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Finiteness of von Neumann algebras and non-commutative L^p-spaces

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0. Introduction

Murray and von Neumann introduced their equivalence relation among projections in a von Neumann algebra and proved that a factor is finite (i.e. every projection is finite) if and only if it has a finite trace. In [2], Cuntz and Pedersen defined another equivalence relation among all positive elements in a C^*-algebra, and proved that the algebra is finite if and only if there is a separating family of finite traces.

In this paper, we introduce an equivalence relation among the positive elements of a non-commutative L^p-space associated with an arbitrary von Neumann algebra, and we study the finiteness of non-commutative L^p-spaces with respect to it.

In §1, we recall the definition of non-commutative L^p-spaces associated with an arbitrary von Neumann algebra defined by Haagerup [4]. For non-commutative L^p-spaces $L^p(N, \tau)$ arising from a semifinite von Neumann algebra N and its trace τ, the intersection $N\cap L^p(N, \tau)$ is L^p-norm dense in $L^p(N, \tau)$. Therefore one may naturally expect some similarity of their order structures between N and $L^p(N, \tau)$ even if there are significant differences, for example, the existence of an order unit. On the other hand, for non-commutative L^p-spaces $L^p(M)$ associated with an arbitrary von Neumann algebra M, it is well-known that any non-zero element in $L^p(M)$ is not bounded and that $M\cap L^p(M)=\{0\}$. Therefore we need some care to deal with them throughout the paper. In §2, we study the monotone order completeness of $L^p(M)$. Applying the result, we show in §3 that $L^p(M)$ has the asymmetric Riesz decomposition property, and we introduce an equivalence relation among the positive elements in $L^p(M)$. In §4, using the equivalence relation introduced in §3, we define a notion of finiteness of $L^p(M)$. Considering bounded linear functionals on $L^p(M)$ which satisfy the property as traces, we show that the finiteness of $L^p(M)$ agrees with that of M for the case of $1<p<\infty$.

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1. Preliminaries

In this section, we will collect definitions and basic facts on the theory of non-commutative L^p-spaces associated with an arbitrary von Neumann algebra. Full details are found in [4] and [12].

Let M be an arbitrary von Neumann algebra. Let N be the crossed product of M by the modular automorphism group $\{\sigma_t\}_{t\in\boldsymbol{R}}$ of a fixed faithful normal semifinite weight on M. Then N admits the dual action $\{\theta_s\}_{s\in\boldsymbol{R}}$ and the faithful normal semifinite trace τ satisfying $\tau\circ\theta_s=e^{-s}\tau$, $s\in\boldsymbol{R}$. We denote by $\tilde{N}$ the set of all τ-measurable operators (affiliated with N). For $0<p\leq\infty$, the Haagerup L^p-space $L^p(M)$ is defined by

$$L^p(M)=\{a\in\tilde{N}\,;\,\theta_s(a)=e^{-s/p}a,\ s\in\boldsymbol{R}\}.$$

It is well-known that there exists a linear order isomorphism $\varphi\longrightarrow h_\varphi$ from the predual M_* onto $L^1(M)$. We thus get a positive linear functional tr on $L^1(M)$ defined by $tr(h_\varphi)=\varphi(1)$, $\varphi\in M_*$. The (quasi-)norm of $L^p(M)$ for $0<p<\infty$ is defined by $\|a\|_p=tr(|a|^p)^{1/p}$, $a\in L^p(M)$. When $1\leq p<\infty$, $L^p(M)$ is a Banach space, and its dual space is $L^q(M)$, where $\frac{1}{p}+\frac{1}{q}=1$. The duality is given by the following bilinear form:

$$(a,b)\longrightarrow tr(ab)\ (=tr(ba)),\ a\in L^p(M),\ b\in L^q(M).$$

The space $L^p(M)$ is independent of the choice of a faithful normal semifinite weight on M up to isomorphism. Furthermore, when M has a faithful normal semifinite trace τ_0, $L^p(M)$ can be identified with the non-commutative L^p-space $L^p(M,\tau_0)$ introduced in [9].

