs seems to be first introduced and studied by H. Nagao and others ([1], [4]. See Corollary 3.3).

In this paper we give Fisher type inequalities for two kinds of ' t-designs '. As special cases they include the Ray-Chaudhuri and Wilson inequality for (combinatorial) t-designs and the Rao bound for orthognal arrays. In the final section we give several constructions of ' t-designs ' and also give a series of examples which attain the bound of the Fisher-type inequality. Our method of proof is standard and uses higher incidence matrices, so in that sense it follows the method of R. Wilson [5].

2. ' t-designs ' and its incidence matrices

We begin with the definition of a semilattice $H(d, q)$. Throughout this paper ' t-designs ' are considered in this semilattice unless we specify.

DEFINITION 2.1. Let d and q be positive integers, D a d-element set and Q a q-element set. Then $H(d, q)=(L, \leq)$ is a semilattice defined as follows.

(1) $L=\bigcup_{J\subset D} Q^J=\{\alpha=(\alpha, J): J\rightarrow Q$, a mapping $|J\subset D\}$
$=$the set of all partial mappings from D to Q.

(2) $(\alpha_1, J_1)\leq(\alpha_2, J_2)$ if $J_1\subset J_2$ and $\alpha_{2|J_1}=\alpha_1$.

(3) $(\alpha, J)=(\alpha_1, J_1)\wedge(\alpha_2, J_2)$,

where $J=\{j\in J_1\cap J_2|\alpha_1(j)=\alpha_2(j)\}$, and $\alpha=\alpha_{1|J}(=\alpha_{2|J})$.

Let $X_i=\bigcup_{J\subset D,|J|=i} Q^J$. For $\alpha=(\alpha, J)\in L$, $D(\alpha)$ denotes the domain of α, i.e., J. Let $L_k=\bigcup_{i=0}^{k} X_i$.

By definition L is a disjoint union of X_0, $X_1, \dots, X_d$.

DEFINITION 2.2. Let t, k be integers with $0\leq t\leq k\leq d$.

(1) A nonempty subset Y of X_k is a $[t]$-$((d, q), k, \lambda)$ *design* or simply a $[t]$-*design*, provided that, for an element α in X_t, the number

$$\lambda_t(\alpha)=|\{y\in Y|\alpha\leq y\}|$$

is a constant λ (independent of the choice of α in X_t).

(2) A nonempty subset Y of L is a $\{t\}$-$((d, q), \lambda_1, \dots, \lambda_t)$ design or simply a $\{t\}$-*design*, provided that, for an element α in X_i, the number

$$\lambda_i(\alpha)=|\{y\in Y|\alpha\leq y\}|$$

is a constant λ_i (independent of the choice of α in X_i) with $i=1, 2, \dots, t$.

REMARK. (1) As it is remarked in the introduction, the concept of $[t]$-design is not new. ([1], [4])

(2) A (combinatorial) t-design, or t-(d, k, λ) design is a $[t]$-$((d, 1), k, \lambda)$ design, and an orthogonal array of strength t is a $[t]$-$((d, q), d, \lambda)$ design.

In addition to the definitions we mention few more notational conventions and terminologies.

Let α and β be elements of a semilattice defined in Definition 2.1.

(1) α and β are said to be *disjoint* if $D(\alpha)\cap D(\beta)=\phi$.

(2) α and β are said to be *consistent* if there exists an element $x\in X_d$ such that $\alpha\leq x$, $\beta\leq x$.

Let A, B and C be finite sets.

(3) Let $Mat(A, B)$ denote the set of all matrices over the real numbers $\mathbf{R}$ having A and B as row and column labeling sets, and for $X\in Mat(A, B)$, $(\alpha, \beta)\in A\times B$, $X[\alpha, \beta]$ denote the (α, β) entry of X. Let

$V(A)$ denote the set of all column vectors over $\mathbf{R}$ having A as a row labeling set.

Let $0 \leq u,\ i,\ j \leq k \leq d$.

(4) W_{ij} denotes a matrix in $Mat(X_i, X_j)$ whose (α, β) entry $W_{ij}[\alpha, \beta]$ is 1, if $\alpha \leq \beta$ and is 0 otherwise.

(5) W_{ij}^u denotes a matrix in $Mat(X_i, X_j)$ whose (α, β) entry $W_{ij}^u[\alpha, \beta]$ is 1, if α and β are consistent and $\alpha \wedge \beta \in X_u$ and is 0 otherwise.

