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Seminormal composition operators on L^2 spaces induced by matrices

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Abstract

The present paper deals with bounded seminormal composition operators on L^2 spaces induced by invertible linear transformations of the d-dimensional Euclidean space $\boldsymbol{R}^d$. A class of density functions on $\boldsymbol{R}^d$ for which seminormal composition operators can be completely characterized in terms of their symbols is distinguished. The spectrum and some related topics are discussed for particular case of Gaussian density function.

1. Introduction

Operators of the form $C_A f = f\circ A$ acting on certain function spaces are called composition operators (cf. [4], [12]). The present paper deals with a special class of composition operators induced by linear transformations of $\boldsymbol{R}^d$, acting on the function space $L^2(\boldsymbol{R}^d, r(x)dx)$, where r is a positive density function on $\boldsymbol{R}^d$. The most satisfactory description of seminormal composition operators of this form appears when the density function r is given by $r(\cdot)=\phi(\|\cdot\|^2)$, where $\|\cdot\|$ is a Hilbert norm on $\boldsymbol{R}^d$ and ϕ is a continuous function operating on positive definite matrices (i. e.) if $(a_{i,j})$ is a positive definite real $n\mathrm{x}n$ matrix, then the matrix $(\phi(a_{i,j}))$ is also positive definite). By the Schoenberg theorem [15] (see also [1] and [14]) such a function ϕ must have a power series representation with all non-negative Taylor's coefficients at 0. It turns out that subnormality of C_A does not depend on the function ϕ, though it depends on the transformation A. Namely C_A is subnormal if and only if A is normal in $(\boldsymbol{R}^d, \|\cdot\|)$. This fact enables us to describe the spectrum of C_A in particular case when $\phi=\exp$ and $\|\cdot\|$ is the usual norm on $\boldsymbol{C}^d$. Section 5 deals with another class of composition operators induced by restrictions of linear transformations of $\boldsymbol{R}^d$. In this case some new phenomena appears.

Recall that an operator C is said to be: *hyponormal* if $C^*C-CC^*\geq 0$, *subnormal* if it is a restriction of a normal operator, *quasinormal* if C commutes with C^*C. C is called *cohyponormal* (resp. *cosubnormal*, *co-*

quasinormal, *coisometry*) if C^* is *hyponormal* (resp. *subnormal*, *quasinormal*, *isometry*). *Seminormal* operators are those which are either hyponormal or cohyponormal.

2. Seminormal composition operators on $L^2(\boldsymbol{R}^d, \mathscr{B}(\boldsymbol{R}^d), rd\omega_d)$

In all what follows $\boldsymbol{R}^d$, $d \geq 1$, stands for real d-dimensional Euclidean space with norm $|\cdot|$ given by $|x|^2 = \sum_{k=1}^{d} |x_k|^2$, $x = (x_1, \ldots, x_d) \in \boldsymbol{R}^d$. The same symbol $|\cdot|$ will also denote norm of complex d-dimensional Euclidean space $\boldsymbol{C}^d$ as well as moduli in $\boldsymbol{R}$ and $\boldsymbol{C}$. The set of all non-negative integers will be denoted by $\boldsymbol{N}$. The symbol ω_d stands for the d-dimensional Lebesgue measure on $\mathscr{B}(\boldsymbol{R}^d)$, the σ-algebra of Borel subsets of $\boldsymbol{R}^d$.

Let r be a Borel function on $\boldsymbol{R}^d$ such that $r(x) \in (0, \infty)$ for almost every $x \in \boldsymbol{R}^d$. Define the Borel measure μ_r on $\boldsymbol{R}^d$ by $d\mu_r = rd\omega_d$. Then ω_d and μ_r are mutually absolutely continuous σ-finite measures. Suppose we are given an invertible linear transformation A of $\boldsymbol{R}^d$. Then the measure $\mu_r A^{-n}$ defined by $\mu_r A^{-n}(\sigma) = \mu_r(A^{-n}(\sigma))$ is absolutely continuous with respect to μ_r for every $n \geq 0$ and the Radon-Nikodym derivative $\chi_n = d\mu_r A^{-n}/d\mu_r$ is given by

$$(2.1) \qquad \chi_n = |\det A|^{-n} \cdot r \circ A^{-n}/r, \quad n \in \boldsymbol{N},$$

where $\det A$ stands for the determinant of a matrix associated with A. This implies that the operator C_A given by the formula $C_A f = f \circ A$ is well defined in $L^2(r) = L^2(\boldsymbol{R}^d, \mathscr{B}(\boldsymbol{R}^d), \mu_r)$. Call it a *composition operator* induced by a *symbol* A. To indicate that the operator C_A acts in $L^2(r)$, we will write $C_{A,r}$ instead of C_A. One can prove that within the class of bounded operators distinct symbols induce distinct composition operators i.e. $A = B$ if and only if $C_{A,r} = C_{B,r}$. Since $C_{A,r} C_{B,r} = C_{BA,r}$ and $C_{I,r}$ is the identity operator on $L^2(r)$, the mapping sending A into $C_{A,r}$ is a semigroup monomorphism between suitable multiplicative semigroups.

It follows from Theorem 1 in [12] (see also [4]) that $C_{A,r}$ is bounded if and only if $r/r \circ A$ belongs to $L^\infty(\omega_d)$ and in this case

$$(\mathrm{BC}) \qquad \|C_{A,r}\| = |\det A|^{-1/2} \|r/r \circ A\|_\infty^{1/2},$$

where $\|\cdot\|_\infty$ indicates the essential supremum norm in $L^\infty(\omega_d)$. If $C_{A,r}$ is bounded, then $C_{A,r}^*$ acts according to the following formula

$$(\mathrm{AD}) \qquad C_{A,r}^* f = |\det A|^{-1} \cdot f \circ A^{-1} \cdot r \circ A^{-1}/r, \quad f \in L^2(r).$$

It is obvious that the map $U_r : L^2(r) \longrightarrow L^2(1/r)$ defined by $U_r f = fr$, $f \in L^2(r)$, is unitary and it satisfies the following equality

(UE) $\quad U_r|\det A|C^*_{A,r}=C_{A^{-1},1/r}U_r.$

The condition (UE) justifies why some results of the paper are formulated for the operators $C_{A,r}$ and $C_{A,1/r}$, though they are proved only for $C_{A,r}$.

Denote by $\mathscr{H}$ the family of all entire functions ϕ of the form

$$\phi(z)=\sum_{k=0}^{\infty}a_k z^k,\ z\in \boldsymbol{C}, \tag{2.2}$$

such that $a_n\geq 0$ for every $n\in \boldsymbol{N}$ and $a_{k_0}>0$ for some $k_0\geq 1$. Thus $\phi\cdot\psi$, $\phi\circ\psi$, $\alpha\phi+\beta\psi\in\mathscr{H}$, provided $\phi,\psi\in\mathscr{H}$, $\alpha,\beta\geq 0$ and $\alpha+\beta>0$. Moreover if $\phi\in\mathscr{H}$, then

$$\phi(s)<\phi(t),\ 0\leq s<t \tag{2.3}$$

and

$$|\phi(zw)|^2\leq\phi(|z|^2)\phi(|w|^2),\ z,w\in\boldsymbol{C}. \tag{2.4}$$

The latter follows from the Cauchy-Schwarz inequality for a weighted l^2 space with weights a_n, $n\in\boldsymbol{N}$.

In the sequel we need more information about behaviour of members of $\mathscr{H}$ at ∞. Given $\phi\in\mathscr{H}$, we define the function $q_\phi:(0,\infty)\longrightarrow(0,\infty]$ by $q_\phi(\vartheta)=\sup_{t>0}Q_\vartheta(t)$ for $\vartheta>0$, where $Q_\vartheta(t)=\phi(\vartheta t)/\phi(t)$ for $t>0$. The following lemma gives an explicit formula for the function q_ϕ.