2. Monotone order completeness of measure topology

In this section we study the monotone order completeness of measure topology associated with a semifinite von Neumann algebra. The result does not seem to have been pointed out in the literature, though it may be well-known probably. As an immediate consequence, we also have the monotone order completeness of non-commutative L^p-spaces to introduce an equivalence relation in L^p-spaces. It may be useful to state these results in the form of a theorem and its corollaries.

Suppose that N is a semifinite von Neumann algebra with a faithful normal semifinite trace τ. We denote by $\tilde{N}$ the set of all τ-measurable operators, which becomes a complete Hausdorff topological $*$-algebra with the measure topology (cf. [7], [12]). For $\varepsilon, \delta>0$, we set

$$N(\varepsilon, \delta)=\{a\in\tilde{N};\ \textit{there exists a projection } e \textit{ in } N \textit{ with } \|ae\|\leq\varepsilon,\ \tau(1-e)\leq\delta\}.$$

Then the family $\{N(\varepsilon, \delta); \varepsilon, \delta>0\}$ is a fundamental system of neighborhoods around 0 with respect to the measure topology. We also denote by $\tilde{N}_+$ the set of all positive self-adjoint elements in $\tilde{N}$. Recall that an operator a in $\tilde{N}$ is to be defined τ-compact if a satisfies that $\tau(E_{(s,\infty)}(|a|))<\infty$ for all $s>0$, where $E_{(s,\infty)}(|a|)$ is the spectral projection of $|a|$ corresponding to the interval (s, ∞). This definition of τ-compactness is equivalent to that the generalized s-number $\mu_t(a)$ of a converges to 0 as $t\longrightarrow\infty$ (cf. [3; Proposition 3.2]).

LEMMA 2.1. *Let a be a τ-compact operator. Let $\{y_n\}_{n=1}^{\infty}$ be a sequence in N which converges to 0 strongly. Then the sequence $\{y_na\}_{n=1}^{\infty}$ converges to 0 in the measure topology.*

PROOF. Considering the polar decomposition of a, we may assume that a is positive self-adjoint. Let $a=\int_{[0,\infty)}\lambda de_\lambda$ be the spectral decomposition of a. Fix any positive numbers ε and δ. Let $\gamma=\sup\|y_n\|(<\infty)$ and $\alpha=\frac{\varepsilon}{\gamma}$. Since a is τ-measurable, we can take a β $(>\alpha)$ such that $\tau\left(\int_{(\beta,\infty)}de_\lambda\right)\leq\delta$. We write $y_na=y_n\int_{[0,\alpha]}\lambda de_\lambda+y_n\int_{(\alpha,\beta]}\lambda de_\lambda+y_n\int_{(\beta,\infty)}\lambda de_\lambda$. Then the first and the last terms are in $N(\varepsilon, \delta)$. For the second term, since a is τ-compact and $\int_{(\alpha,\beta]}\lambda de_\lambda\leq\beta\int_{(\alpha,\infty)}de_\lambda$, it follows that $\int_{(\alpha,\beta]}\lambda de_\lambda\in L^2(N, \tau)$. Hence, representing N on $L^2(N, \tau)$, we have $\|y_n\int_{(\alpha,\beta]}\lambda de_\lambda\|_2\longrightarrow0$ as $y_n\longrightarrow0$ strongly. This completes the proof. □

THEOREM 2.2. *Let $\{a_n\}_{n=1}^{\infty}$ be an increasing sequence in $\tilde{N}_+$. Assume that there is a τ-compact operator a in $\tilde{N}$ satisfying $a_n\leq a$ for all $n\in\boldsymbol{N}$. Then there exists a unique element a_∞ in $\tilde{N}$ such that a_n converges to a_∞ in the measure topology.*

PROOF. By [8; Lemma 2.2], for each $n\in\boldsymbol{N}$, there is a unique $x_n\in N$ such that $0\leq x_n\leq s(a)$ and $a_n=a^{1/2}x_na^{1/2}$. The same lemma shows that the sequence $\{x_n\}_{n=1}^{\infty}$ is increasing. The x_n converges strongly to an element x in N. We put $a_\infty=a^{1/2}xa^{1/2}$. Since $x-x_n$ converges to 0 strongly, we conclude from the previous lemma that $x_n a^{1/2}$ converges to $xa^{1/2}$ in the measure topology. This yields the result and completes the proof. □

REMARK 2. 3. In the preceding theorem, we can not drop the condition that a is τ-compact.

