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The Grothendieck ring of linear representations of a finite category

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Dedicated to Professor Tosiro Tsuzuku
on his sixtieth birthday

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Introduction

A finite category is a category whose objects and morphisms form finite sets. Yoshida proved the following theorem in his attempt to define the Burnside ring of a finite category [4].

THEOREM. *Suppose that a finite category C satisfies the following conditions.*

(a) *C has the unique epi-mono factorization property (see Section* 4 *for the precise definition).*

(b) *For any object x of C and any cyclic subgroup H of* $\mathrm{Aut}(x)$, *a quotient object $H\backslash x$ exists.*

Let I be a set of representatives for isomorphism classes of objects of C. Denote by $\mathbf{Z}[I]$ and $\mathbf{Z}^{I}$ the free abelian group on I and the ring of **Z***-valued functions on I respectively. Define a group homomorphism φ: $\mathbf{Z}[I]\longrightarrow\mathbf{Z}^{I}$ by $\varphi(x)(y)=\#\mathrm{Hom}_{C}(y, x)$ for $x, y\in I$. Then*

(i) *φ is injective.*

(ii) $\#\mathrm{Coker}(\varphi)=\prod_{x\in I}\#\mathrm{Aut}(x)$.

(iii) $\mathrm{Image}(\varphi)$ *is a subring of $\mathbf{Z}^{I}$ (with the common identity).*

Thus, for such a category C, $\mathbf{Z}[I]$ has a unique ring structure such that φ is a ring homomorphism. Yoshida called $\mathbf{Z}[I]$ the abstract Burnside ring of C. When C is the category of transitive G-sets for a finite group G, the ring $\mathbf{Z}[I]$ is just the Burnside ring of G, i. e., the Grothendieck ring of the category of finite G-sets.

In this paper we prove a linear version of the above theorem. Let k be a field of characteristic $p>0$ and C a finite category. A $k[C]$-module means a functor $C^{\mathrm{op}}\longrightarrow\{k\text{-modules}\}$. Let $G_{0}(k[C])$ (resp. $K_{0}(k[C])$) be the Grothendieck group of the category of finite dimensional (resp. finite dimensional projective) $k[C]$-modules with respect to exact sequences. Tensor product makes $G_{0}(k[C])$ a commutative ring. Let $c: K_{0}(k[C])$

$\longrightarrow G_0(k[C])$ be the Cartan map, namely the map induced by viewing projective $k[C]$-modules simply as $k[C]$-modules. For an object x of C, let $c_x: K_0(k[\mathrm{Aut}(x)]) \longrightarrow G_0(k[\mathrm{Aut}(x)])$ be the Cartan map for $k[\mathrm{Aut}(x)]$-modules.

THEOREM A. *Suppose that* C *satisfies the following conditions.*

(a) C *has the unique epi-mono factorization property.*

(b) *For any object* x *of* C *and any* p*-subgroup* Q *of* $\mathrm{Aut}(x)$, *a quotiont object* $Q\backslash x$ *exists.*

Let I *be as in the previous theorem. Then*

(i) *The map* c *is injective.*

(ii) $\#\mathrm{Coker}(c)=\prod_{x\in I}\#\mathrm{Coker}(c_x)$.

(iii) $\mathrm{Image}(c)$ *is a subring of* $G_0(k[C])$).

The proof of this theorem is based on the next theorem. Denote by k_C the constant functor on C^{op} with value k.

THEOREM B. *Let* C *be as in Theorem A. Then the* $k[C]$*-module* k_C *has a finite projective dimension.*

These theorems are proved in Sections 3-5. As preparation we classify simple $k[C]$-modules for an arbitary finite category C and determine the ring $G_0(k[C])$ in Section 2.

1. Notation and conventions

We fix a field k throughout and put $p=\mathrm{char}(k)$ if $\mathrm{char}(k)>0$, $p=1$ if $\mathrm{char}(k)=0$. Our modules are right and finitely generated, unless specified otherwise. The category of such modules over a ring A is denoted by A-$\mathcal{M}od$. The Grothendieck group of the category of A-modules (resp. projective A-modules) with respect to exact sequences is denoted by $G_0(A)$ (resp. $K_0(A)$). An (resp. a projective) A-module M has its class $[M]$ in $G_0(A)$ (resp. $K_0(A)$).

Let C be a finite category. We denote by $\mathrm{ob}(C)$ and $\mathrm{mor}(C)$ the set of objects and the set of morphisms of C respectively. We often write $\mathrm{Hom}_C(x, y)=C(x, y)$ for objects x, y of C. We denote by C^{op} the dual category of C and by $C^\wedge$ the category of functors $C^{\mathrm{op}}\longrightarrow\{\mathrm{sets}\}$. For $x\in \mathrm{ob}(C)$, we set $h_x=\mathrm{Hom}_C(-, x)\in\mathrm{ob}(C^\wedge)$. Given $F\in\mathrm{ob}(C^\wedge)$, the category C/F is defined as follows. Objects are pairs (x, a) with $x\in\mathrm{ob}(C)$, $a\in F(x)$, and $\mathrm{Hom}_{C/F}((x, a), (y, b))=\{f\in\mathrm{Hom}_C(x, y)\mid F(f)(b)=a\}$. When $F=h_x$, we write $C/h_x=C/x$. Dually $x\backslash C$ denotes the category of morphisms $x\longrightarrow y$.