(6) For a subset $Y \subset L$, N_i denotes a matrix in $Mat(X_i, Y)$ whose (α, y) entry $N_i[\alpha, y]$ is 1, if $\alpha \leq y$ and is 0 otherwise.

(7) For a subset $Y \subset X_k$, $C_{k,Y}$ denotes a matrix in $Mat(X_k, Y)$ whose (α, y) entry $C_{k,Y}[\alpha, y]$ is 1, if $\alpha = y$ and is 0 otherwise. In particular, we have $W_{ik}C_{k,Y} = N_i$.

(8) For a subset $A \subset L$, $\boldsymbol{1}_A$ denotes the all one vector in $V(A)$, and $\boldsymbol{1}_i$ denotes $\boldsymbol{1}_{X_i}$.

(9) For a subset $Y \subset L$, N_i^u denotes a matrix in $Mat(X_i, Y)$ whose (α, y) entry $N_i^u[\alpha, u]$ is 1, if α and y are consistent and $\alpha \wedge y \in X_u$ and is 0 otherwise.

We collect several basic lemmas, which we need later.

LEMMA 2.1. *For* $0 \leq i \leq j \leq t \leq k \leq d$,

$$W_{ij} W_{jk} = \binom{k-i}{j-i} W_{ik}.$$

PROOF. Let $(\alpha, \beta) \in X_i \times X_k$. Then

$$W_{ij} W_{jk}[\alpha, \beta] = |\{\gamma \in X_j | \alpha \leq \gamma \leq \beta)| = \binom{k-i}{j-i} W_{ik}.$$

Hence we have the relation.

LEMMA 2.2. *For* $0 \leq i \leq j \leq t \leq k \leq d$, *we have the following.*

(1) *For a subset* $Y \subset X_k$, Y *is a* $[t]$-$((d, q), k, \lambda)$ *design if and only if* $N_t \boldsymbol{1}_Y = \lambda \boldsymbol{1}_t$.

(2) *If a subset* $Y \subset X_k$ *is a* $[t]$-$((d, q), k, \lambda)$ *design,* Y *is a* $[i]$-$((d, q), k, \lambda_i)$ *design with*

$$\lambda_i = \binom{d-i}{t-i} q^{t-i} \lambda \Big/ \binom{k-i}{t-i} = \binom{d-i}{k-i} q^{t-i} \lambda \Big/ \binom{d-t}{k-t}$$

So in particular, it is a $\{t\}$-$((d, q), \lambda_1, \ldots, \lambda_t)$ *design with*

$$\lambda_i = |Y| \binom{d-i}{k-i} \Big/ \binom{d}{k} q^i.$$

PROOF. (1) It is clear from the definition.
(2) Suppose $Y\subset X_k$ is a $[t]$-$((d, q), k, \lambda)$ design. Then

$$\lambda\binom{k-i}{j-i}q^{t-i}\boldsymbol{1}_i=\lambda W_{it}\boldsymbol{1}_t=W_{it}N_t\boldsymbol{1}_Y=W_{it}W_{tk}C_{k,Y}\boldsymbol{1}_Y$$
$$=\binom{k-i}{t-i}W_{ik}C_{k,Y}\boldsymbol{1}_Y=\binom{k-i}{t-i}N_i\boldsymbol{1}_Y..$$

So by (1), Y is a $[i]$-$((d, q), k, \lambda_i)$ design with

$$\lambda_i=\binom{d-i}{t-i}q^{t-i}\lambda\Big/\binom{k-i}{t-i}=\binom{d-i}{k-i}q^{t-i}\lambda\Big/\binom{d-t}{k-t}.$$

Here, the last equality follows from a well-known identity of binomial coefficients. In particular, $\lambda_0=|Y|$. So

$$\lambda=|Y|\binom{d-i}{k-i}\Big/\binom{d}{k}q^t, \text{ and } \lambda_i=|Y|\binom{d-i}{k-i}\Big/\binom{d}{k}q^i.$$

LEMMA 2.3. *Let $Y\subset L$ be a $\{t\}$-$((d, q), \lambda_1, \dots, \lambda_t)$ design, and $(\alpha, \beta)\in X_i\times X_j$ with $i+j\leq t$. If α and β are disjoint then the following hold.*

(1) *$\lambda_i^j=|\{y\in Y|\alpha\leq y,\ \beta$ and y are disjoint$\}|$ is a constant independent of the choices of α and β.*

(2) *$\lambda_i^j+q\lambda_{i+1}^{j-1}=\lambda_i^{j-1}$ with $j\geq 1$, and $\lambda_i=\lambda_i^0$.*