LEMMA 2.1. *Let $\phi\in\mathscr{H}$. If 0 is a zero of ϕ of multiplicity $m\in\boldsymbol{N}$ and ∞ is a pole of ϕ of order $n\in\boldsymbol{N}\cup\{\infty\}$, then*[1]

$$q_\phi(\vartheta)=\vartheta^m\max\{1,\vartheta^{n-m}\},\ \vartheta>0, \tag{2.5}$$

where $\vartheta^\infty=\infty$ for $\vartheta>1$, $1^\infty=1$ and $\vartheta^\infty=0$ for $\vartheta\in(0,1)$.

PROOF. Let $\{a_k\}_{k=0}^{\infty}$ represent ϕ via (2.2) with $a_{k_0}>0$ for some $k_0\geq 1$. We split the proof into two steps.

Step 1. *If $\vartheta>1$ and $q_\phi(\vartheta)<\infty$, then ϕ is a polynomial i. e. $n\in\boldsymbol{N}$.*

Indeed the assumption $q_\phi(\vartheta)<\infty$ implies that there exists $M>1$ such that

$$\sum_{k=0}^{\infty}a_k(M-\vartheta^k)t^k\geq 0,\ t>0. \tag{2.6}$$

Since $\vartheta>1$, there exists $j\in\boldsymbol{N}$ such that $M-\vartheta^k\geq 0$ for $k=0,1,\ldots,j$ and $\vartheta^k-M>0$ for $k>j$. It follows from (2.6) that

$$\psi(t)\leq p(t),\ t>0$$

[1]By definition ϕ has a pole of order ∞ at ∞ if ϕ is not a polynomial.

where $\psi(z)=\sum_{k=j+1}^{\infty}a_k(\vartheta^k-M)z^k$ and $p(z)=\sum_{k=0}^{j}a_k(M-\vartheta^k)z^k$, $z\in\boldsymbol{C}$. Thus ψ must be a polynomial of degree not exceeding j. Consequently the degree of ϕ is less or equal to j.

Step 2. *If* $m=0$ *and* $\vartheta>1$, *then the function* Q_ϑ *is strictly increasing.*

To see this take s and t such that $0<s<t$. Then the following equality holds

(2.7) $$\phi(s)\phi(\vartheta t)-\phi(\vartheta s)\phi(t)=\sum_{0\le k<l}a_k a_l(\vartheta^l-\vartheta^k)t^{k+l}((s/t)^k-(s/t)^l).$$

Since $a_0>0$, $a_{k_0}>0$, $s/t<1$ and $\vartheta>1$, Step 2 is a consequence of (2.7).

Suppose that $m=0$ and $n=\infty$. Then, in virtue of Step 1, the equality (2.5) holds for $\vartheta>1$. If $0<\vartheta\le 1$, then (2.3) implies that $Q_\vartheta(t)\le Q_1(t)\le 1$ for $t>0$. Since $\lim_{t\to 0+}Q_\vartheta(t)=1$, (2.5) holds for $0<\vartheta\le 1$.

Assume that $m=0$ and $n<\infty$. If $\vartheta>1$, then Q_ϑ is strictly increasing, because of Step 2. Since $\lim_{t\to 0+}Q_\vartheta(t)=1$ and $\lim_{t\to\infty}Q_\vartheta(t)=\vartheta^n$, (2.5) holds for $\vartheta>1$. Similarly as in the previous paragraph we show that (2.5) holds for $0<\vartheta\le 1$.

If $m>0$, then there exists an entire function $\psi\in\mathscr{H}\cup\boldsymbol{C}$ such that $\phi(z)=z^m\psi(z)$ for $z\in\boldsymbol{C}$, $\psi(0)>0$ and ψ has a pole of order $n-m$ at ∞. Since $q_\phi(\vartheta)=\vartheta^m q_\psi(\vartheta)$ for $\vartheta\in(0,\infty)$, the case $m>0$ can be easily reduced to $m=0$. ■

Let $\|\cdot\|$ be a Hilbert norm on $\boldsymbol{R}^d$. Denote by $\|A\|$ the norm of the operator A in $(\boldsymbol{R}^d,\|\cdot\|)$. It is well known that each Hilbert norm $\|\cdot\|$ on $\boldsymbol{R}^d$ has the form $\|x\|=|Vx|$, $x\in\boldsymbol{R}^d$, where V is some positive invertible operator in $(\boldsymbol{R}^d,|\cdot|)$. Thus the adjoint of A in $(\boldsymbol{R}^d,\|\cdot\|)$ is equal to $V^{-2}A^*V^2$ and consequently A is normal in $(\boldsymbol{R}^d,\|\cdot\|)$ if and only if VAV^{-1} is normal in $(\boldsymbol{R}^d,|\cdot|)$.

The following theorem gives necessary and sufficient conditions for $C_{A,r}$ to be bounded on $L^2(r)$ for $r(\cdot)=\phi(\|\cdot\|^2)$ and $\phi\in\mathscr{H}$.

PROPOSITION 2.2. *Let* $r(\cdot)=\phi(\|\cdot\|^2)$ *with* $\phi\in\mathscr{H}$. *If* ϕ *is a polynomial, then* $C_{A,r}$ *and* $C_{A,1/r}$ *are bounded. If* ϕ *is not a polynomial, then* $C_{A,r}$ (*resp.* $C_{A,1/r}$) *is bounded if and only if* $\|A^{-1}\|\le 1$ (*resp.* $\|A\|\le 1$). *In both cases we have*

$$\|C_{A,r}\|=|\det A|^{-1/2}q_\phi(\|A^{-1}\|)\ (\textit{resp. } \|C_{A,1/r}\|=|\det A|^{-1/2}q_\phi(\|A\|)),$$

where $m\in\boldsymbol{N}$ *is the multiplicity of a zero which* ϕ *has at* 0 *and* $n\in\boldsymbol{N}\cup\{\infty\}$ *is the order of a pole which* ϕ *has at* ∞.

PROOF. It follows from Lemma 2.1 that

$$
\begin{aligned}
&\sup_{x\neq 0} r(x)/r(Ax) = \sup_{x\neq 0}\sup_{t>0}\phi(t\|A^{-1}x\|^2)/\phi(t\|x\|^2) \\
(2.8)\quad &= \sup_{x\neq 0}\sup_{t>0}\phi(t\|A^{-1}x\|^2/\|x\|^2)/\phi(t) = \\
&= \sup_{x\neq 0}(\|A^{-1}x\|/\|x\|)^{2m}\max\{1,(\|A^{-1}x\|/\|x\|)^{2(n-m)}\}.
\end{aligned}
$$

In virtue of (BC), this part of conclusion which concerns $C_{A,r}$ follows from equalities (2.8). The other one concerning $C_{A,1/r}$ is a consequence of (UE). ■

Our next goal is to characterize seminormal composition operators $C_{A,r}$ in terms of their symbols. We will make use of results from papers [19], [6] and [9] in which this question has been settled in more goneral context. Assume that $C_{A,r}$ is bounded. Using Corollary 1 from [9] and Theorem 3 from [6] we get

(HY) *$C_{A,r}$ (resp. $C^*_{A,r}$) is hyponormal if and only if $r^2 \leq r\circ A^{-1}\cdot r\circ A$ a.e. $[\omega_d]$(resp. $r^2 \geq r\circ A^{-1}\cdot r\circ A$ a.e. $[\omega_d]$).*

Consequently (see also [19], Lemma 2)

(NO) *$C_{A,r}$ is normal if and only if $r^2 = r\circ A^{-1}\cdot r\circ A$ a.e. $[\omega_d]$.*

Finally

(UN) *$C_{A,r}$ is unitary if and only if $|\det A| = r\circ A^{-1}/r$ a.e. $[\omega_d]$.*

Notice that within the class of composition operators of the form $C_{A,r}$, coquasinormal and quasinormal operators coincide with normal ones (see [6], Corollary 5 and [19], Lemmas 2 and 3). Similarly coisometries and isometries coincide with unitary operators (use (UE) and Lemma 1 in [19]). It follows from what is in [3] that if $C_{A,r}$ is hyponormal, then all its powers are hyponormal too. On the other hand if the measure μ_r is finite, then $C_{A,r}$ is hyponormal if and only if $C_{A,r}$ is unitary (cf. [6], Lemma 7). So only if μ_r is not finite there is a chance of finding hyponormal but not subonormal operators of the form $C_{A,r}$. At the end of Section 2 we present examples of this sort. Another example of this kind has been found by Campbell and Dibrell [3].

