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On the behavior of solutions of elliptic and parabolic equations at a crack*

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In [2], [4] we studied the initial-Dirichlet problem for parabolic equations in n-dimensional domains with non-smooth boundaries and investigated the behavior of the solutions near the edges of the boundary. In these papers, the "angles" $\omega(P)$ at the edges were always considered to be less than 2π. The case of cracks (or slits), which corresponds to the value $\omega=2\pi$, is of great practical importance, cf [6], [8] and the references mentioned there.

In this paper, we consider domains with cracks, which correspond to angles of value 2π on the boundary. We investigate the behavior at the tips of these cracks, of solutions of the Dirichlet problem for elliptic equations, as well as the initial-Dirichlet problem for parabolic equations. The plan in this paper will be as follows. We first consider the Dirichlet problem for an elliptic equation in a domain G with cracks on the boundary. The full details of the proofs will be given. We then state the result for the initial-Dirichlet problem for a parabolic equation in G X $[0, T]$, and to establish the result in this case, we only indicate the necessary modifications on the proofs given in the elliptic case.

We describe first the domain $G\subset \boldsymbol{R}^n$, $n\geq 2$ in which we consider the problem. The boundary ∂G of G consists of a finite number of $(n-1)$-dimensional surfaces Γ_i; $i=1,2,\ldots,k$ of class $C^{2+\alpha}$. The surface Γ_i may intersect only with Γ_{i-1} and Γ_{i+1} across $(n-2)$-dimensional manifolds S_{i-1} and S_i. The surface Γ_i may also be isolated; does not intersect with any of the other surfaces. Let $P\in S_i$; $S_i=\Gamma_i\cap\Gamma_{i+1}$ and let the angle at P between Γ_i and Γ_{i+1} be $\gamma(P)$, where $0<\gamma(P)<2\pi$. In [2], [4] we studied the smoothness properties of solutions of the initial-Dirichlet problem for parabolic equations near the boundary point P. The case when $\gamma(P)=2\pi$ was not studied there. In this paper we confine ourselves with this case.

THEOREM 1. *Let* $G\subset \boldsymbol{R}^n$, $n\geq 2$, *and let* $\Gamma\subset\partial G$ *be an* $(n-1)$-*dimensional surface with edge* S. *Let* $\partial G\backslash\Gamma$, Γ *and* S *be of class* $C^{2+\alpha}$. *In* G *we consider the Dirichlet problem*

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(1) $Lu \equiv a_{ij}(x) u_{x_i x_j} + a_i(x) u_{x_i} + a(x) u = f(x)$, *in* G
(2) $u = 0$ *on* ∂G

where $x = (x_1, \dots, x_n)$ *and we use the summation convention. We assume that* (1) *is uniformly elliptic in* G. *If* a_{ij}, a_i, a *and* f *belong to* $C^{\alpha}(\bar{G})$, $0 < \alpha < 1$, *then* $u \in C^{\frac{1}{2}-\epsilon}(\bar{G})$, *where* $\epsilon > 0$ *is arbitrarily small.*

We first simplify the problem through the following remarks.

REMARK 1. Under the assumptions of the theorem, it follows that $u \in C^{2+\alpha}(G_1)$,, where G_1 is any compact subregion of $\bar{G}$ with positive distance from the edge S, [1]. Thus it is sufficient to prove that $u \in C^{\frac{1}{2}-\epsilon}(B(P, \rho))$, where P is any point on S and $B(P, \rho) \subset G$ is a ball with center at P and radius ρ, $\rho > 0$.

REMARK 2. We can assume that the surface Γ coincides with the hyperplane $x_k = 0$, $k = 3, \dots, n$ and that the crack around P has the equation $x_2 = 0$. This can be always accomplished using invertible $C^{2+\alpha}$ maps.