We mean by a $k[C]$-module a functor $C^{op} \longrightarrow k\text{-}\mathcal{M}od$. The category of $k[C]$-modules is denoted by $k[C]\text{-}\mathcal{M}od$. If $F: C^{op} \longrightarrow$ {finite sets} is a functor, $k[F]$ denotes the $k[C]$-module taking $x \in \mathrm{ob}(C)$ to $k[F(x)]$, the free k-module on the set $F(x)$. Then the $k[C]$-modules $k[h_x]$, $x \in \mathrm{ob}(C)$, are projective.

2. Simple modules

Let C be a finite category. Let A be a ring object of $C^\wedge$, i.e., a functor $C^{op} \longrightarrow$ {rings}. An A-module is an abelian group object F of $C^\wedge$ together with a morphism $F \times A \longrightarrow F$ in $C^\wedge$ satisfying the same commutative diagrams as in the definition of usual modules. The category of A-modules is denoted by $A\text{-}\mathcal{M}od$. We aim to classify simple objects of this category. Though our main concern lies in the case where A is the constant ring functor k_C, the general case does not require more effort.

Before doing it, we make a slight reduction. A category C is said to be Karoubien if every idempotent endomorphism e in C has a factorization $e = ip$ such that pi is an identity morphism (Grothendieck and Verdier, [2]). For any category C it is known that there is a Karoubien category C' with $C'^\wedge$ being equivalent to $C^\wedge$. Here is a construction of C'. Objects of C' are pairs (x, e) where $x \in \mathrm{ob}(C)$ and $e^2 = e \in \mathrm{End}(x)$, and $\mathrm{Hom}_{C'}((x, e), (x', e')) = \{f \in \mathrm{Hom}_C(x, x') \mid e'f = f = fe\}$. Composition of morphisms of C' is restriction of that of C. If C is finite, so is C'. Since $C^\wedge \simeq C'^\wedge$, there is a ring object A' of $C'^\wedge$ so that $A\text{-}\mathcal{M}od \simeq A'\text{-}\mathcal{M}od$. Thus, for our purpose we may replace C by C'. Until the end of this section we assume that C is Karoubien.

When a monoid M acts on a ring R on the right, $R[M]$ denotes the twisted monoid ring. Elements of $R[M]$ are of the form $\sum \sigma r$, with $\sigma \in M$, $r \in R$, and product is defined by $\sigma r \cdot \tau s = \sigma\tau r^\tau s$. If $x \in \mathrm{ob}(C)$, the monoid $\mathrm{End}(x)$ and the group $\mathrm{Aut}(x)$ act on the ring $A(x)$, so we have the rings $A(x)[\mathrm{End}(x)]$, $A(x)[\mathrm{Aut}(x)]$.

LEMMA 2.1. *Let F be an A-module and $x \in \mathrm{ob}(C)$ such that $F(x) \neq 0$. Then the following are equivalent.*

(i) *F is a simple A-module.*

(ii) *$F(x)$ is a simple $A(x)[\mathrm{End}(x)]$-module, and for any $y \in \mathrm{ob}(C)$ we have*

$$\sum_{f: y \to x} \mathrm{Im}(F(f))A(y) = F(y),$$

$$\bigcap_{g: x \to y} \mathrm{Ker}(F(g)) = 0.$$

PROOF. (i)$\Rightarrow$(ii): Let M be an $A(x)[\mathrm{End}(x)]$-submodule of $F(x)$. Define an A-submodule F' of F by

$$F'(y)=\sum_{f:y\to x}F(f)(M)A(y)$$

for $y\in\mathrm{ob}(C)$. Then $F'(x)=M$. Since F is simple, $F'=F$ or $F'=0$. Hence $M=F(x)$ or $M=0$. Thus $F(x)$ is a simple $A(x)[\mathrm{End}(x)]$-module. Let $M=F(x)$. Then $F'=F$. This proves the first equality in (ii). The second one follows similarly, by considering a submodule F'' of F defined by

$$F''(y)=\bigcap_{g:x\to y}\mathrm{Ker}(F(g))$$

for $y\in\mathrm{ob}(C)$. (ii)$\Rightarrow$(i): Obvious. Q.E.D.

Let $x, y\in\mathrm{ob}(C)$. We write $x\leq y$ if x is a direct summand of y, i.e., if there are morphisms $i:x\to y$ and $p:y\to x$ such that $pi=\mathrm{id}_x$. Note that $x\leq y\leq x$ implies $x\cong y$. The following lemma is well known.