Below we show that cohyponormal operators of the form $C_{A,r}$, where $r(\cdot) = \phi(\|\cdot\|^2)$ and $\phi \in \mathscr{H}$, must be normal. Notice that the measure $\mu_{1/r}$ may be infinite (e.g. when ϕ is a polynomial of degree not exceeding $d/2$), so Proposition 2.3 does not follow from Lemma 7 in [6].

PROPOSITION 2.3. *Let $\|\cdot\|$ be a Hilbert norm on $\boldsymbol{R}^d$ and let $\phi \in \mathscr{H}$. If $r(\cdot) = \phi(\|\cdot\|^2)$ and $C_{A,r}$ is bounded, then*

(i) *$C^*_{A,r}$ is hyponormal if and only if either ϕ is a monomial and tA is* *unitary in* $(\boldsymbol{R}^d, \|\cdot\|)$ *for some* $t>0$ *or* ϕ *is not a monomial and* A *is unitary in* $(\boldsymbol{R}^d, \|\cdot\|)$.

(ii) $C_{A,r}$ *is unitary if and only if* A *is unitary in* $(\boldsymbol{R}^d, \|\cdot\|)$.

If $r(\cdot)=1/\phi(\|\cdot\|^2)$ *and* $C_{A,r}$ *is bounded, then* (i) *and* (ii) *hold replacing* $C_{A,r}$ *by* $C_{A,r}^*$.

PROOF. Concentrate on the case $r(\cdot)=\phi(\|\cdot\|^2)$. We split the proof into a few steps.

Step 1. *If* $C_{A,r}^*$ *is hyponormal, then both* A *and* $C_{A,r}$ *are normal.* Indeed, the condition (HY) implies that

$$\phi(\|x\|^2)^2 \geq \phi(\|A^{-1}x\|^2)\phi(\|Ax\|^2), \quad x\in\boldsymbol{R}^d. \tag{2.9}$$

On the other hand (2.3) and (2.4) lead to the following inequalities

$$\begin{aligned}\phi(\|x\|^2)^2 &= \phi((A^{-1}x, A^*x))^2 \leq \phi(\|A^{-1}x\|\,\|A^*x\|)^2 \\ &\leq \phi(\|A^{-1}x\|^2)\phi(\|A^*x\|^2), \quad x\in\boldsymbol{R}^d.\end{aligned} \tag{2.10}$$

Thus both (2.9) and (2.10) give us

$$\phi(\|Ax\|^2) \leq \phi(\|A^*x\|^2), \quad x\in\boldsymbol{R}^d.$$

Since ϕ has the property (2.3), we have $\|Ax\|^2 \leq \|A^*x\|^2$ for every $x\in\boldsymbol{R}^d$. Thus A^* is hyponormal in $(\boldsymbol{R}^d, \|\cdot\|)$. Consequently A is normal in $(\boldsymbol{R}^d, \|\cdot\|)$ (cf. [5], p. 105). Since $\|Ax\|=\|A^*x\|$ for every $x\in\boldsymbol{R}^d$, the inequalities (2.9) and (2.10) turn to equalities. Thus

$$\phi(\|x\|^2)^2 = \phi(\|A^{-1}x\|^2)\phi(\|Ax\|^2), \quad x\in\boldsymbol{R}^d. \tag{2.11}$$

This and (NO) imply that $C_{A,r}$ is normal.

Step 2. *If* $C_{A,r}$ *is normal, then we have*

$$\frac{d^n}{dz^n}\phi(z\|x\|^2)^2\Big|_{z=0} = \frac{d^n}{dz^n}(\phi(z\|A^{-1}x\|^2)\phi(z\|Ax\|^2))\Big|_{z=0}, \quad x\in\boldsymbol{R}^d,\ n\in\boldsymbol{N}. \tag{2.12}$$

This follows from (NO) and the identity principle for holomorphic functions.

Step 3. *If* $C_{A,r}$ *is normal and* $\phi(z)=\sum_{k=j}^{\infty}a_k z^k$ *with* $a_j>0$ *for some* $j\geq 1$, *then* tA *is unitary in* $(\boldsymbol{R}^d, \|\cdot\|)$ *for some* $t>0$.

Indeed, it follows from Step 2 that (2.12) holds for $n=2j$. Thus

$$\|x\|^2 = \|A^{-1}x\|\,\|Ax\|, \quad x\in\boldsymbol{R}^d. \tag{2.13}$$

On the other hand

$$\|x\|^2 = (A^{-1}x, A^*x) \leq \|A^{-1}x\|\,\|A^*x\|, \quad x\in\boldsymbol{R}^d. \tag{2.14}$$

Since — by Step 1 — A is normal in $(\boldsymbol{R}^d, \|\cdot\|)$, one can deduce from (2.13) and (2.14) that the Cauchy-Schwarz inequality (2.14) turns to equality. This is possible if and only if for every $x \neq 0$ there exsits $z_x \in \boldsymbol{C}$ such that $A^*x = z_x A^{-1}x$, or equivalently if

$$(2.15)\qquad AA^*x = z_x x,\ x \neq 0.$$

This means that z_x is a nonzero eigenvalue of the positive operator AA^*. Since eigenspaces corresponding to distinct eigenvalues of AA^* are pairwise orthogonal, the equality (2.15) may hold if and only if $x \longrightarrow z_x$ is constant, say $z_x = t^{-2}$ for some $t > 0$. Combining this fact with (2.15) we get tA is a coisometry on $(\boldsymbol{R}^d, \|\cdot\|)$. Thus tA is unitary.

Step 4. *If $C_{A,r}$ is normal and $\phi(z) = a_j z^j + \sum_{k=m}^{\infty} a_k z^k$ with $a_j, a_m > 0$ for some $m > j \geq 1$, then A is unitary in $(\boldsymbol{R}^d, \|\cdot\|)$.*

Indeed, it follows from Step 2 that (2.12) holds for $n = j + m$. Thus

$$(2.16)\qquad 2\|x\|^{2(j+m)} = \|A^{-1}x\|^{2j}\|Ax\|^{2m} + \|A^{-1}x\|^{2m}\|Ax\|^{2j},\ x \in \boldsymbol{R}^d.$$

On the other hand Step 3 implies that for some $t > 0$ the operator $t^{-1}A$ is unitary in $(\boldsymbol{R}^d, \|\cdot\|)$. Thus (2.16) turns to the equality $(t^{2(m-j)} - 1)^2 = 0$, which holds only if $t = 1$. Consequently A is unitary.