PROOF. (1) follows form (2). For the proof of (2), we proceed by induction on j. Let $\gamma\in X_{j-1}$, $\gamma\leq\beta$ *and* $a\in D(\alpha)\setminus D(\gamma)$. For each c in Q, let

$$\Lambda_c=\{y\in Y|\alpha\leq y,\ D(\gamma)\cap D(y)=\phi,\ y(a)=c\}$$
$$\Lambda_0=\{y\in Y|\alpha\leq y,\ D(\beta)\cap D(y)=\phi\}.$$

Then

$$\{y\in Y|\alpha\leq y,\ D(\gamma)\cap D(y)=\phi\}$$
$$=\Lambda_0\cup(\bigcup_{c\in Q}\Lambda_c),\ \text{(disjoint union)}.$$

Since the left hand side is equal to λ_i^{j-1} by induction hypothesis, and $|\Lambda_c|=\lambda_{i+1}^{j-1}$, for each $c\in Q$, we have the assertion.

LEMMA 2.4. $\lambda_i^j=\sum_{u=0}^{j}(-1)^u\binom{j}{u}q^u\lambda_{i+u}$.

PROOF. We proceed by induction on j. If $j=0$, there is nothing to prove. Using the recurrence relation in the previous lemma, and the induction hypothesis, we have

$$\begin{aligned}\lambda_i^{j+1}&=\lambda_i^j-q\lambda_{i+1}^j\\&=\sum_{u=0}^{j}(-1)^u\binom{j}{u}q^u\lambda_{i+u}-\sum_{v=0}^{j}(-1)^v\binom{j}{v}q^{v+1}\lambda_{i+v+1}\\&=\lambda_i^0+\sum_{u=1}^{j}\left((-1)^u\binom{j}{u}q^u\lambda_{i+u}+(-1)^u\binom{j}{u-1}q^u\lambda_{i+u}\right)\\&\quad+(-1)^{j+1}q^{j+1}\lambda_{i+j+1}\\&=\sum_{u=0}^{j+1}(-1)^u\binom{j+1}{u}q^u\lambda_{i+u}.\end{aligned}$$

LEMMA 2.5. $\sum_{u=0}^{j}(-1)^u\binom{j}{u}\binom{d-i-u}{k-i-u}=\binom{d-i-j}{k-i}$, *if* $i+j\geq k$.

PROOF. In $H(d, q)$, let $q=1$. Let $Y=X_k$. Then Y in a $[t]$-$((d, 1), k, \binom{d-t}{k-t})$ design, for $t=0, 1, \ldots, k$. So by LEMMA 2.3 and 2.4, the left hand side becomes λ_i^j for this trivial t-design. Hence $\lambda_i^j=\binom{d-i-j}{k-i}$.

LEMMA 2.6. *Let Y be a $[t]$-$((d, q), k, \lambda)$ design. If $i+j\leq t$, then*

$$\lambda_i^j=\binom{d-i-j}{k-i}q^{t-i}\lambda/\binom{d-t}{k-t}.$$

PROOF. Since $\lambda_{i+u}=\binom{d-i-u}{k-i-u}q^{t-i-u}\lambda/\binom{d-t}{k-t}$,

$$\begin{aligned}\lambda_i^j&=\sum_{u=0}^{j}(-1)^u\binom{j}{u}q^u\lambda_{i+u}\\&=\sum_{u=0}^{j}(-1)^u\binom{j}{u}q^u\binom{d-i-u}{k-i-u}q^{t-i-u}\lambda/\binom{d-t}{k-t}\\&=\frac{q^{t-i}\lambda}{\binom{d-t}{k-t}}\sum_{u=0}^{j}(-1)^u\binom{j}{u}\binom{d-i-u}{k-i-u}\\&=\binom{d-i-j}{k-i}q^{t-1}\lambda/\binom{d-t}{k-t}.\end{aligned}$$

In [2], Delsarte defined a concept of regular semilattices and developed a theory of t-designs in these semilattices. As for $[t]$-$((d, q), k, \lambda)$ designs, the underlying semilattice is L_k, which is a union of $X_0, \ldots, X_k$. By inspection, we can show the following:

(*) If L_k is a regular semilattice then either $q=1$ or $k=d$.

The case $q=1$ corresponds to the Johnson scheme and the (combinatorial)

t-designs, while the case $k=d$ corresponds to the Hamming scheme and the orthogonal arrays. In these cases we have regular semilattices, so we may fully apply Delsarte's theory of t-designs. As we do not depend on his theory, we shall not give the proof of the statement (*) in this paper.