LEMMA 2.2. *Let S be a finite monoid. If $f\in S$, then f^n is an idempotent for some integer $n>0$.*

LEMMA 2.3. *Let F be a simple A-module and $x\in\mathrm{ob}(C)$. Suppose that $F(x)\neq 0$ and $F(y)=0$ for all $y<x$. Then every non-unit of* End(x) *annihilates $F(x)$.*

PROOF. Put $S=\mathrm{End}(x)$ and let S_0 be the set of non-units of S. If $f\in S_0$, then $F(f)$ is nilpotent. Indeed, take $n>0$ such that $f^n=e$ is an idempotent. Since C is Karoubien, we can write $e=ip:x\to y\to x$ with $pi=\mathrm{id}_y$. By $e\in S_0$, i is not an isomorphism, hence $y<x$. Then $F(y)=0$, so $F(f)^n=F(e)=0$ as asserted. Now let I be the two-sided ideal of $A(x)[S]$ generated by S_0. Suppose that $F(x)I\neq 0$. Then $F(x)I=F(x)$ because $F(x)$ is a simple $A(x)[S]$-module. Put $T=\{t\in S_0|F(x)t\neq 0\}$. Then $T\neq\emptyset$ and if $t\in T$, then $st\in T$ for some $s\in S_0$. Take $t_0\in T$ such that the subset St_0 of S is minimal among St for all $t\in T$. Take $s_0\in S_0$ such that $s_0t_0\in T$. Then $St_0=Ss_0t_0$, so $t_0=ss_0t_0$ with $s\in S$. By the earlier observation, $F(x)(ss_0)^n=0$ for some $n>0$. Then $t_0=(ss_0)^nt_0$ also annihilates $F(x)$. This is a contradiction. Thus $F(x)I=0$, which proves the lemma.

LEMMA 2.4. *Let F, x be as in Lemma 2.3. For $y\in\mathrm{ob}(C)$, we have that $F(y)\neq 0$ if and only if $x\leq y$.*

PROOF. If $x\leq y$, then $F(x)$ is a direct summand of $F(y)$, so $F(y)\neq$

0. Define an A-submodule F' of F by

$$F'(z)=\sum_{f}\mathrm{Im}(F(f))A(z)$$

for $z\in\mathrm{ob}(C)$, where f ranges over all morphisms $z\to x$ which do not have sections. By Lemma 2.3, $F'(x)=0$. Since F is simple, $F'=0$. On the other hand, we have by Lemma 2.1 that

$$0\neq F(y)=\sum_{f:y\to x}\mathrm{Im}(F(f))A(y).$$

Therefore some $f:y\to x$ must have a section. This proves the lemma.

Let $x\in\mathrm{ob}(C)$. We denote by $A[h_x]$ the right A-module taking $y\in\mathrm{ob}(C)$ to the free right $A(y)$-module on the set $C(y,x)$. The ring $A(x)[\mathrm{End}(x)]$ acts naturally on $A[h_x]$ on the left. Therefore, if V is a right $A(x)[\mathrm{End}(x)]$-module, we have a right A-module $F'=V\otimes_{A(x)[\mathrm{End}(x)]}A[h_x]$. Define a right A-module $S_{x,V}$ by

$$S_{x,V}(y)=F'(y)/\bigcap_{g:x\to y}\mathrm{Ker}(F'(g))$$

for $y\in\mathrm{ob}(C)$.

LEMMA 2.5. *If V is a simple $A(x)[\mathrm{End}(x)]$-module, then $S_{x,V}$ is a simple A-module.*

PROOF. This follows immediately from Lemma 2.1.

Let I be a representative system of isomorphism classes of objects of C. For each $x\in I$, take a representative system R_x of isomorphism classes of simple $A(x)[\mathrm{Aut}(x)]$-modules. Let R be the set of pairs (x, V) with $x\in I$, $V\in R_x$. Any $A(x)[\mathrm{Aut}(x)]$-module can be viewed as an $A(x)[\mathrm{End}(x)]$-module on which the non-units of $\mathrm{End}(x)$ act as zero. Thus, for each $(x, V)\in R$, we have the simple A-module $S_{x,V}$.

PROPOSITION 2.6. *$\{S_{x,V}|(x, V)\in R\}$ is a representative system of isomorphism classes of simple A-modules.*

PROOF. Let F be a simple A-module. Minimal elements of the set $\{x\in\mathrm{ob}(C)|F(x)\neq 0\}$ with respect to the preorder $\leq$ are isomorphic to each other by Lemma 2.4. We call these elements vertices of F. If x is a vertex of F, then by lemma 2.3, the non-units of $\mathrm{End}(x)$ annihilate $F(x)$. Hence $F(x)$ is a simple $A(x)[\mathrm{Aut}(x)]$-module. By the definition of $S_{x,F(x)}$ and Lemma 2.1 applied to F, there is a nonzero A-homomorphism $S_{x,F(x)}\to F$. Both sides being simple, we have $S_{x,F(x)}\cong F$.

We next claim that if $(x, V)\in R$, then x is a vertex of $S_{x,V}$. Let $y<x$. Then $\mathrm{id}_y=pi$ where $i: y\to x$ and $p: x\to y$ are not isomorphisms. Let F' be as in the definition of $S_{x,V}$. If $a\in A(y)$, $f: y\to x$ and $v\in V$, then we have that $v\otimes fa=v\otimes fpia=vfp\otimes ia$ in $F'(y)$. Since $fp\in\mathrm{End}(x)$ is not a unit, $vfp=0$. Thus $F'(y)=0$ and so $S_{x,V}(y)=0$. This proves the claim.