Step 5. *If $C_{A,r}$ is normal and $\phi(z) = a_0 + \sum_{k=j}^{\infty} a_k z^k$ with $a_0, a_j > 0$ for some $j \geq 1$, then A is unitary in $(\boldsymbol{R}^d, \|\cdot\|)$.*

Indeed, it follows from Step 2 that (2.12) holds for $n = j$. Thus

$$(2.17)\qquad 2\|x\|^{2j} = \|A^{-1}x\|^{2j} + \|Ax\|^{2j},\ x \in \boldsymbol{R}^d.$$

In virtue of Step 1, A is normal in $(\boldsymbol{R}^d, \|\cdot\|)$. Thus

$$(2.18)\qquad \|x\|^{2j} = (A^{-1}x, A^*x)^j \leq \|A^{-1}x\|^j\|A^*x\|^j = \|A^{-1}x\|^j\|Ax\|^j,\ x \in \boldsymbol{R}^d.$$

Combining the inequalities (2.17) and (2.18) we get $(\|A^{-1}x\|^j - \|Ax\|^j)^2 \leq 0$ for $x \in \boldsymbol{R}^d$. Therefore $\|A^{-1}x\| = \|Ax\|$ for $x \in \boldsymbol{R}^d$. This and (2.17) implies that A is unitary.

Step 6. *If $C_{A,r}$ is unitary and $\phi(z) = a_j z^j$ with $a_j > 0$ for some $j \geq 1$, then A is unitary in $(\boldsymbol{R}^d, \|\cdot\|)$.*

Indeed, it follows from (UN) that the operator $C_{A,r}$ is unitary if and only if $\|A^{-1}x\|^2 = \varepsilon\|x\|^2$ for $x \in \boldsymbol{R}^d$, where $\varepsilon = |\det A|^{1/j}$. The latter means that $U = (\varepsilon^{1/2}A)^{-1}$ is a unitary operator in $(\boldsymbol{R}^d, \|\cdot\|)$. Since $|\det U| = 1$, we get $\varepsilon^j = |\det A| = \varepsilon^{-1/2}$. Therefore $\varepsilon = 1$ and A is unitary.

The condition (i) follows from Steps 1, 3, 4 and 5. The condition (ii) is a consequence of Steps 4, 5 and 6. ■

[2]Characterization (SU) was presented, among other facts from the paper, at the "18^{th} Seminar in Functional Analysis", held at Stara Lubovna, Czechoslovakia, June **2**-**10**, 1987.

The question of subnormality of bounded composition operators with measurable symbols has been investigated by Lambert in [10]. Using a version of Corollary 4 in [10] and (2.1) we get[2]

(SU) *$C_{A,r}$ (resp. $C^*_{A,r}$) is subnormal if and only if for almost every $x\in \boldsymbol{R}^d$, the sequence $\{r(A^{-n}(x))\}_{n=0}^{\infty}$ (resp. $\{1/r(A^n(x))\}_{n=0}^{\infty}$) is a Hamburger moment sequence*[3].

Since the operator $C_{A,r}$ is one-to-one, we can improve the characterization (SU) as follows

PROPOSITION 2.4. *Assume $C_{A,r}$ is bounded. If $C_{A,r}$ (resp. $C^*_{A,r}$) is subnormal, then for almost every $x\in \boldsymbol{R}^d$, the sequence $\{r(A^{-n}(x))\}_{n=-\infty}^{\infty}$ (resp. $\{1/r(A^n(x))\}_{n=-\infty}^{\infty}$) is a two-sided Stieltjes moment sequence. Moreover if σ is an open subset of $\boldsymbol{R}^d$ such that r is continuous on σ, $\omega_d(\boldsymbol{R}^d\setminus\sigma)=0$ and $A(\sigma)=\sigma$, then the above holds replacing "for almost every $x\in \boldsymbol{R}^d$" by "for every $x\in\sigma$".*

PROOF. Suppose that $C_{A,r}$ is subnormal and r is continuous on σ which satisfies the assumptions of Proposition 2.4 (if r is not continuous, then the proof is based on (SU)). Then all functions χ_n, $n\geq 0$, are continuous on σ. Using Theorem 2 in [10] we conclude that the sequence $\{r(A^{-n}(y))\}_{n=0}^{\infty}$ is a Stieltjes moment sequence for every $y\in\sigma$. Take $x\in\sigma$. Since $A^k(x)\in\sigma$ for $k\in\boldsymbol{Z}$, we get that for each $k\in\boldsymbol{Z}$ the sequence $\{r(A^{-(n+k)}(x))\}_{n=0}^{\infty}$ is a Stieltjes moment sequence. Thus the conclusion can be derived from the following characterization of two-sided Stieltjes moment sequences (see [1], p. 202)

a sequence $\{a_n\}_{n=-\infty}^{\infty}$ of real numbers is a two-sided Stieltjes moment sequence if and only if $\sum a_{n+m}z_n\bar{z}_m\geq 0$ and $\sum a_{n+m+1}z_n\bar{z}_m\geq 0$ for every sequence $\{z_n\}_{n=-\infty}^{\infty}\subset \boldsymbol{C}$ vanishing off a finite set. ■

Now we can characterize bounded subnormal composition operators of the form $C_{A,r}$, where $r(\cdot)=\phi(\|\cdot\|^2)$ and $\phi\in\mathscr{H}$. To do this we need the following characterization of normal algebraic contractions (cf. [17], Theorem 6.3)

(*) *If N is an algebraic contraction on a Hilbert space H and $\phi\in\mathscr{H}$, then N is normal if and only if $\{\phi(\|N^n f\|^2)\}_{n=0}^{\infty}$ is a Hamburger moment sequence for every $f\in H$.*

Then our result is

THEOREM 2.5. *Let $\|\cdot\|$ be a Hilbert norm on $\boldsymbol{R}^d$ and let $\phi\in\mathscr{H}$. If*

[3]All facts concerning moment sequences needed in this paper can be found in [1].

$r(\cdot)=\phi(\|\cdot\|^2)$ and $C_{A,r}$ (resp. $C_{A,1/r}$) is bounded, then $C_{A,r}$ (resp. $C^*_{A,1/r}$) is subnormal if and only if A is normal in $(\boldsymbol{R}^d, \|\cdot\|)$.

PROOF. Assume $C_{A,r}$ is bounded. Suppose that ϕ is not a polynomial. Then it follows from Proposition 2.2 that $\|A^{-1}\|\leq 1$. On the other hand, by the Cayley-Hamilton theorem, the operator A^{-1} is algebraic. In virtue of Proposition 2.4, the operator $C_{A,r}$ is subnormal if and only if $\{\phi(\|A^{-n}x\|^2)\}_{n=0}^{\infty}$ is a Hamburger moment sequence for every $x\in\boldsymbol{R}^d$. Combining this fact with $(*)$ we get that A is normal in $(\boldsymbol{R}^d, \|\cdot\|)$.

Assume now that ϕ is a polynomial of degree j with its leading coefficient $a_j>0$. If A is normal in $(\boldsymbol{R}^d, \|\cdot\|)$, then subnormality of $C_{A,r}$ follows from the Schur theorem (cf. [1], Theorem 3.1.12) and the characterization (SU). Conversely if $C_{A,r}$ is subnormal, then due to Proposition 2.4, $\{\phi(t\|A^{-n}x\|^2)\}_{n=0}^{\infty}$ is a Hamburger moment sequence for all $x\in\boldsymbol{R}^d$ and $t>0$. Since

$$\|A^{-n}x\|^{2j}=\lim_{t\to\infty}\phi(t\|A^{-n}x\|^2)(a_jt^j)^{-1},\quad n\in\boldsymbol{N},\ x\in\boldsymbol{R}^d,$$

the sequence $\{\|A^{-n}x\|^{2j}\}_{n=0}^{\infty}$ is a Hamburger moment sequence for every $x\in\boldsymbol{R}^d$. But A^{-1} is algebraic, so Proposition 6.2 in [17] guarantees normality of A in $(\boldsymbol{R}^d, \|\cdot\|)$. ■

Notice now that for some functions ϕ from $\mathscr{H}$ there are hyponormal operators $C_{A,r}$ with $r(\cdot)=\phi(\|\cdot\|^2)$ that are not subnormal.