Let l^2 be the usual sequence space. We denote an increasing sequence $\{a_n\}_{n=1}^{\infty}$ of bounded operators on l^2 by matrices with respect to its canonical basis $e_n=(0,\cdots,0,\overset{n}{1},0,\cdots)$ as follows;

$a_n=\begin{bmatrix} E_n & 0 \\ 0 & 0 \end{bmatrix}$, where E_n is the identity matrix of degree n. Then $\{a_n\}_{n=1}^{\infty}$ is dominated by the identity operator. However, it is impossible that $\{a_n\}_{n=1}^{\infty}$ forms a Cauchy sequence in the measure topology.

We assume that $0<p<\infty$ throughout the rest of this section. It is well-known that non-commutative L^p-spaces $L^p(N, \tau)$ associated with a semifinite von Neumann algebra N and its trace τ are included in the class of τ-compact operators (cf. [3; Remark 3. 3). From Theorem 2. 2 and [3; Theorem 3. 6], we have the following result.

COROLLARY 2. 4. *Let $\{a_n\}_{n=1}^{\infty}$ be an increasing sequence in $L^p(N, \tau)_+$. Assume that there is an element a in $L^p(N, \tau)$ satisfying $a_n \leq a$ for all $n \in \boldsymbol{N}$. Then there exists a unique element a_∞ in $L^p(N, \tau)$ such that $\|a_n - a_\infty\|_p \longrightarrow 0$.*

Moreover, we can also obtain a corresponding result for non-commutative L^p-spaces $L^p(M)$ associated with an arbitrary von Neumann algebra M. For any a in $L^p(M)$, it is known that $\mu_t(a)=t^{-1/p}\|a\|_p$ for all $t>0$, where $\mu_t(a)$ is the generalized s-number relative to the canonical trace on the crossed product (cf. [3; Lemma 4. 8]). This implies that $L^p(M)$ is included in the class of τ-compact operators.

COROLLARY 2. 5. *Let $\{a_n\}_{n=1}^{\infty}$ be an increasing sequence in $L^p(M)_+$. Assume that there is an element a in $L^p(M)$ satisfying $a_n \leq a$ for all $n \in \boldsymbol{N}$. Then there exists a unique element a_∞ in $L^p(M)$ such that $\|a_n - a_\infty\|_p \longrightarrow 0$.*

PROOF. From the assumption, we conclude by Theorem 2. 2 that a_n converges to an element a_∞ in the measure topology. Since $L^p(M)$ is closed in the measure topology (cf. [4; Definition 1. 7]), a_∞ is included in $L^p(M)$. Moreover, the norm topology of $L^p(M)$ is exactly the induced measure topology (cf. [4; Proposition 1. 17] or [12; ChapterII, Proposition 26]), we conclude that $\|a_n - a_\infty\|_p \longrightarrow 0$. □

3. Asymmetric decomposition and equivalence relation in L^p-spaces

Let M be an arbitrary von Neumann algebra. We introduce an equivalence relation in $L^p(M)_+$ as in the theory of C^*-algebra to study a functional on L^p-spaces which satisfies the property as a trace. For a, b in $L^p(M)_+$, we define $a\sim b$ if there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in $L^{2p}(M)$ such that $a=\sum_{n=1}^{\infty}x_n^*x_n$, $b=\sum_{n=1}^{\infty}x_nx_n^*$ in the sense of L^p-(quasi-)norm convergence. Also, we define $a<b$ if there exists an element c in $L^p(M)_+$ such that $a\sim c\leq b$. Then we have the countably asymmetric decomposition for $L^p(M)$.