3. Fisher's inequality

In this section we give two generalizations of the Fisher's inequality. One is obtained by the determination of the condition that an incidence matrix of a $\{t\}$-design is of full rank. This leads to a generalization of the inequlity of Ray-Chaudhuri and Wilson. On the other hand, for a $[t]$-design we obtained a lower bound of the rank of an incidence matrix to get a Fisher type inequality, which is equivalent with the Rao bound, if it is an orthogonal array, i. e., a $[t]$-$((d, q), d, \lambda)$ design.

LEMMA 3.1. *Let $Y\subset L$ be a $\{t\}$-$((d, q), \lambda_1, \dots, \lambda_t)$ design. If $i+j \leq t$,*

$$N_i(N_j^u)^T=\sum_{v=0}^{\min\{i,j\}}\binom{j-v}{u-v}\lambda_{i+u-v}^{j-u}W_{ij}^v,$$

where T denotes the transpose of the matrix.

PROOF. Let $(\alpha, \beta)\in X_i\times X_j$ with $\alpha\wedge\beta\in X_v$. Then

$$N_i(N_j^u)^T[\alpha, \beta]=|\{y\in Y|\alpha\leq y,\ \beta\wedge y\in X_u,\ \beta \text{ and } y \text{ are consistent}\}|.$$

So $N_i(N_j^u)^T[\alpha, \beta]$ equals 0, if α and β are not consistent. Suppose α and β are consistent. Then by counting the number above, we have

$$N_i(N_j^u)^T[\alpha, \beta]=\binom{j-v}{u-v}\lambda_{i+u-v}^{j-u},$$

and the relation follows.

THEOREM 3.2 *Let $Y\subset L$ be a $\{2a\}$-$((d, q), \lambda_1, \dots, \lambda_{2s})$ design. If $\lambda_s^s\neq 0$, then*

$$|Y|\geq rank\ N_s=\binom{d}{s}q^s.$$

PROOF. Since $0\leq\lambda_i^j=\lambda_i^{j-1}-q\lambda_{i+1}^{j-1}\leq\lambda_i^{j-1}$ by Lemma 2.3, $\lambda_s^i\neq 0$ for $i=0, 1, \dots, s$. By the previous lemma, letting $i=j=s$, we have

$$N_s(N_s^0)^T=\lambda_s^s W_{ss}^0$$
$$N_s(N_s^1)^T=s\lambda_{s+1}^{s-1}W_{ss}^0+\lambda_s^{s-1}W_{ss}^1$$
$$\cdots\cdots\cdots\cdots\cdots$$

$$N_s(N_s^u)^T=\sum_{v=0}^{u-1}\binom{s-v}{u-v}\lambda_{s+u}^{s-u}W_{ss}^v+\lambda_s^{s-u}W_{ss}^u$$

....

$$N_s(N_s^s)^T=\sum_{v=0}^{s-1}\binom{s-v}{s-v}\lambda_{2s-v}^0 W_{ss}^v+\lambda_s^0 W_{ss}^u$$

Hence $I=W_{ss}^s$ can be written as a linear combination of

$$N_s(N_s^0)^T,\ N_s(N_s^1)^T, \dots, N_s(N_s^s)^T.$$

Thus there exists a matrix M in $Mat(Y, X_s)$ such that

$$I=N_sM,$$

where I denotes $|X_s|$ by $|X_s|$ identity matrix. Hence

$$|Y|\geq rank\ N_s=|X_s|=\binom{d}{s}q^s.$$

COROLLARY 3.3 (Nagao [4]). *Let Y be a $[2s]$-$((d, q), k, \lambda)$-design. If $k+s\leq d$, then*

$$|Y|\geq\binom{d}{s}q^s.$$

PROOF. Suppose $k+s\leq d$. Let $\alpha\in X_s$. Since $\lambda_s\neq 0$, there exists $y\in Y$ such that $\alpha\leq y$. Since $y\in X_k$, there exists $\beta\in X_s$, where y and β are disjoint. Hence $\lambda_s^s\neq 0$. Thus we may apply the previous theorem. Also see Lemma 2.6.

REMARK. The proofs of Theorem 3.2 and 3.3 are essentially same as that of Nagao for $[t]$-design. Since Theorem 3.2 is stated for $\{t\}$-designs, it includes a slight generalization of the Ray-Chandhuri and Wilson inequality. In order to show that an incidence matrix N_s is of full rank, we used a condition $k+s\leq d$. So Corollary 3.3 says nothing on an orthogonal array, where $k=d$.