EXAMPLE 2.6. Put $r(x)=\exp(|x|^2)$ for $x\in\boldsymbol{R}^d$. Let A be the linear transformation of $\boldsymbol{R}^d$, $d>1$, defined by

$$A(x)=(c_1x_2, c_2x_3, \ldots, c_{d-1}x_d, c_dx_1),\quad x=(x_1, \ldots, x_d)\in\boldsymbol{R}^d,$$

where $c_1, \ldots, c_d$ are nonzero real numbers such that

(2.19) $\quad |c_j|\geq 2^{1/2},\ j=1, \ldots, d,$

and

(2.20) $\quad |c_k|\neq|c_l|$ for some distinct k and l.

It is easy to see that A is invertible and A^{-1} is a contraction. Moreover, by (2.20), A is not normal in $(\boldsymbol{R}^d, |\cdot|)$. Thus, in virtue of Proposition 2.2 and Theorem 2.5, the operator $C=C_{A,r}$ is bounded and not subnormal. We show that C is hyponormal. Indeed, if $x=(x_1, \ldots, x_d)\in\boldsymbol{R}^d$, then (2.19) implies

$$|Ax|^2+|A^{-1}x|^2\geq|c_d|^2|x_1|^2+|c_1|^2|x_2|^2+\cdots+|c_{d-1}|^2|x_d|^2\geq 2|x|^2,$$

for every $x\in \boldsymbol{R}^d$, or equivalently

$$r^2\leq r\circ A\cdot r\circ A^{-1}.$$

It follows from (HY) that C is hyponormal.

Concluding the whole section, we can say that within the class of seminormal composition operators of the form $C_{A,r}$, where $r(\cdot)=\phi(\|\cdot\|^2)$ and $\phi\in\mathscr{H}$, only hyponormal ones can not be completely characterized in terms of their symbols. In other words the fact that $C_{A,r}$ is hyponormal depends essentially on shape of ϕ, which is not the case for subnormal, normal, unitary and cohyponormal operators. Indeed if $\phi(z)=z\exp(z)$, $\|x\|=|x|$ for $x\in\boldsymbol{R}^d$ and A is the same as in Example 2.6, then for $c_d=2^{1/2}$ and sufficiently large c_1 the operator $C_{A,r}$ is not hyponormal, though it is for $\phi=\exp$.

3. Seminormal composition operators on L^2 $(\boldsymbol{C}^d, \mathscr{B}(\boldsymbol{C}^d), r d\omega_{2d})$

We begin with an observation that all results of Section 2 remain true for composition operators induced by invertible $\boldsymbol{C}$-linear transformations of $\boldsymbol{C}^d$. We have only to replace the quantity $|\det A|$ by a new one $|\det A|^2$, where in this case $\det A$ stands for the determinant of a complex matrix associated with A. The reason is that for any pair of real $d\times d$ matrices C and D we have

$$|\det(C+iD)|^2=\left|\det\begin{bmatrix} C & -D \\ D & C\end{bmatrix}\right|.$$

The complex case has some advantage over the real one. To see this consider a bounded composition operator $C_{A,r}$ with $r(\cdot)=1/\phi(\|\cdot\|^2)$, where $\|\cdot\|$ is a Hilbert norm on $\boldsymbol{C}^d$, $\phi\in\mathscr{H}$ and A is an invertible $\boldsymbol{C}$-linear transformation of $\boldsymbol{C}^d$. Then $C_{A,r}^*$ is subnormal if and only if A is normal in $(\boldsymbol{C}^d, \|\cdot\|)$. In this case there exists a unitary operator U in $(\boldsymbol{C}^d, \|\cdot\|)$ such that the operator $D=U^{-1}AU$ is diagonal. In consequence $C_{U,r}$ is unitary and, by (AD), $C_{D,r}=C_{U,r}C_{A,r}C_{U,r}^*$. This means that a subnormal $C_{A,r}^*$ is unitarily equivalent to $C_{D,r}^*$ with some diagonal normal operator D. The same is true for subnormal composition operators of the form $C_{A,1/r}$.

Throughout the remaining part of Section 3 the symbol r (resp. ρ) stands for a density function on $\boldsymbol{C}^d$ (resp. $\boldsymbol{C}$) defined by $r(z)=1/\exp(|z|^2)$, $z\in\boldsymbol{C}^d$ (resp. $\rho(z)=1/\exp(|z|^2)$, $z\in\boldsymbol{C}$). In this particular case $L^2(r)=L^2(\rho)\otimes\cdots\otimes L^2(\rho)$. So if the diagonal operator D from the previous paragraph is chosen to be of the form

$$D(z)=(a_1z_1, \dots, a_dz_d),\ \ z=(z_1, \dots, z_d)\in\boldsymbol{C}^d,$$

then $C_{D,r}=C_{a_1,\rho}\otimes\cdots\otimes C_{a_d,\rho}$, where a is idientified with the transformation $\boldsymbol{C}\ni z\longrightarrow az\in\boldsymbol{C}$. Summing up we have proved the following

THEOREM 3.1. *If $C_{A,r}$ is bounded, then the following conditions are equivalent*
(i) *$C^*_{A,r}$ is subnormal,*
(ii) *A is normal in $(\boldsymbol{C}^d, |\cdot|)$,*
(iii) *there are $a_1,\dots,a_d\in\boldsymbol{C}$ such that $C_{A,r}$ is unitarily equivalent to $C_{a_1,\rho}\otimes\cdots\otimes C_{a_d,\rho}$ and $0<|a_j|\leq 1$ for $1\leq j\leq d$.*
The sequence $a_1,\dots,a_d$ appearing in (iii) *can be always chosen to be composed of the eigenvalues of A listed in an order taking account of their multiplicities.*

Theorem 3.1 can be used to describe the spectrum of cosubnormal composition operators of the form $C_{A,r}$ with r as above. In the sequel $\sigma_p(W)$, $\sigma_r(W)$ and $\sigma_c(W)$ stand for the point, the residual and the continuous parts of the spectrum $\sigma(W)$ of an operator W.

To begin with consider the case $d=1$. The following result is an extension of Theorem 5.0 from [11].

PROPOSITION 3.2. *Let $a\in\boldsymbol{C}$ be such that $C=C_{a,\rho}$ is bounded.*
(i) *If $0<|a|<1$, then $\sigma(C)=\{w\in\boldsymbol{C}:|w|\leq|a|^{-1}\}$, $\sigma_p(C)=\sigma_r(C^*)=\{w\in\boldsymbol{C}:0<|w|<|a|^{-1}\}$, $\sigma_r(C)=\sigma_p(C^*)=\emptyset$ and $\sigma_c(C)=\sigma_c(C^*)=\{w\in\boldsymbol{C}:|w|=|a|^{-1}\}\cup\{0\}$.*
(ii) *If there exists $n\geq 1$ such that $a^n=1$ and $a^{n-1}\neq 1$, then $\sigma(C)=\sigma_p(C)=\sigma_p(C^*)=\{1,a,\dots,a^{n-1}\}$.*
(iii) *If $|a|=1$ and $a^n\neq 1$ for every $n\geq 1$, then $\sigma(C)=\{w\in\boldsymbol{C}:|w|=1\}$, $\sigma_p(C^*)=\sigma_p(C)=\{a^n:n\in\boldsymbol{Z}\}$, $\sigma_c(C)=\sigma_c(C^*)=\sigma(C)\setminus\sigma_p(C)$ and $\sigma_r(C)=\sigma_r(C^*)=\emptyset$.*

PROOF. Part (i) of Proposition 3.2 can be proved in the same way as Theorem 5.0 in [11]. We mention only that the eigenfunctions f_z appearing in the proof of this theorem have to be replaced by the following ones

$$f_z(w)=\exp(z\ln|w|),\quad w\in\boldsymbol{C}\setminus\{0\},$$

where z is any complex number with $\operatorname{Re}z>-1$.