PROPOSITION 3. 1. *Let $0<p<\infty$. If $\{x_i\}_{i=1}^{\infty}$, $\{y_j\}_{j=1}^{\infty}$ are sequences in $L^{2p}(M)$ such that $\sum_{i=1}^{\infty} x_i^*x_i=\sum_{j=1}^{\infty} y_jy_j^*$. Then there exists a double sequence $\{z_{i,j}\}_{i,j=1}^{\infty}$* in $L^{2p}(M)$ such that *$x_ix_i^*=\sum_{j=1}^{\infty}z_{i,j}z_{i,}^*{}_j$ and $y_j{}^*y_j=\sum_{i=1}^{\infty}z_{i,}^*{}_jz_{i,j}$.*

PROOF. Put $a=\sum x_i^*x_i=\sum y_jy_j^*$. As in the proof of [8; Lemma 2. 21], we can find a unique operator s_i in N satisfying the following conditions; $0\leq s_i^*s_i\leq s(|x_i|)\leq s(a)$, $x_i=s_ia^{1/2}$ in $\tilde{N}$. If follows from the uniqueness that s_i is fixed under the dual action and that $s_i\in M$. Similarly, there exists an element t_j in M such that $y_j^*=t_j^*a^{1/2}$. Since $\sum_{i=1}^{n} x_i^*x_i=a^{1/2}(\sum_{i=1}^{n}s_i^*s_i)a^{1/2}$ increases to a in the measure topology, we can conclude by the uniqueness part of [8; Lemma 2. 2] that the sequence $\{\sum_{i=1}^{n}s_i^*s_i\}_{n=1}^{\infty}$ increases strongly to the range projection of a. Then the sequence $\{t_j^*a^{1/2}(\sum_{i=1}^{n}s_i^*s_i)a^{1/2}t_j\}_{n=1}^{\infty}$ increases to $t_j^*at_j=y_j^*y_j$ in L^p-norm topology. Putting $z_{i,j}=s_ia^{1/2}t_j$, we complete the proof. □

By deleting some of the x_i and corresponding $z_{i,j}$, we immediately conclude the following corollay.

COROLLARY 3. 2. *Let $0<p<\infty$. If $\{x_i\}_{i=1}^{\infty}$, $\{y_j\}_{j=1}^{\infty}$ are sequences in $L^{2p}(M)$ such that $\sum_{i=1}^{\infty} x_i^*x_i\leq\sum_{j=1}^{\infty} y_jy_j^*$. Then there exists a double sequence $\{z_{i,j}\}_{ij=1}^{\infty}$ in $L^{2p}(M)$ such that $x_ix_i^*=\sum_{j=1}^{\infty}z_{i,j}z_{i,}^*{}_j$ and $\sum_{i=1}^{\infty}z_{i}^*{}_{,j}z_{i,j}\leq y_j^*y_j$.*

THEOREM 3. 3. *Let $0<p<\infty$. The relation " $\sim$ " becomes an equivalence relation in $L^p(M)_+$. It is countably additive in the sense that $\sum_{i=1}^{\infty} a_i\sim\sum_{i=1}^{\infty} b_i$ when the sum exists and $a_i\sim b_i$ in $L^p(M)_+$. The relation " $<$ " satisfies the transitivity and the Riesz decomposition property: if $\sum_{i=1}^{\infty}a_i<\sum_{j=1}^{\infty}b_j$ then there exists a double sequence $\{c_{i,j}\}_{i,j=1}^{\infty}$ in $L^p(M)_+$ such that $a_i=\sum_{j=1}^{\infty}c_{i,j}$ and $\sum_{i=1}^{\infty}c_{i,j}<b_j$.*

PROOF. To see that the relation " $\sim$ " is an equivalence relation, it is enough to show the transitivity. For elements a, b and c in $L^p(M)_+$, suppose that $a\sim b$ and $b\sim c$. From the above proposition there is a double sequence $\{z_{i,j}\}_{i,j=1}^\infty$ in $L^{2p}(M)$ such that

$$a=\sum_{i=1}^\infty\left(\sum_{j=1}^\infty z_{i,j}z_{i,j}^*\right) \quad and \quad c=\sum_{j=1}^\infty\left(\sum_{i=1}^\infty z_{i,j}^* z_{i,j}\right).$$

Suppose that $\boldsymbol{K}$ is any bijective map $\boldsymbol{K}: \boldsymbol{N}\ni n\longrightarrow(i(n), j(n))\in \boldsymbol{N}\times\boldsymbol{N}$. By the monotone order completeness, it is straightforward to see that the sequence $\{\sum_{n=1}^N z_{\boldsymbol{K}(n)}z_{\boldsymbol{K}(n)}^*\}_{N=1}^\infty$ converges to a in the L^p-norm topology. Moreover, the series $\sum_{j=1}^\infty(\sum_{i=1}^\infty z_{i,j}z_{i,j}^*)$ also converges to a. Thus we have $a=\sum_{n=1}^\infty z_{\boldsymbol{K}(n)}z_{\boldsymbol{K}(n)}^*$ and $c=\sum_{n=1}^\infty z_{\boldsymbol{K}(n)}^* z_{\boldsymbol{K}(n)}$, hence the relation " $\sim$ " becomes an equivalence relation in $L^P(M)_+$.