REMARK 3. We can assume that P is located at the origin $x = 0$. We can also assume that $a_{ij}(0) = \delta_{ij}$, the Kronecker delta, $i, j = 1, 2$. This can be reached by using the following nonsingular transformation

$$y_1 = \frac{1}{\Lambda \sqrt{a_{22}(0)}} [a_{22}(0) x_1 - a_{12}(0) x_2]$$

$$y_2 = \frac{1}{\sqrt{a_{22}(0)}} x_2$$

$$y_k = x_k, \quad k > 2$$

where

$$\Lambda = [a_{11}(0) a_{22}(0) - a_{12}^2(0)]^{\frac{1}{2}}$$

REMARK 4. In our proof we assume that the solution u vanishes outside a small sphere with center at O and of radius $3r_0$ say. This situation may be reached by introducing first the cut-off function $\xi(|x|) \epsilon C^3(\boldsymbol{R}^n)$, that satisfies

$$\xi(|x|) = \begin{cases} 1 & 0 \le |x| \le 2r_0 \\ 0 & |x| \ge 3r_0 \end{cases}$$

and then considering the function $v(x) = \xi(|x|)\ u(x)$, which will satisfy an equation of the form(1) with $v(x) \equiv 0$ for $r \ge 3r_0$.

To prove the theorem, we first need an estimate for the solution. This is accomplished by constructing a barrier function.

LEMMA. *There exists* $\rho>0$ *such that in* $B(O, \rho)$ *we have*

(3) $|u(x)|\leq Mr^{\frac{1}{2}-\epsilon}$,

where $r^2=x_1^2+x_2^2$, $\epsilon>0$ *is arbitrarily small, and* $M>0$ *is a constant independent of* r.

PROOF. We first fix ϵ, $0<\epsilon<\frac{1}{2}$ and we consider positive numbers β, λ and ν that satisfy

$$\beta<\frac{2\epsilon\pi}{1-2\epsilon},\quad \lambda=\frac{\pi}{2\pi+2\beta},\quad \nu<\lambda<\frac{1}{2},\quad \nu=\frac{1}{2}-\epsilon$$

then we define the function $v(x)$ as follows

$$v(x)=-Mr^{\nu}\cos\lambda(\theta-\pi),\quad M>0$$

where θ is given by $x_1=r\cos\theta$ and $x_2=r\sin\theta$. We write

$$Lu\equiv\Delta u+[a_{ij}(x)-\delta_{ij}]u_{x_ix_j}+a_iu_{x_i}+au.$$

Now Lv is given by

$$Lv=M(\lambda^2-\nu^2)r^{\nu-2}\cos\lambda(\theta-\pi)+o(r^{\nu-2}).$$

Noting that $\nu-2<0$, and that for any $\theta\in[0, 2\pi]$ we have $\cos\lambda(\theta-\pi)\geq\cos\lambda\pi>0$, we can make $Lv\geq|f(x)|$ in $B(O, \rho)$ by taking $\rho>0$ sufficiently small. Thus in $B(O, \rho)\backslash\Gamma$ we have

$$L(u-v)\leq 0.$$

Since $u\equiv 0$ on the boundary of $B(O, \rho)\backslash\Gamma$, we have $u-v\geq 0$ there. Taking ρ sufficiently small to apply the Maximum Principle, we finally reach $u-v\geq 0$ in $B(O, \rho)$ i.e.,

$$u\geq -Mr^{\nu}\cos\lambda(\theta-\pi)\geq -Mr^{\nu}.$$

Similarly we can prove the other part of inequality (3). The lemma is proved.

We now prove Theorem 1, taking into consideration Remarks 1–4.

PROOF OF THEOREM 1. Consider any two points P and Q in $\bar{B}(O, \rho)$ with distances r_1 and r_2 from the crack line $x_k=0$, $k\geq 2$, where $0\leq r_2\leq r_1\leq \rho$. If $r_2\leq\frac{1}{2}r_1$ then $d(P, Q)\geq\frac{1}{2}r_1$ and from the previous lemma, it follows that

$$|u(P)-u(Q)|\le Mr_1^{\frac{1}{2}-\epsilon}+Mr_2^{\frac{1}{2}-\epsilon}\le$$
$$\le 2Mr_1^{\frac{1}{2}-\epsilon}\le M_0[d(P,Q)]^{\frac{1}{2}-\epsilon},$$

where M_0 depends on M and ϵ.