Now suppose that $S_{x,V}\cong S_{x',V'}$ for (x, V), $(x', V')\in R$. Considering vertices of both sides, we have $x=x'$. By evaluation at x, we get $V=V'$. This completes the proof.

We consider the case where $A=k_C$, the constant ring functor. Then A-modules are simply functors $C^{\mathrm{op}}\longrightarrow k\text{-}\mathcal{M}od$, i.e., $k[C]$-modules. The tensor product $F\otimes G$ of $k[C]$-modules F, G is the $k[C]$-module defined by $(F\otimes G)(x)=F(x)\otimes_k G(x)$ for $x\in\mathrm{ob}(C)$. The Grothendieck group $G_0(k[C])$ of $k[C]$-modules becomes a commutative ring with multiplication induced by tensor product and identity element $[k_C]$.

PROPOSITION 2.7. *The homomorphism*

$$G_0(k[C])\longrightarrow \prod_{x\in I} G_0(k[\mathrm{Aut}(x)])$$

taking the classes $[F]$ of $k[C]$-modules F to $\{[F(x)]\}_{x\in I}$ is a ring isomorphism.

PROOF. Let $(x, V)\in R$. If $S_{x,V}(y)\neq 0$, then $x\leq y$, and $S_{x,V}(x)\cong V$ as $k[\mathrm{Aut}(x)]$-modules. Hence the above homomorphism takes the basis $\{[S_{x,V}]\}_{(x,V)\in R}$ of $G_0(k[C])$ to a basis of $\prod_{x\in I} G_0(k[\mathrm{Aut}(x)])$ and so it is an isomorphism.

3. A category without non-isomorphic endomorphisms

Throughout this section C is a finite category such that $\mathrm{End}(x)=\mathrm{Aut}(x)$ for all $x\in\mathrm{ob}(C)$. We denote by $\mathrm{pd}\,F$ the projective dimension of a $k[C]$-module F. The finitistic dimension of $k[C]$ is by definition the supremum of finite projective dimensions of $k[C]$-modules, and denoted by $\mathrm{f.dim}\,k[C]$. See Bass [1].

REMARK 3.1. Define $\dim C$ to be the supremum of lengths n of chains $x_0\to x_1\to\ldots\to x_n$ of non-isomorphisms of C. Then it is not difficult to prove that $\mathrm{f.dim}\,k[C]\leq\dim C$.

LEMMA 3.2. *Suppose that C satisfies the following conditions.*

(i) $\mathrm{Aut}(x)$ *is a p-group for any $x\in\mathrm{ob}(C)$.*

(ii) *If $f: x\to y$ is not an isomorphism, then $f=fg$ for some $g\in$* $\mathrm{Aut}(x)$ *with $g\neq 1$.*

Then $\mathrm{f.dim}\, k[C]=0$.

PROOF. For $x\in \mathrm{ob}(C)$, define a $k[C]$-module I_x by $I_x(y)=\mathrm{Map}(C(x, y), k)$ for $y\in\mathrm{ob}(C)$. Then I_x is an injective hull of a simple $k[C]$-module S_x. By a result of Bass [1, Theorem 6.3], it suffices to show that $\mathrm{Hom}(I_x, S_x)\neq 0$ for any $x\in\mathrm{ob}(C)$. This amounts to saying that $\sum_f \mathrm{Im}(I_x(f))\neq I_x(x)$, where $f: x\to y$ runs over all non-isomorphisms. For such a morphism f, the map $C(x, x)\longrightarrow C(x, y): g\longmapsto fg$ is not injective by (ii), so the map $I_x(f)$ is not surjective. Since $I_x(x)$ has a unique maximal left $k[\mathrm{Aut}(x)]$-submodule by (i), we have that $\sum_f \mathrm{Im}(I_x(f))\neq I_x(x)$ as required.

Let G be a finite group. Let $S(G)$ be the category whose objects are the right G-sets $H\backslash G:=\{Hg|g\in G\}$ for subgroups H of G and whose morphisms are G-maps.

PROPOSITION 3.3. *If G is a p-group,* $\mathrm{f.dim}\, k[S(G)]=0$.

PROOF. It is enough to verify that $S(G)$ satisfies conditions (i), (ii) of the previous lemma. (i) is obvious. Let $f: Q\backslash G\longrightarrow Q'\backslash G$ be a non-isomorphism with Q, Q' subgroups of G. We may assume that $Q<Q'$ and f is the projection. Then $Q<N_{Q'}(Q)$ and any element $w\in N_{Q'}(Q)-Q$ induces an automorphism $\overline{w}: Q\backslash G\longrightarrow Q\backslash G$ such that $f\overline{w}=f$. Thus $S(G)$ satisfies (ii) and the proposition is proved.