To show the Riesz decomposition property, suppose that $\sum_{i=1}^\infty a_i\sim c\leq \sum_{j=1}^\infty b_j$ for some c in $L^p(M)$. Then there exists a sequence $\{u_n\}_{n=1}^\infty$ in $L^{2p}(M)$ such that $\sum_{i=1}^\infty a_i=\sum_{n=1}^\infty u_n^* u_n$ and $\sum_{n=1}^\infty u_n u_n^*=c\leq\sum_{j=1}^\infty b_j$. By the first equation, we can take a double sequence $\{v_{i,n}\}_{i,n=1}^\infty$ in $L^{2p}(M)$ such that $a_i=\sum_{n=1}^\infty v_{i,n}v_{i,n}^*$ and $\sum_{i=1}^\infty v_{i,n}^* v_{i,n}=u_n u_n^*$. Then we have $\sum_{i,n=1}^\infty v_{i,n}^* v_{i,n}\leq \sum_{j=1}^\infty b_j$, hence there is a triple sequence $\{w_{i,j,n}\}_{i,j,n=1}^\infty$ in $L^{2p}(M)$ such that $v_{i,n}v_{i,n}^*=\sum_{j=1}^\infty w_{i,j,n}^* w_{i,j,n}$ and $\sum_{i,n=1}^\infty w_{i,j,n}^* w_{i,j,n}\sim\sum_{i,n=1}^\infty w_{i,j,n}w_{i,j,n}^*\leq b_j$. Putting $c_{i,j}=\sum_{n=1}^\infty w_{i,j,n}^* w_{i,j,n}$, we have $a_i=\sum_{j=1}^\infty c_{i,j}$ and $\sum_{i=1}^\infty c_{i,j}< b_j$. It is easy to establish the rest of the theorem, and the proof is omitted. □

4. Finiteness of L^p-spaces

As an application of the preceding results, we study a certain finiteness of non-commutative L^p-spaces associated with an arbitrary von Neumann algebra, and we shall see that the notion of finiteness of L^p-spaces for $1<p<\infty$ is coincides with that of von Neumann algebras. Let M be an arbitrary (not necessarily semifinite) von Neumann algebra. Once Theorem 3.3 has been established, we can consider a quotient space of L^p-space with respect to the relation " $\sim$ ". We denote by L_{sa}^p the set of all self-adjoint elements in $L^p(M)$ and denote by L_0^p the real linear subspace of L_{sa}^p consisting of elements of the form $a-b$, where a, $b\in L^p(M)_+$ and $a\sim b$. Moreover, we denote by Q the quotient map $Q: L_{sa}^p \longrightarrow L_{sa}^p/L_0^p$. As in the proof of [2; Theorem 2.6], it is straightforward to verify that the subspace L_0^p is closed in L_{sa}^p. Therefore, there is a canonical linear isometry between the dual of the quotient space $Q(L_{sa}^p)$ and the space $(L_0^p)^\perp$ consisting of elements f in $(L_{sa}^p)^*$ such that $f(a)=0$ for all a in L_0^p. Note that $f\in(L_0^p)^\perp$ if and only if $f(x^*x)=f(xx^*)$ for all $x\in L^{2p}(M)$.