Now we turn to a proof of an inequality for a $[t]$-design. The following seems to be a well-known technique to reduce the problem of computing the rank of an incidence matrix N_s to that of W_{sk}, but it gives us a starting point for our proof.

LEMMA 3.4. *Let $L\subset X$ be a $[t]$-$((d, q), k, \lambda)$ design, and λ_i $(i=0, 1, \dots, t)$ be the constants in* Lemma 2.2. *For $0\leq i, j\leq k$ with $i+j\leq t$,*

$$N_i(N_j)^T=\sum_{u=0}^{\min\{i,j\}}\lambda_{i+j-u}W_{ij}^u=\frac{\lambda}{\binom{d-t}{k-t}_q q^{k-t}}W_{ik}(W_{jk})^T,$$

PROOF. Let $\alpha\in X_i$ and $\beta\in X_j$ with $\alpha\wedge\beta\in X_u$. Then

$$\begin{aligned}N_i(N_j)^T[\alpha,\beta]&=|\{y\in Y|\alpha\le y,\ \beta\le y\}|\\&=\begin{cases}\lambda_{i+j-u} & \text{if } \alpha \text{ and } \beta \text{ are consistent}\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

$$\begin{aligned}N_i(N_j)^T&=\sum_{u=0}^{\min\{i,j\}}\lambda_{i+j-u}W_{ij}^u\\&=\frac{\lambda}{\binom{d-t}{k-t}_q q^{k-t}}\sum_{u=0}^{\min\{i,j\}}\binom{d-i-j+u}{k-i-j+u}_q q^{k-i-j+u}W_{ij}^u\\&=\frac{\lambda}{\binom{d-t}{k-t}_q q^{k-t}}W_{ik}(W_{jk})^T,\end{aligned}$$

LEMMA 3.5. *Let $Y\subset X_k$ be a $[2s]$-$((d,q),k,\lambda)$ design. Then $|Y|\ge rank(N_{2s})\ge rank(N_s)=rank(W_{sk})$.*

PROOF. Since the columns of an incidence matrix N_{2s} is indexed by the elements of Y, the rank of the matrix N_{2s} does not exceed $|Y|$. By Lemma 2.1,

$$W_{s,2s}N_{2s}=\binom{k-s}{s}N_s.$$

So the row space of N_{2s} is in that of N_s. Since N_s and W_{sk} are real matrices, the row space of N_s (resp. W_{sk}) is equal to that of $N_s^TN_s$ (resp. $W_{sk}^TW_{sk}$). Moreover, since nonzero eigenvalues with multiplicities are the same both in A^TA and AA^T for any matrix A, we may apply the previous lemma to obtain the following.

$$\begin{aligned}|Y|\ge rank(N_{2s})\ge rank(N_s)&=rank(N_s^TN_s)=rank(N_sN_s^T)\\&=rank\left(\frac{\lambda}{\binom{d-t}{k-t}_q q^{k-t}}W_{sk}W_{sk}^T\right)\\&=rank(W_{sk}W_{sk}^T)=rank(W_{sk}^TW_{sk})=rank(W_{sk}).\end{aligned}$$

LEMMA 3.6. *In $H(d,1)$ with $0\le s\le k\le d$, the following hold.*

(1) $rank(W_{sk})=\binom{d}{s}$, *if* $s+k\le d$.

(2) $rank(W_{sk})=\binom{d}{k}$, *if* $s+k\geq d$.

PROOF. See Theorem 11 in [3].

Let $A=W_{sk}$ with $0\leq s\leq k\leq d$ and x_0 be an element in X_d. $X_{ji}=\{\alpha\in X_j | \alpha\wedge x_0\in X_{j-i}\}$.

For μ, $\nu\in X_{ii}$ let

$$X_{ji}^{\mu}=\{\alpha\in X_{ji}|\alpha\geq\mu\},$$

$A_i^{\mu\nu}$ denote the restriction of A to $X_{is}^{\mu}\times X_{ki}^{\nu}$ and A_{ij} denote the restriction of A to $X_{si}\times X_{kj}$. Then it is easy to see that $A_{ij}=0$ if $i>j$ and we may arrange the rows and the columns of A so that the diagonal blocks are A_{ii}, s and the below diagonals are all zero matrices. Hence

$$rank(A)\geq\sum_{i=0}^{s} rank(A_{ii}).$$

Since μ, $\nu\in X_{ii}$, $\mu\wedge x_0\in X_0$, $\nu\wedge x_0\in X_0$, and $A_i^{\mu\nu}=0$ if $\mu\neq\nu$. So again we may arrange the rows and the columns of A_{ii} so that the diagonal blocks are $A_i^{\mu\nu}$'s and the offdiagonals are all zero matrices. Hence

$$rank(A_{ii})\geq\sum_{\mu\subset X_{ii}} rank(A_i^{\mu\nu}).$$

By inspection it is easy to see that $A_i^{\mu\nu}$ is nothing but an incidence matrix $W_{s-i,k-i}$ in $H(d-i, 1)$.