Suppose given $x\in\mathrm{ob}(C)$ and a subgroup Q of $\mathrm{Aut}(x)$. For $y\in\mathrm{ob}(C)$, $C(x, y)^Q$ denotes the set of fixed elements of $C(x, y)$ under the natural action of Q. If there is an object of C which represents the functor $y\longmapsto C(x, y)^Q$, such an object is called a quotient of x by Q and denoted by $Q\backslash x$. Now suppose that C satisfies the following condition.

(3.4) For any $x\in\mathrm{ob}(C)$ and any p-subgroup Q of $\mathrm{Aut}(x)$, a quotient object $Q\backslash x$ exists.

Then, for $x\in\mathrm{ob}(C)$ and a p-Sylow subgroup P of $\mathrm{Aut}(x)$, we can define a functor

$$q_{x,P}: k[C]\text{-}\mathcal{Mod}\longrightarrow k[S(P)]\text{-}\mathcal{Mod}$$

by

$$q_{x,P}(F)(Q\backslash P)=F(Q\backslash x)$$

for $k[C]$-modules F and subgroups Q of P.

LEMMA 3.5. *Suppose that C satisfies* (3.4). *Let F be a $k[C]$-*

module. Then the following are equivalent.

(i) pd $F<\infty$.

(ii) $q_{x,P}(F)$ *is a projective* $k[S(P)]$*-module for any* $x\in \mathrm{ob}(C)$ *and any* p*-Sylow subgroup* P *of* $\mathrm{Aut}(x)$.

PROOF. Fix a pair (x, P) as in (ii). The functor $q_{x,P}$ is clearly exact. It preserves projective modules. To see this, it is enough to show that for any $y\in \mathrm{ob}(C)$, the functor $Q\backslash P\longmapsto k[C(x, y)^{Q}]$ on $S(P)^{\mathrm{op}}$ is projective. But this follows from the isomorphisms $C(x, y)^{Q}\cong \mathrm{Hom}_{P}(Q\backslash P, C(x, y))\cong \coprod_{i}\mathrm{Hom}_{P}(Q\backslash P, H_{i}\backslash P)$, where $C(x, y)\cong\coprod_{i}H_{i}\backslash P$ as P-sets. Now suppose pd $F<\infty$. Then pd $q_{x,P}(F)<\infty$. By Proposition 3.3, $q_{x,P}(F)$ is projective. This proves (i)$\Rightarrow$(ii).

For the converse, we first observe the following fact. If P is a p-group and M is a projective $k[S(P)]$-module such that $M(Q\backslash P)=0$ for all nontrivial subgroups Q of P, then $M(1\backslash P)$ is a free left $k[P]$-module. Indeed, M must be isomorphic to a direct sum of copies of $k[\mathrm{Hom}_{P}(-, 1\backslash P)]$. Suppose that F satisfies (ii) and $F\neq 0$. Take $x\in \mathrm{ob}(C)$ such that $F(x)\neq 0$ and that $F(y)=0$ if there is a non-isomorphism $x\rightarrow y$. Take a p-Sylow subgroup P of $\mathrm{Aut}(x)$. If $1<Q\leq P$, the projection $x\rightarrow Q\backslash x$ is not an isomorphism. Applying the above observation to $M=q_{x,P}(F)$, we see that $M(1\backslash P)=F(x)$ is a free left $k[P]$-module, and hence a projective left $k[\mathrm{Aut}(x)]$-module. So there is an exact sequence of $k[C]$-modules

$$0\rightarrow F'\rightarrow\bigoplus_{y}(U_{y}\underset{\mathrm{Aut}(y)}{\otimes}k[h_{y}])\oplus(F(x)\underset{\mathrm{Aut}(x)}{\otimes}k[h_{x}])\rightarrow F\rightarrow 0$$

where y runs over objects of C such that $F(y)\neq 0$ and $y\not\cong x$, and U_{y} is a projective $k[\mathrm{Aut}(y)]$-module. The middle term is projective and $F'(x)=0$. Applying $q_{x',P'}$ to this sequence for any pair (x', P'), we see that F' also satisfies (ii). By induction we may assume pd $F'<\infty$. Then pd $F<\infty$. This proves the lemma.

Let $k_{C}: C^{\mathrm{op}}\longrightarrow k\text{-}\mathcal{M}od$ be the constant functor with value k.

PROPOSITION 3.6. *Suppose that* C *satisfies* (3.4). *Then the* $k[C]$*-module* k_{C} *has a finite projective dimension.*

PROOF. For any x and P as in (ii) of the previous lemma, $q_{x,P}(k_{C})=k_{S(P)}\cong k[\mathrm{Hom}_{P}(-, P\backslash P)]$ is projective. Hence the conclusion follows from the lemma.

When G is a finite group, the full subcategory of $S(G)$ consisting of the objects $Q\backslash G$ for p-subgroups Q of G satisfies (3.4).

4. A category having the unique epi-mono factorization property

A category C is said to have the unique epi-mono factorization property if there are two classes $E(C)$ and $M(C)$ of morphisms of C satisfying the following conditions.

(1) If $f \in E(C)$, f is an epimorphism.