Suppose that $a^n=1$ and $a^{n-1}\neq 1$. Then C^n is the identity operator on $L^2(\rho)$. It follows from the spectral mapping theorem that $\sigma(C)^n=\sigma(C^n)=\{1\}$. Consequently $\sigma(C)\subset\{1,a,\dots,a^{n-1}\}$. But $Cf_k=a^kf_k$ for $k\in\boldsymbol{N}$, where $f_k(z)=z^k$, $z\in\boldsymbol{C}$, so $\{1,a,\dots,a^{n-1}\}\subset\sigma_p(C)$.

Assume that $|a|=1$ and $a^n\neq 1$ for every $n\geq 1$. Then C is unitary and consequently $\sigma(C)\subset\{z\in\boldsymbol{C}:|z|=1\}$ and $\sigma_r(C)=\sigma_r(C^*)=\emptyset$. Notice that

$\{a^n : n\in \boldsymbol{N}\}\subset\sigma_p(C)$. Indeed, if $k, l\geq 0$ and the function $h_{k,l}$ is defined by $h_{k,l}(z)=z^k\bar{z}^l$, $z\in \boldsymbol{C}$, then $h_{k,l}\in L^2(\rho)$ and $Ch_{k,l}=a^{k-l}h_{k,l}$. Since $a^n\neq 1$ for every $n\geq 1$, the set $\{a^n : n\in \boldsymbol{Z}\}$ is dense in the unit circle. Thus $\sigma(C)=\{z\in \boldsymbol{C} : |z|=1\}$.

Now we show that

$$(3.1)\qquad (\boldsymbol{C}\backslash\{a^n : n\in \boldsymbol{Z}\})\cap\sigma_p(C^*)=\emptyset.$$

Indeed, if $w\in \boldsymbol{C}\backslash\{a^n : n\in \boldsymbol{Z}\}$, then for $k, l\geq 0$ we have $(w-C)h_{k,l}=(w-a^{k-l})h_{k,l}$. Since the set $\{h_{k,l} : k, l\geq 0\}$ is total in $L^2(\rho)$, the range of $w-C$ is dense in $L^2(\rho)$. Thus $\bar{w}\in \boldsymbol{C}\backslash\sigma_p(C^*)$, which proves (3.1). It follows from (3.1) and Proposition I. 3.1 in [18] that $(\boldsymbol{C}\backslash\{a^n : n\in \boldsymbol{Z}\})\cap\sigma_p(C)=\emptyset$. This completes the proof. ■

Now we can turn back to the case $d>1$.

PROPOSITION 3.3. *If $C=C_{A,r}$ is bounded, cosubnormal and not unitary, then*

$$\sigma\{C)=\{z\in \boldsymbol{C}^d : |z|\leq|\det A|^{-1}\}$$

and

$$\{z\in \boldsymbol{C}^d : 0<|z|<|\det A|^{-1}\}\subset\sigma_p(C).$$

PROOF. Since C is cosubnormal and not unitary, the transformation A is normal and not unitary (use Proposition 2.3 and Theorem 2.5). This means that A has at least one eigenvalue a such that $0<|a|<1$. By Theorem 3.1, the operator C is unitarily equivalent to $C_{a_1,\rho}\otimes\cdots\otimes C_{a_d,\rho}$ with $a_1=a$. Thus, in virtue of Proposition 3.2, $\{z\in \boldsymbol{C}^d : 0<|z|<|\det A|^{-1})\subset\sigma_p(C)$ and consequently $\sigma(C)=(z\in \boldsymbol{C}^d : |z|\leq|\det A|^{-1}\}$ (see also [2]).

4. Related operators

In this section we draw a few consequences from Theorem 2.5. Two of them are inspired by the paper of Mlak [11]. The first one shows that composition operators induced by some linear transformations of $\boldsymbol{R}^\infty$ are cosubnormal. Denote by ν the Gaussian measure on $\boldsymbol{R}$ with density function $\frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}|\cdot|^2)$. Let ν_d (resp. ν_∞) stand for the tensor product of d (resp. countable many) copies of ν.

THEOREM 4.1. *Let $\boldsymbol{a}=\{a_k\}_{k=1}^\infty$ be a sequence of real numbers such that $0<|a_k|\leq 1$ for $k\geq 1$ and $c=\sup_{n\geq 1}\prod_{k=1}^n|a_k|^{-1/2}<\infty$. Then the formula*

$$(C_{\boldsymbol{a}}f)(x)=f(a_1x_1, a_2x_2,\ldots),\ x=(x_1, x_2, \ldots)\in \boldsymbol{R}^\infty,$$

determines a well defined bounded cosubnormal operator on $L^2(\nu_\infty)$. Moreover if $|a_k|<1$ for some $k\geq 1$, then $\sigma(C_a)=\{z\in \boldsymbol{C}:|z|\leq c\}$.

PROOF. It follows from Theorem 3.1 in [11] that the operator C_a is well defined and bounded on $L^2(\nu_\infty)$. To prove that C_a^* is subnormal notice first that functions of the form

$$f(x)=\tilde{f}(x_1, x_2, \dots, x_d),\ x=(x_1, x_2, \dots)\in \boldsymbol{R}^\infty, \tag{4.1}$$

where $\tilde{f}\in L^2(\nu_d)$ and $d\geq 1$, are dense in $L^2(\nu_\infty)$ (cf. [16]). Thus, in virtue of the Lambert characterization of subnormal operators (cf. [8], Theorem 3.1), it is enough to show that the sequence $\{\|C_a^{*n}f\|^2\}_{n=0}^\infty$ is a Stieltjes moment sequence for each function f of the form (4.1). Take such a function f. Then, using Proposition 3.0 from [11], one can check that

$$(C_a^*f)(x)=(C_{a_1}^*\otimes\cdots\otimes C_{a_d}^*\tilde{f})(x_1, \dots, x_d),\ x=(x_1, x_2, \dots)\in \boldsymbol{R}^\infty,$$

where C_a is a composition operator on $L^2(\nu)$ induced by the transformation $\boldsymbol{R}\ni x\longrightarrow ax\in \boldsymbol{R}$. Thus

$$\|C_a^{*n}f\|^2=\|C_{a_1}^{*n}\otimes\cdots\otimes C_{a_d}^{*n}\tilde{f}\|^2,\ n\in \boldsymbol{N}. \tag{4.2}$$

By Theorem 2.5, the operators $C_{a_j}^*$, $j=1, \dots, d$, are subnormal in $L^2(\nu)$. Consequently $C_{a_1}^*\otimes\cdots\otimes C_{a_d}^*$ is subnormal in $L^2(\nu_d)$. This and (4.2) imply that $\{\|C_a^{*n}f\|^2\}_{n=0}^\infty$ is a Stieltjes moment sequence, which completes the proof of subnormality of C_a^*. The formula $\sigma(C_a)=\{z\in \boldsymbol{C}:|z|\leq c\}$ has been proved by Janas in [7]. ■

In [11] Mlak has proved that a composition operator C_a $(0<a<1)$ acting on $L^2(\nu)$ is unitarily equivalent to an integral operator R_a on $L^2(\omega_1)$ acting according to the following formula

$$(R_af)(x)=((1-a^2)\Pi)^{-1/2}\int_{\boldsymbol{R}}f(u)\exp((x-au)^2(1-a^2)^{-1})\,du,$$
$$f\in L^2(\omega_1).$$

It follows from Theorem 2.5 that R_a is a cosubnormal integral operator.