Since $s-i+k-i\leq d-i$, if and only if $i\geq s+k-d$, Lemma 3.6 implies the following.

(1) $rank(A_i^{\mu\nu})=\binom{d-i}{s-i}$, *if* $i\geq s+k-d$.

(2) $rank(A_i^{\mu\nu})=\binom{d-i}{k-i}$, *if* $i\leq s+k-d$.

Combined with Lemma 3.5, we proved the following theorem.

THEOREM 3.7. (1) *Let* $0\leq s\leq k\leq d$. *Then*

$$rank(W_{sk})\geq\sum_{i=0}^{s+k-d-1}\binom{d-i}{k-i}\binom{d}{i}(q-1)^i+\sum_{i=s+k-d}^{s}\binom{d-i}{s-i}\binom{d}{i}(q-1)^i$$

(2) *Let* $Y\subset L$ *be a* $[2s]$-$((d, q), k, \lambda)$ *design. Then*

$$|Y|\geq\sum_{i=0}^{s+k-d-1}\binom{d-i}{k-i}\binom{d}{i}(q-1)^i+\sum_{i=s+k-d}^{s}\binom{d-i}{s-i}\binom{d}{i}(q-1)^i$$

REMARK. (1) With a little more effort we can show that the equality holds in Theorem 3.7. (1), if either $s+k\leq d$ or $k=d$.

(2) If $s+k\leq d$, the first summand yields zero in the right hand side of the inequality. So

$$|Y|\geq\sum_{i=0}^{s}\binom{d-i}{s-i}\binom{d}{i}(q-1)^{i}=\sum_{i=0}^{s}\binom{d}{s}\binom{s}{i}(q-1)^{i}=\binom{d}{s}q^{s}.$$

This gives an alternative proof of Corollary 3.3.

(3) If $k=d$, the second summand has only one term. So

$$|Y|\geq\sum_{i=0}^{s}\binom{d}{i}(q-1)^{i}.$$

This gives the Rao's bound of an orthogonal array of strength $2s$, which is the dual of the Hamming bound in Coding theory when the space is linear and Q is a field of q-elements. Note that $Y^{\perp}$ is a $(2s+1)$-code in that case.

4. Examples

A) Let x_0 be an element of X_d in $H(d, q)$, $q>1$, and $\Delta_{q,x_0}=\Delta$ a surjective mapping from a semilattice $H(d, q)$ to $H(d, q-1)$ defined as follows.

For $\alpha\in L$ (i.e., an element of $H(d, q)$),
$D(\Delta\alpha)=\{s\in D(\alpha)|\alpha(s)\neq x_0(s)\}$, and
$(\Delta\alpha)(s)=\alpha(s)$ if $s\in D(\Delta\alpha)$.

We employ the bar notation for the images of Δ. For example, $\Delta\alpha=\bar{\alpha}$, $\Delta L=\bar{L}$ and $\Delta Y=\bar{Y}$ if Y is a subset of L. We identify $H(d, q-1)$ or $\bar{L}$ as a subset of $H(d, q)$ or L.

DEFINITION 4.1. A subset $\bar{Y}$ of $\bar{L}$ is a $\{t\}$-design of type q, if $\bar{Y}$ is a $\{t\}$-$(d, q-1)$, $\bar{\lambda}_1, \dots, \bar{\lambda}_t)$ design with $\bar{\lambda}_i=q^{-i}|\bar{Y}|$.

LEMMA 4.1. *A subset $\bar{Y}$ of $\bar{L}$ is a $\{t\}$-design of type q, if and only if*

$$\lambda_i(\alpha)=|\{y\in Y|\alpha\leq y\}|=q^{-i}|Y|$$

for each element α in X_i satisfying $\alpha\wedge x_0\in X_0$ with $i=0, \dots, t$, where $Y=\Delta^{-1}(\bar{Y})\cap X_d$.