(2) If $f \in M(C)$, f is a monomorphism.

(3) $E(C)$ and $M(C)$ contain all isomorphisms and are closed under composition.

(4) Any morphism f of C is factorized as $f = gh$ with $g \in M(C)$, $h \in E(C)$. This factorization is unique in the sense that if $f = g'h'$ with $g' \in M(C)$, $h' \in E(C)$, then $g' = gu$, $h' = u^{-1}h$ for some isomorphism u.

The unique epi-mono factorization property is called FAC in [4]. We call elements of $E(C)$ and $M(C)$ admissible epimorphisms and admissible monomorphisms respectively. Troughout this section C is a finite category having the unique epi-mono factorization property. The following are easy consequences of (1)—(4).

(5) If $f \in E(C) \cap M(C)$, f is an isomorphism.

(6) If $gh \in E(C)$, then $g \in E(C)$.

(7) If $gh \in M(C)$, then $h \in M(C)$.

(8) C is Karoubien.

(9) If $F \in \mathrm{ob}(C^{\wedge})$, then the category C/F defined in Section 1 has also the unique epi-mono factorization property. More precisely, let $p: C/F \longrightarrow C$ be the canonical functor $(x, a) \longmapsto x$. Then the classes $p^{-1}E(C)$, $p^{-1}M(C)$ satisfy conditions (1)—(4) for C/F.

Define subcategories C_e, C_m of C by $\mathrm{ob}(C_e) = \mathrm{ob}(C_m) = \mathrm{ob}(C)$ and $\mathrm{mor}(C_e) = E(C)$, $\mathrm{mor}(C_m) = M(C)$. Both C_e and C_m have no nonisomorphic endomorphisms. Let $j_e: C_e \longrightarrow C$, $j_m: C_m \longrightarrow C$ be the inclusion functors. Define functors $j_e^*: k[C]\text{-}\mathcal{Mod} \longrightarrow k[C_e]\text{-}\mathcal{Mod}$, $j_m^*: k[C]\text{-}\mathcal{Mod} \longrightarrow k[C_m]\text{-}\mathcal{Mod}$ by $j_e^*(F) = F \circ j_e$, $j_m^*(F) = F \circ j_m$.

LEMMA 4.1. (i) *A right adjoint* $j_e^*: k[C_e]\text{-}\mathcal{Mod} \longrightarrow k[C]\text{-}\mathcal{Mod}$ *to* j_{e*} *is given by*

$$j_{e*}(F)(x) = \prod_{z \to x} F(z)$$

for $k[C_e]$*-modules* F *and* $x \in \mathrm{ob}(C)$*, where* $z \to x$ *runs over representatives for isomorphism classes of objects of the category* C_m/x.

(ii) *A left adjoint* $j_{m!}: k[C_m]\text{-}\mathcal{Mod} \longrightarrow k[C]\text{-}\mathcal{Mod}$ *to* j_m^* *is given by*

$$j_{m!}(F)(x) = \bigoplus_{x \to z} F(z)$$

where $x\to z$ runs over representatives for isomorphism classes of objects of the category $x\backslash C_e$.

This is a consequence of condition (4). We omit the proof.

Let $x\in \mathrm{ob}(C)$. Let $r_x\in \mathrm{ob}(C^\wedge)$ be the subobject of h_x defined by $r_x(y)=\{f\in C(y, x)|f\notin E(C)\}$ for $y\in\mathrm{ob}(C)$. Define a $k[C]$-module T_x by

$$T_x=\mathrm{Coker}(k[r_x] \hookrightarrow k[h_x]).$$

Then $T_x(y)\cong k[C_e(y, x)]$ for $y\in\mathrm{ob}(C)$. The group $\mathrm{Aut}(x)$ acts naturally on T_x on the left.

LEMMA 4.2. pd $T_x<\infty$.

PROOF. Note that the category C_m/x is essentially a partially ordered set. Let $i: C_m/x\longrightarrow C_m$ be the canonical functor $(z\to x)\longmapsto z$ and let k_x be a simple $k[C_m/x]$-module supported on final objects of C_m/x. The restriction functor $i^*: k[C_m]\text{-}\mathcal{Mod}\longrightarrow k[C_m/x]\text{-}\mathcal{Mod}$ has a left adjoint $i_!$ given by

$$i_!(F)(y)=\bigoplus_{y\to x} F(y\to x)$$

for $k[C_m/x]$-modules F and $y\in\mathrm{ob}(C_m)$, where $y\to x$ runs over all objects of C_m/x. We see that

$$\begin{aligned} i_!(k_x)(y)&=k[\mathrm{Aut}(x)] &&\text{if } y=x,\\ &=0 &&\text{if } y\ncong x,\end{aligned}$$

for $y\in\mathrm{ob}(C_m)$. By Lemma 4.1 it follows easily that $j_{m!}i_!(k_x)\cong T_x$. Since both $j_{m!}$ and $i_!$ are exact and preserve projectives, pd $T_x\leq$ pd $k_x<\infty$. This proves the lemma.