If $r_2>\frac{1}{2}r_1$, we consider the domain

$$D_P=\{x\in B(O,\rho),\frac{1}{2}r_1\le r\le r_1,\ |x_i-x_i^0|\le\frac{1}{2}r_1,\ i=3,\dots,n\},$$

where $(x_1^0,\dots,x_n^0)$ are the coordinates of P. The transformation

(4. a) $\quad x_i=\dfrac{2r_1}{\rho}x_i',\ i=1,2$

(4. b) $\quad x_i-x_i^0=\dfrac{2r_1}{\rho}(x_i'-x_i^0),\ i>2,$

transforms D_P into

$D_P'=\{\frac{\rho}{4}\le r'\le\frac{\rho}{2},\ |x_i'-x_i^0|\le\frac{\rho}{4},\ i>2\}$. $r'^2=x_1'^2+x_2'^2$. In D_P' the function $v(x')=u(x)$ satisfies the elliptic equation

$c_{ij}(x')v_{x_i'x_j'}+\dfrac{2r_1}{\rho}c_i(x')v_{x_i'}+\left(\dfrac{2r_1}{\rho}\right)^2c(x')v=\left(\dfrac{2r_1}{\rho}\right)^2h(x')$, where c_{ij}, c_i, c and h are the coefficients of (1) after the transformation (4). Consider

$$D_P''=\{\frac{\rho}{8}\le r'\le\rho,\ |x_i'-x_i^0|\le\frac{\rho}{4},\ i>2\}.$$

In D_P' and D_P'' we apply the Shauder estimate [1], to get

$$\|v\|_{2+\alpha}^{D_P'}\le C_0[\|v\|_0^{D_P''}+\left(\frac{2r_1}{\rho}\right)^2\|h\|_\alpha^{D_P''}],$$

We note that C_0 is independent of r_1, since it depends on the maximum norms of the coefficients of the equation and in our problem $r_1/\rho<1$. The constant C_0 also depends on α and the ellipticity of the equation ($\inf c_{ij}(x')\xi_i\xi_j$). Since $r=\dfrac{2r_1}{\rho}r'$, thus from the previous lemma, it follows that, in D_P'',

$$\|v\|_0^{D_P'}\le M_0r_1^{\frac{1}{2}-\epsilon}.$$

Thus

(5) $\|v\|_{2+\alpha}^{D'_P} \leq C_1 r_1^{\frac{1}{2}-\epsilon}.$

where C_1 depends on C_0 and M_0.

Let $H_\gamma^\Omega(W)$ *be the Hölder coefficient of exponent* γ *of the function* W in the domain Ω, then since

(6) $H_{\frac{1}{2}-\epsilon}^{D'_P}(v) \leq k\|v\|_{2+\alpha}^{D'_P},$

it follows from (4), (5) and (6) that

$$H_{\frac{1}{2}-\epsilon}^{D_P}(u) \leq k_0,$$

where k_0 depends on k and C_1. This completes the proof of the theorem.

We now turn to the parabolic case. Let G, ∂G, Γ and S be as given in Theorem 1. In $\Omega = G\ X\ J$, $J=[0, T]$ we consider the initial-Dirichlet problem

(7) $Lu \equiv a_{ij}(x)u_{x_ix_j} + a_i(x, t)u_{x_i} + a(x, t)u - u_t = f(x, t)$

where the solution $u(x, t)$ satisfies the initial condition

(8. a) $u(x, 0)=0,\ x \in \bar{G},$

and the Dirichlet boundary condition

(8. b) $u(x, t)|_{\partial GXJ}=0,$

THEOREM 2. *Let* $u(x, t)$ *be a solution of the parabolic equation* (7) *in* Ω, *that satisfies the initial-Dirichlet conditions* (8). *If* a_{ij}, a_i, a *and* f $\in C^\alpha(\bar{\Omega})$, *then*

(9) $u \in C^{\frac{1}{2}-\epsilon}(\bar{\Omega}),$

where $\epsilon > 0$ *is arbitrarily small.*

We note that, in [4], *we studied the smoothness of solutions of* (7) *—*(8) *in domains with edges of " angles "* $\omega(P)$ *that are less than* 2π. *The result there was* $u \in C^{\frac{\pi}{\omega}-\epsilon}$. *The result* (8) *in the given crack case coincides with that result for* $\omega = 2\pi$.

As mentioned in the introduction, we conclude by indicating here the modifications needed on the proof given for the elliptic case.

REMARK 1′. The case of a smooth boundary was studied in great details, cf [7]. So it remains to prove our claim in $B(P, \rho)X\ J$; cf Remark 1.

REMARK 2'. Remarks 2−4 are still valid here.

REMARK 3'. A bound of the form (3) for the solution $v(x, t)$ in $B(P, \rho) X \bar{J}$ may be found using the same barrier function, as in the lemma.

REMARK 4'. The proof of Theorem 2 goes along the same lines as that of Theorem 1, but here we use the Shauder-type estimates for solutions of parabolic equations as given in [7].

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