PROOF. Let α be an element in X_i satisfying $\alpha\wedge x_0\in X_0$. Then $\Delta\alpha=\bar{\alpha}=\alpha$. Moreover for an element x in X_d, $\alpha\leq x$ is equivalent to $\Delta(\alpha)\leq\Delta(x)$. Hence $|\{\bar{y}\in\bar{Y}|\bar{\alpha}\leq\bar{y}\}|=|\{y\in Y|\alpha\leq y\}|$ as the restriction of the mapping Δ to X_d *is a bijection onto* $\bar{L}$.

Hence $\lambda_i(\alpha)=\bar{\lambda}_i=q^{-i}|\bar{Y}|=q^{-i}|Y|$.

PROPOSITION 4.2. (1) *If a subset* $\bar{Y}$ *of* $\bar{L}$ *is a* $\{t\}$*-design of type* q, *then* $\Delta^{-1}(\bar{Y})\cap X_d$ *is a* $[t]$*-design with* $\lambda=q^{-t}|Y|$, *i. e., an orthogonal array of strength* t.

(2) *Conversely, if a subset* Y *of* X_d *is a* $[t]$*-design, then* $\Delta(Y)$ *is a* $\{t\}$*-design of type* q.

PROOF. (2) Since Y is a subset of X_d, $Y=\Delta^{-1}(\bar{Y})\cap X_d$. By Lemma 2.2, $\lambda_i=q^{t-1}\lambda$ and $|Y|=\lambda_0=q^t\lambda$. We have $\lambda_i=q^{-i}|Y|$. Hence the assertion follows from Lemma 4.1.

(1) Let $Y=\Delta^{-2}(\bar{Y})\cap X_d$ and α be an element of X_t with $\alpha\wedge x_0\in X_s$. We show by double induction on t and s that

$$\lambda_t(\alpha)=|\{y\in Y|\alpha\leq y\}|=q^{-t}|Y|.$$

If $s=0$, the assertion holds by Lemma 4.1. Now assume that $\lambda_u(\beta)=q^{-u}|Y|$ for any element β in X_u such that $\beta\wedge x_0\in X_j$, with $0\leq u<t$, or $u=t$ and $0\leq j<s$. Choose $a\in D$ so that $\alpha(a)=x_0(a)$. Let $D_1=D\backslash\{a\}$, $Q_1=Q\backslash\{\alpha(a)\}$ and α^0 be an element of X_{t-1} defined by $\alpha^0=\alpha_{|D_1}$. For each $c\in Q_1$, let α^c be an element of X_t defined by $\alpha^c_{|D^1}$ and $\alpha^c(a)=c$. Then $\{y\in Y|\alpha^0\leq y\}$ is a disjoint union of $\{y\in Y|\alpha^0\leq y\}$ and $\bigcup_{c\in Q_1}\{y\in Y|\alpha^c\leq y\}$. Since $\alpha^c(a)\neq x_0(a)$, while $\alpha(a)=x_0(a)$, we have $\alpha^c\wedge x_0\in X_{s-1}$. Hence we have

$$\lambda_t(\alpha)=\lambda_{t-1}(\alpha^0)-\sum_{c\in Q_1}\lambda_t(\alpha)^c)=q^{-t+1}|Y|-(q-1)q^{-t}|Y|=q^{-t}|Y|.$$

The construction above, especially when $q=2$, may be known to many, who are interested in t-designs with multiple block sizes.

We give two types of examples of $[t]$-designs. The first is a trivial example called a product type. The second is less trivial and we give a series of [2]-designs which attain the bound in Corollary 3.3.

B) Let Y_1 be a $[t]$-$((d, 1), k, \lambda_1)$ design, i. e., a combinatorial t-design, and $\{Y_\alpha\}_{\alpha\in Y_1}$ be a collection of $[t]$-$((k, q), k, \lambda_2)$ designs, i. e., orthogonal arrays of strength t. Let $\{D_\alpha\}_{\alpha\in Y_1}$ be a collection of the domain sets of Y_α's of size k. Since each $\alpha\in Y_1$ is a k-element subset, we fix a bijection f_α from α to C_α. Let $Y=\{(\beta, \alpha)|\alpha\in Y_1, \beta\in Y_\alpha\}$. Let $D((\beta, \alpha))=\alpha$, and for $a\in\alpha$, $(\beta, \alpha)(a)=\beta(f_\alpha(a))$. Then Y yields a subset of a semilattice $H(d, q)$. Now it is easy to see that Y is a $[t]$-$((d, q), k, \lambda_1\lambda_2)$ design. We call this $[t]$-design, a product type.