Let $K_0(k[C])$ be the Grothendieck group of projective $k[C]$-modules. The Cartan map $c: K_0(k[C])\longrightarrow G_0(k[C])$ is defined by $c[F]=[F]$ for projective $k[C]$-modules F. For each $x\in\mathrm{ob}(C)$ we have also the Cartan map $c_x: K_0(k[\mathrm{Aut}(x)])\longrightarrow G_0(k[\mathrm{Aut}(x)])$ of the algebra $k[\mathrm{Aut}(x)]$. See Serre [3]. Now we prove assertions (i), (ii) of Theorem A in Introduction.

PROPOSITION 4.3. *The Cartan map c is injective and*

$$\#\mathrm{Coker}(c)=\prod_{x\in I}\#\mathrm{Coker}(c_x)$$

where I is a representative system of isomorphism classes of $\mathrm{ob}(C)$.

PROOF. For $x, y\in\mathrm{ob}(C)$ we write $x\leq_m y$ if $C_m(x, y)\neq\emptyset$, and $x\geq_e y$ if

$C_e(x, y) \neq \emptyset$. In the paragraph preceding Proposition 2.6 we defined the sets R_x, R and the simple $k[C]$-modules $S_{x,V}$ for $(x, V) \in R$. Let $S_{x,V}^{\sim}$ be a projective cover of $S_{x,V}$. Then $\{[S_{x,V}^{\sim}]\}_{(x,V)\in R}$, $\{[S_{x,V}]\}_{(x,V)\in R}$ are bases of $K_0(k[C])$, $G_0(k[C])$ respectively.

Fix $(x, V) \in R$ for a moment. Let $V^{\sim}$ be a projective cover of the simple $k[\mathrm{Aut}(x)]$-module V. In the proof of Lemma 4.2 we defined the adjoint functors

$$k[C_m/x]\text{-}\mathcal{Mod} \underset{i^* j_m{}^*}{\overset{j_{m!} i_!}{\rightleftarrows}} k[C]\text{-}\mathcal{Mod}$$

and showed that $T_x \cong j_{m!} i_!(k_x)$. If $(y, W) \in R$, then

$$\begin{aligned}\mathrm{Ext}_C^q(V^{\sim} \otimes_{\mathrm{Aut}(x)} T_x, S_{y,W}) &\cong \mathrm{Hom}_{\mathrm{Aut}(x)}(V^{\sim}, \mathrm{Ext}_C^q(T_x, S_{y,W})) \\ &\cong \mathrm{Hom}_{\mathrm{Aut}(x)}(V^{\sim}, \mathrm{Ext}_{C_m/x}^q(k_x, i^* j_m{}^* S_{y,W}))\end{aligned}$$

for any $q \in \mathbf{N}$. Since $S_{y,W}$ is supported on objects containing y as a direct summand, and since split monomorphisms belong to $M(C)$, we have that $i^* j_m{}^* S_{y,W} = 0$ unless $y \leq_m x$. If $y = x$, then $i^* j_m{}^* S_{y,W} \cong W \otimes k_x$ is injective. Hence $\mathrm{Ext}_C^q(V^{\sim} \otimes_{\mathrm{Aut}(x)} T_x, S_{x,W}) \neq 0$ if and only if $q = 0$ and $W = V$. We know also that $V^{\sim} \otimes_{\mathrm{Aut}(x)} T_x$ has a finite projective resolution by Lemma 4.2. From these facts it follows that $[V^{\sim} \otimes_{\mathrm{Aut}(x)} T_x] - [S_{x,V}^{\sim}]$ is a linear combination of $[S_{y,W}^{\sim}]$ with $y <_m x$ in $G_0(k[C])$. Therefore the classes $[V^{\sim} \otimes_{\mathrm{Aut}(x)} T_x]$, for all $(x, V) \in R$, span $\mathrm{Im}(c)$.

If $(V^{\sim} \otimes_{\mathrm{Aut}(x)} T_x)(y) \neq 0$, then $y \geq_e x$. So we can write

$$[V^{\sim} \otimes_{\mathrm{Aut}(x)} T_x] = \sum_{\substack{(y, W) \in R \\ y \geq_e x}} m_{x,V;y,W} [S_{y,W}]$$

in $G_0(k[C])$ with $m_{x,V;y,W} \in \mathbf{N}$. Evaluating both sides at x, we have

$$[V^{\sim}] = \sum_{W \in R_x} m_{x,V;x,W} [W]$$

in $G_0(k[\mathrm{Aut}(x)])$. Namely, $(m_{x,V;x,W})_{V,W \in R_x}$ is the Cartan matrix of $k[\mathrm{Aut}(x)]$, whose determinant is known to be nonzero [3]. Hence

$$\begin{aligned}&\det(m_{x,V;y,W})_{(x,V),(y,W)\in R} \\ &\quad = \prod_{x \in I} \det(m_{x,V;x,W})_{V,W \in R_x} \\ &\quad = \prod_{x \in I} \#\mathrm{Coker}(c_x).\end{aligned}$$

Thus c is injective and $\#\mathrm{Coker}(c) = \prod_{x \in I} \#\mathrm{Coker}(c_x)$ as required.