Consider a semigroup $\{T_a: a\geq 0\}$ of bounded operators on $L^2(\nu)$ defined by $T_a=C_{\exp(-a)}^*$, $a\geq 0$. It is easy to see that $\{T_a: a\geq 0\}$ is a continuous semigroup of subnormal operators. Thus, by the Nussbaum theorem (cf. [13], Proposition 3), its infinitesimal generator $x\frac{d}{dx}+(1-x^2)$ is subnormal too. Full details and proofs of these and related facts will appear elsewhere.

5. Generalized composition operators

Let $\|\cdot\|$ be a Hilbert norm on $\boldsymbol{R}^d$ and let X be the open unit ball in $(\boldsymbol{R}^d, \|\cdot\|)$. Given a Borel function r on X such that $r(x)\in(0,\infty)$ for almost every $x\in X$, we define the σ-finite Borel measure μ_r on X by $d\mu_r = r d\omega_d$. Let A be an invertible linear transformation of $\boldsymbol{R}^d$ such that $AX\subset X$, or equivalently that $\|A\|\leq 1$. Denote by T the restriction of A to X. Since the measure $\mu_r T^{-n}$ is absolutely continuous with respect to μ_r and

$$\varkappa_n = d\mu_r T^{-n}/d\mu_r = |\det A|^{-n}\cdot\chi_{T^n X}\cdot r\circ T^{-n}/r,\quad n\in\boldsymbol{N},$$

where χ_σ stands for the indicator function of the set σ, the transformation T induces the composition operator $C_{T,r}$ in $L^2_X(r) = L^2(X, \mathscr{B}(X), \mu_r)$ via $C_{T,r}f = f\circ T$. The operator $C_{T,r}$ is bounded if and only if $\varkappa_1\in L^\infty(X, \omega_d)$ and in this case $\|C_{T,r}\|^2 = \|\varkappa_1\|_\infty$ (cf. [12]).

Denote by $\mathscr{H}(\varDelta)$ the class of all holomorphic functions ϕ on the open unit disc $\varDelta$ such that $\frac{d^n\phi}{dz^n}(0)\geq 0$ for every $n\geq 0$ and $\frac{d^j\phi}{dz^j}(0)>0$ for at least one $j\geq 1$. The class $\mathscr{H}(\varDelta)$ shares the properties (2.3) and (2.4) on $\varDelta$ with the class $\mathscr{H}$. Moreover if the function ϕ has a power series representation $\phi(z) = \sum_{k=0}^\infty a_k z^k$, $z\in\varDelta$, then

$$\phi(1-) = \lim_{t\to 1-}\phi(t) = \sum_{k=0}^\infty a_k.$$

Now we focus our interest on the question of boundedness of $C_{T,r}$ in two special cases.

PROPOSITION 5.1. *Assume that $\phi\in\mathscr{H}(\varDelta)$ has a zero of multiplicity $m\geq 0$ at 0. If $r(\cdot) = 1/\phi(\|\cdot\|^2)$, then $C_{T,r}$ is bounded and $\|C_{T,r}\| = |\det A|^{-1/2}\|A\|^m$. If $r(\cdot) = \phi(\|\cdot\|^2)$ and A is not unitary in $(\boldsymbol{R}^d, \|\cdot\|)$, then $C_{T,r}$ is bounded if and only if $\phi(1-)<\infty$. In this case $\|C_{T,r}\| = |\det A|^{-1/2}(\phi(1-)/\phi(\|A^{-1}\|^{-2}))^{1/2}$. If $r(\cdot) = \phi(\|\cdot\|^2)$ and A is unitary in $(\boldsymbol{R}^d, \|\cdot\|)$, then $C_{T,r}$ is bounded and $\|C_{T,r}\| = |\det A|^{-1/2}$.*

PROOF. Since $\|C_{T,r}\|^2 = \|\varkappa_1\|_\infty$, it is enough to determine the quantity $|\det A|\|\varkappa_1\|_\infty$.

Using the equality (2.7) one can prove that if $0<\vartheta<1$, then the function $t\longrightarrow\phi(\vartheta t)/\phi(t)$ is decreasing on the open interval (0, 1). This in turn implies that

(5.1) $$\sup_{0<t<1}\phi(\vartheta t)/\phi(t) = \lim_{t\to 0+}\phi(\vartheta t)/\phi(t) = \vartheta^m$$

and

(5.2) $\qquad \sup_{0<t<1}\phi(t)/\phi(\vartheta t)=\lim_{t\to 1-}\phi(t)/\phi(\vartheta t)=\phi(1-)/\phi(\vartheta),$

for $0<\vartheta<1$.

Assume that $r(\cdot)=1/\phi(\|\cdot\|^2)$. Then, by (5.1), we have

$$\begin{aligned}|\det A|\ \|\chi_1\|_\infty &= \sup_{0<\|x\|<1}\phi(\|Ax\|^2)/\phi(\|x\|^2)\\ &= \sup_{0<\|x\|<1}\sup_{0<t<1}\phi(t\|Ax\|^2/\|x\|^2)/\phi(t)\\ &= \sup_{0<\|x\|<1}(\|Ax\|/\|x\|)^{2m}=\|A\|^{2m}<\infty,\end{aligned}$$

which proves boundedness of $C_{T,r}$ and shows $\|C_{T,r}\|=|\det A|^{-1/2}\|A\|^m$.

Suppose now that $r(\cdot)=\phi(\|\cdot\|^2)$ and A is not unitary in $(\boldsymbol{R}^d, \|\cdot\|)$. Then the set $\{x\in X\setminus\{0\}: \|Ax\|/\|x\|<1\}$ is nonempty and, by (5.2), the following equalities hold

$$\begin{aligned}|\det A|\ \|\chi_1\|_\infty &= \sup_{0<\|x\|<1}\ \phi(\|x\|^2)/\phi(\|Ax\|^2)\\ &= \sup_{0<\|x\|<1}\ \sup_{0<t<1}\phi(t)/\phi(t\|Ax\|^2/\|x\|^2)\\ &= \sup\{\phi(1-)/\phi(\|Ax\|^2/\|x\|^2): 0<\|x\|<1 \text{ and } \|Ax\|/\|x\|<1\}\\ &= \phi(1-)/\phi(\inf_{0<\|x\|<1}\|Ax\|^2/\|x\|^2)=\phi(1-)/\phi(\|A^{-1}\|^{-2}),\end{aligned}$$

so $C_{T,r}$ is bounded if and only if $\phi(1-)<\infty$. If A is unitary in $(\boldsymbol{R}^d, \|\cdot\|)$, then $\chi_1\in L^\infty(X, \omega_d)$ and consequently the operator $C_{T,r}$ is bounded whatever $\phi(1-)$ would be. ■

The next proposition shows that in most cases bounded hyponormal composition operators of the form $C_{T,r}$ must be unitary. This fact does not follow from Lemma 7 in [6], because the measure μ_r may be infinite.