By a little computation it is not hard to see that any [2s]-design of product type with $s\geq 1$ and $k\neq d$ does not attain the bound given in Theorem 3.7.

C) Let x_0 be an element of X_d and Y be a $[t]$-$((d, q), d, \lambda)$ design satisfying the following.

(*) There is a constant k such that $\alpha \wedge x_0 \in X_{d-k}$ for every α in $Y \setminus \{x_0\}$.

Assume $q>1$, then we have $k \geq t$. Now it follows from Proposition 4.2 that $\Delta(Y \setminus \{x_0\})$ is a $[t]$-$((d, q-1), k, \lambda)$ design.

LEMMA 4.3. *Let C be a $[d, m, t+1]$ linear code over a field with q-elements, where d is the dimension of the underlying vector space, m is the dimension of C and $t+1$ is the minimum weight of C. If every nonzero vector of $C^{\perp}$ has a constant weight k, then $C^{\perp}$ is a $[t]$-$((d, q), d, \lambda)$ design satisfying the condition (*). Here $\lambda = q^{d-m-t}$. In partiqular, $\Delta(C^{\perp} \setminus \{0\})$ is a $[t]$-$((d, q-1), k, \lambda)$ design.*

PROOF. Let H be a generator matrix of $C^{\perp}$. Since the minimum weight of C is $t+1$, any choice of t-columns of H are linearly independent. Hence $C^{\perp}$ is a $[t]$-$((d, q), d, \lambda)$ design and the rest of the assertions are straight forward.

Let C be a $[q+1, q-1]$ Hamming code over a field $GF(q)$ with q-elements. Then C is a $[q+1, q-1, 3]$ code and a generator matrix of $C^{\perp}$, or equivalently a parity check matrix of C has columns which are pairwise linearly independent. For example the first q columns are $\begin{pmatrix} 1 \\ a \end{pmatrix}$, $a \in GF(q)$, and the last column is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then it is easy to check that every nonzero vector of $C^{\perp}$ has weight q. Thus by Lemma 4.3, $\Delta(C^{\perp} \setminus \{0\})$ is a $[2]$-$((q+1, q-1), q, 1)$ design. Hence this design attains the bound in Theorem 3.2 or equivalently, the bound in Corollary 3.3. The following is the smallest example, $[2]$-$((4, 2), 3, 1)$ design by this construction.

$$
\begin{aligned}
&y_1 = (1, 1, 1, *), \ y_2 = (2, 2, 2, *) \\
&y_3 = (1, 2, *, 1), \ y_4 = (2, 1, *, 2) \\
&y_5 = (1, *, 2, 2), \ y_6 = (2, *, 1, 1) \\
&y_7 = (*, 1, 2, 1), \ y_8 = (*, 2, 1, 2)
\end{aligned}
$$

Here $Q = \{1, 2\}$, and $*$ denotes the point where the value is not defined.

D) The following is an example of $[2]$-$((7, 2), 4, 1)$ design which does not come from the construction given in C. This design also attains the bound.

$$
\begin{aligned}
&y_1 = (2, 1, 1, *, 1, *, *), \ y_2 = (1, 2, 2, *, 2, *, *), \\
&y_3 = (*, 2, 1, 1, *, 1, *), \ y_4 = (*, 1, 2, 2, *, 2, *),
\end{aligned}
$$

$y_5=(*, *, 2, 1, 1, *, 1)$, $y_6=(*, *, 1, 2, 2, *, 2)$,
$y_7=(1, *, *, 2, 1, 1, *)$, $y_8=(2, *, *, 1, 2, 2, *)$,
$y_9=(*, 1, *, *, 2, 1, 1)$, $y_{10}=(*, 2, *, *, 1, 2, 2)$,
$y_{11}=(1, *, 1, *, *, 2, 1)$, $y_{12}=(2, *, 2, *, *, 1, 2)$,
$y_{13}=(1, 1, *, 1, *, *, 2)$, $y_{14}=(2, 2, *, 2, *, *, 1)$.

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References

[1] T. ATSUMI, T. ITO, H. NAGAO, The theorems of Gross type for (t)-designs, preprint.
[2] P. DELSARTE, Association schemes and t-designs in regular semilattices, J. Combinatorial Theory, Ser. A, 20 (1976), 230-243.
[3] D. KREHER, An incidence algebra for t-designs with automorphisms, J. Combinatorial Theory, Ser. A, 42 (1986), 239-251.
[4] H. NAGAO, Personal communication.
[5] R. WILSON, Incidence matrices of t-designs, Linear Algebra and its Applications, 46 (1982), 73-82.

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