LEMMA 4.4. *If F is a $k[C]$-module, then*

$$\mathrm{pd}\, j_e^*(F) \leq \mathrm{pd}\, F \leq \mathrm{pd}\, j_e^*(F) + \dim C_m$$

where $\dim C_m$ *is as defined in Remark* 3.1.

PROOF. By Lemma 4.1, j_e^* is exact and preserves projectives. So $\mathrm{pd}\, j_e^*(F) \leq \mathrm{pd}\, F$. To prove the second innequality, it is enough to show that if $j_e^*(F)$ is projective, then $\mathrm{pd}\, F \leq \dim C_m$. For $(x, V) \in R$, let $S_{x,V}^e$ be a simple $k[C_e]$-module whose value at x is V. Then $\mathrm{Ext}^q(F, j_{e*}(S_{x,V}^e)) \cong \mathrm{Ext}^q(j_e^*(F), S_{x,V}^e) = 0$ for $q > 0$. There is an injection $g: S_{x,V} \longrightarrow j_{e*}(S_{x,V}^e)$ and the composition factors of $\mathrm{Coker}(g)$ consist of $S_{y,W}$ with $(y, W) \in R$ and $x <_m y$. Using induction on $\dim x \backslash C_m$, we see that $\mathrm{Ext}^q(F, S_{x,V}) = 0$ for $q > \dim x \backslash C_m$. Hence, if $q > \dim C_m$, then $\mathrm{Ext}^q(F, S) = 0$ for any simple module S. Thus $\mathrm{pd}\, F \leq \dim C_m$ as required.

REMARK 4.5. It follows from this lemma and Remark 3.1 that $\mathrm{f.dim}\, k[C] \leq \dim C_e + \dim C_m$.

5. The main theorems

In the rest we assume that a finite category C has the unique epi-mono factorization property and satisfies (3.4).

THEOREM 5.1. *The constant* $k[C]$*-module* k_C *has a finite projective dimension.*

PROOF. Let x, Q be as in (3.4). One easily sees that the quotient morphism $x \longrightarrow Q \backslash x$ is an admissible epimorphism and that the bijections $C(Q \backslash x, y) \cong C(x, y)^Q$, for $y \in \mathrm{ob}(C)$, restrict to bijections $C_e(Q \backslash x, y) \cong C_e(x, y)^Q$. Therefore the category C_e also satisfies (3.4). The theorem follows from Proposition 3.6 and Lemma 4.4.

LEMMA 5.2. *If* F, G *are projective* $k[C]$*-modules, then* $F \otimes G$ *has a finite projective dimension.*

PROOF. The following fact is easily proved. If $K: C^{\mathrm{op}} \longrightarrow \{\text{finite sets}\}$ is a functor such that for all x, Q as in (3.4) the maps $K(Q \backslash x) \longrightarrow K(x)^Q$ induced by the quotient morphisms $x \longrightarrow Q \backslash x$ are bijections, then the category C/K also satisfies (3.4). This can be applied when K is a product of representable functors.

To prove the lemma, it is enough to show that $\mathrm{pd}(k[h_x] \otimes k[h_y]) < \infty$ for any $x, y \in \mathrm{ob}(C)$. Put $K = h_x \times h_y \in \mathrm{ob}(C^\wedge)$. By the above observation, we can apply Theorem 5.1 to C/K. Thus the $k[C/K]$-module $k_{C/K}$ has a finite projective dimension. Let $p: C/K \longrightarrow C$ be the canonical functor. The restriction functor $p^*: k[C]\text{-}\mathcal{Mod} \longrightarrow k[C/K]\text{-}\mathcal{Mod}$ has a left adjoint

$p_!$ given by

$$p_!(F)(x)=\bigoplus_{a\in K(x)} F(x, a)$$

for $k[C/K]$-modules F and $x\in \mathrm{ob}(C)$. Since $p_!$ is exact and preserves projectives, $\mathrm{pd}\, p_!(k_{C/K})<\infty$. Since $p_!(k_{C/K})\cong k[K]\cong k[h_x]\otimes k[h_y]$, the conclusion follows.

THEOREM 5.3. *The image of the Cartan map* $c: K_0(k[C])\longrightarrow G_0(k[C])$ *is a subring.*

PROOF. By Theorem 5.1 and Lemma 5.2, $\mathrm{Im}(c)$ contains $1=[k_C]$ and is closed under product.

Let us consider the case where $C=S(G)$ for a finite group G. We can describe the ring structure of $K_0(k[C])$ induced by that of $G_0(k[C])$ through the map c. For any finite group H, let $P(H)$ be the free abelian group on the set of isomorphism classes of indecomposable direct summands of permutation $k[H]$-modules. Tensor product makes $P(H)$ a ring. A finite group H is said to be p-perfect if H has no nontrivial factor p-group. Then there is a ring isomorphism

$$K_0(k[C])\cong\prod_H P(N_G(H)/H)$$

where H runs over representatives for conjugacy classes of p-perfect subgroups of G.

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