PROPOSITION 5.2. *If $C_{T,r}$ is a bounded hyponormal operator, then A is unitary in $(\boldsymbol{R}^d, \|\cdot\|)$. If $\phi:[0,\infty)\longrightarrow(0,\infty)$ is a Borel function, $r(\cdot)=\phi(\|\cdot\|)$ and A is unitary in $(\boldsymbol{R}^d, \|\cdot\|)$, then $C_{T,r}$ is unitary.*

PROOF. Assume that T is hyponormal. Then (cf. [6], Corollary 11) $\chi_1\geq\chi_1\circ T$ a.e., or equivalently $0\leq r^2\leq\chi_{TX}\cdot r\circ T\cdot r\circ T^{-1}$ a.e.. This means that $r=0$ a.e. on the set $X\setminus TX$. Suppose that $X\neq TX$. Since TX is an open convex subset of the unit ball X, the set $X\setminus TX$ has nonempty interior. Consequently $\omega_d(X\setminus TX)>0$, which contradicts our assumption $r>0$ a.e.. Thus $X=TX$, or equivalently A is unitary in $(\boldsymbol{R}^d, \|\cdot\|)$. The other part of the conclusion is obvious. ■

Now we determine conditions for the adjoint of $C_{T,r}$ to be hyponormal, quasinormal and isometry. It follows from Theorem 3 in [6] that $C_{T,r}^*$ is hyponormal if and only if

(5.3) $r^2 \geq \chi_{TX} \cdot r \circ T \cdot r \circ T^{-1}$ a.e..

$C^*_{T,r}$ is quasinormal if and only if (cf. [6], Theorem 4)

(5.4) $r^2 = r \circ T \cdot r \circ T^{-1}$ a.e. on the set TX.

Finally $C^*_{T,r}$ is isometry if and only if

(5.5) $r / r \circ T = |\det A|$ a.e..

All these facts permit us to prove the following

PROPOSITION 5.3. *Let* $\|\cdot\|$ *and* T *be as above and* $\phi \in \mathscr{H}(\Delta)$. *If* $r(\cdot) = 1/\phi(\|\cdot\|^2)$, *then*

(i) $C^*_{T,r}$ *is quasinormal if and only if either* ϕ *is a monomial and* tA *is unitary in* $(\boldsymbol{R}^d, \|\cdot\|)$ *for some* $t > 0$ *or* ϕ *is not a monomial and* A *is unitary in* $(\boldsymbol{R}^d, \|\cdot\|)$.

(ii) $C^*_{T,r}$ *is isometry if and only if* A *is unitary in* $(\boldsymbol{R}^d, \|\cdot\|)$. *If* $r(\cdot) = \phi(\|\cdot\|^2)$ *and* $C_{T,r}$ *is bounded, then* (i) *and* (ii) *hold. Moreover*

(iii) $C^*_{T,r}$ *is hyponormal if and only if* $C^*_{T,r}$ *is quasinormal.*

PROOF. Assume $r(\cdot) = 1/\phi(\|\cdot\|^2)$. If $C^*_{T,r}$ is quasinormal, then — in virtue of (5.4) — $\phi(\|x\|^2)^2 = \phi(\|Ax\|^2) \cdot \phi(\|A^{-1}x\|^2)$ for $x \in TX$. Thus we can repeat arguments used in the proof of Steps 1 to 5 of Proposition 2.3 to get the condition (i). If $C^*_{T,r}$ is isometry and ϕ is a monomial, then — by (5.5) — $|\det A| \phi(\|x\|^2) = \phi(\|Ax\|^2)$ for every $x \in X$. So we can repeat the proof of Step 6 of Proposition 2.3 to get the condition (ii).

Assume $r(\cdot) = \phi(\|\cdot\|^2)$. The proof of the conditions (i) and (ii) is similar to that from the previous paragraph. If $C^*_{T,r}$ is hyponormal, then using (5.3) we get $\phi(\|x\|^2)^2 \geq \phi(\|Ax\|^2) \cdot \phi(\|A^{-1}x\|^2)$ for $x \in TX$. Following arguments utilized in the proof of Step 1 of Proposition 2.3 we conclude that $C^*_{T,r}$ is quasinormal. This finishes the proof. ■

Below we show that the question of subnormality of $C^*_{T,r}$ is equivalent to the question of subnormality of some generalized composition operators (cf. [12]).

Let B be an invertible linear transformation of $\boldsymbol{R}^d$ such that $B^{-1}X \subset X$, or equivalently $\|B^{-1}\| \leq 1$. Denote by S the measurable transformation within the space X defined by $D(S) = B^{-1}X$ and $Sx = Bx$ for $x \in D(S)$. Let r be a Borel function on X such that $r(x) \in (0, \infty)$ for almost every $x \in X$. Since the measure $\mu_r S^{-n}$ is absolutely continuous with respect to μ_r and

(5.6) $\lambda_n = d\mu_r\, S^{-n}/d\mu_r = |\det B|^{-n} \cdot r\circ S^{-n}/r,\ n\in \boldsymbol{N},$

the transformation S induces the generalized composition operator $C_{S,r}$ in $L^2_X(r)$ via $C_{S,r}\, f = \chi_{D(S)} \cdot f\circ S$. The operator $C_{S,r}$ is bounded if and only if $\lambda_1 \in L^\infty(X, \omega_d)$ (cf. [12], Theorem 1).

Using the equality (5.6) and Corollary 4 in [10] (adapted to generalized composition operators) we get

PROPOSITION 5.4. *A bounded generalized composition operator $C_{S,r}$ is subnormal if and only if for almost every $x\in X$, the sequence $\{r(B^{-n}(x))\}_{n=0}^\infty$ is a Hamburger moment sequence. Moreover if $r: X\backslash\{0\} \longrightarrow (0, \infty)$ is continuous, then the above holds replacing "for almost every $x\in X$" by "for every $x \in X\backslash\{0\}$".*

Turn back to the bounded operator $C_{T,r}$. It is easy to see that its adjoint acts according to the following formula

$$C^*_{T,r}\, f = |\det A|^{-1} \cdot \chi_{TX} \cdot f\circ T^{-1} \cdot r\circ T^{-1}/r,\ f\in L^2_X(r).$$

Moreover the mapping $U: L^2_X(r) \longrightarrow L^2_X(1/\mathrm{r})$ defined by $Uf = fr$ for $f\in L^2_X(r)$ is unitary and satisfies the following condition

(5.7) $U|\det A| C^*_{T,r} = C_{S,1/r} U,$

where S comes from $B=A^{-1}$. Thus the question of subnormality of $C^*_{T,r}$ for $r(\cdot)=1/\phi(\|\cdot\|^2)$ and $\phi\in\mathscr{H}(\Delta)$ can be answered as follows

THEOREM 5.5. *If $r(\cdot)=1/\phi(\|\cdot\|^2)$ with $\phi\in\mathscr{H}(\Delta)$, then the operator $C^*_{T,r}$ is subnormal if and only if A is normal in $(\boldsymbol{R}^d, \|\cdot\|)$.*

PROOF. In virtue of (5.7) and Proposition 5.4, the operator $C^*_{T,r}$ is subnormal if and only if $\{\phi(\|A^n x\|^2)\}_{n=0}^\infty$ is a Hamburger moment sequence for every $x\in X$. Since A is an algebraic contraction, the conclusion of Theorem 5.5 is a consequence of the following version of Theorem 6.3 in [17] (its proof is similar)

If N is an algebraic contraction on a Hilbert space H and $\phi\in\mathscr{H}(\Delta)$, then N is normal if and only if $\{\phi(\|N^n f\|^2)\}_{n=0}^\infty$ is a Hamburger moment sequence for every $f\in H$ such that $\|f\|<1$. ■

Now a few comments are in order. Assume that $\phi\in\mathscr{H}(\Delta)$. If $r(\cdot)= 1/\phi(\|\cdot\|^2)$, then seminormal generalized composition operators of the form $C_{S,r}$ coincide with normal ones (or with unitary ones, provided ϕ is not a monomial). If $r(\cdot)=\phi(\|\cdot\|^2)$, then there are generalized composition operators of the form $C_{S,r}$ which are hyponormal but not subnormal (consider

$\phi=\exp$, $\|x\|=|x|$ for $x\in \boldsymbol{R}^d$ and S attached to $B=A$, where A is as in Example 2. 6).

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