LEMMA 4. 1. *Let $1 \leq p < \infty$ and $\frac{1}{p}+\frac{1}{q}=1$. Suppose that $f \in (L_0^p)^\perp$. Let b be a unique element in L_{sa}^q corresponding to f such that $f = tr(b\cdot)$. If $b=b_+ - b_-$ is the Jordan decomposition of b, then $tr(b_+\cdot)$ and $tr(b_-\cdot)$ are elements of $(L_0^p)^\perp$.*

PROOF. Note that b satisfies $tr(bx^*x)=tr(bxx^*)$, $x \in L^{2p}(M)$. Putting $x=ua^{1/2}$, we have $tr(ba)=tr(u^*bua)$ for any unitary $u \in M$ and any $a \in L_+^p$. This implies that b is affiliated with the commutant M'. By the uniqueness of the Jordan decomposition, it follows that b_+, b_- are also affiliated with M'. Denote by e_1(resp. e_2) the support projection of b_+ (resp. b_-). Then we have $b_+=be_1$, $b_-=-be_2$, and e_1, e_2 are orthogonal projections in the center of M. Hence we have $tr(b_+x^*x)=tr(be_1x^*x)=tr(bx^*e_1x)=tr(be_1xx^*e_1)=tr(b_+xx^*)$ and $tr(b_-x^*x)=tr(b_-xx^*)$. This completes the proof. □

THEOREM 4. 2. *Let $1<p<\infty$ and $\frac{1}{p}+\frac{1}{q}=1$. If $a \in L^p(M)_+$, then the following four constants are equal;*

$\alpha=\inf\{\|a-c\|_p ; c \in L_0^p\}$,

$\beta=\inf\{\|b\|_p ; b > a, b \in L_+^p\}$,

$\gamma=\sup\{f(a) ; f \in (L_{sa}^p)^*, \|f\|=1, f(x^*x)=f(xx^*) \geq 0, x \in L^{2p}(M)\}$, *and*

$\delta=\sup\{tr(h_\varphi^{1/q}a) ; \varphi$ *is a normal tracial state on* $M\}$.

PROOF. A similar argument as in the proof of [2; Theorem 2. 9] shows that $\alpha \geq \beta \geq \gamma$. Suppose $\alpha>0$ to show that $\alpha \leq \gamma$. Since α is the quotient norm of a in $Q(L_{sa}^p)$, there is by Hahn-Banach's theorem an element $\tilde{f}$ in $Q(L_{sa}^p)^*$ with $\|\tilde{f}\|=1$ such that $\tilde{f}(Q(a))=\alpha$. Let b be a unique element in L_{sa}^q corresponding to $\tilde{f}(Q(\cdot))$ such that $\tilde{f}(Q(\cdot))=tr(b\cdot)$. If $b=b_+-b_-$ is the Jordan decomposition of b, then we have $tr(b_+\cdot) \in (L_0^p)^\perp$ by Lemma 4. 1. Since $\|b\|_q^q=\|b_+\|_q^q+\|b_-\|_q^q$, we have $\|b_+\|_q \leq 1$ and $tr(b_+a) \geq \alpha$. It follows that $\|b_+\|_q=1$. Hence we have $b_-=0$ and $b \geq 0$. Put $f=\tilde{f}(Q(\cdot))$. Then we have $f \in (L_{sa}^p)^*$, $\|f\|=1$, and f satisties that $f(x^*x)=f(xx^*) \geq 0$ for any $x \in L^{2p}(M)$. Thus $\alpha \leq \gamma$. To see that $\gamma=\delta$, suppose that f is an element in $(L_{sa}^p)^*$ satisfying $f(x^*x)=f(xx^*) \geq 0$ for any $x \in L^{2p}(M)$. Let b be a unique element in L_+^q corresponding to f such that $f=tr(b\cdot)$. Then b is affiliated with M'. Taking a unique positive element $\varphi \in M_*$ such that $b=h_\varphi^{1/q}$, h_φ is affiliated with M'. It follows from [5; Théorème 2] or [12; ChapterIV, Proposition 4] that the Connes' spatial derivative $\frac{d\varphi}{d\psi_0}$ is affiliated with M', where ψ_0 is a faithful normal semifinite weight on M'. Due to [1; Theorem 9] or [12; ChapterIII, Corollary 27], we conclude that φ is a trace on M. Conversely, for each normal finite trace φ on M, the element h_φ is affiliated with M'. Hence the element $tr(h_\varphi^{1/q}\cdot)$ in $(L_{sa}^p)^*$ satisfies that $tr(h_\varphi^{1/q}x^*x)=tr(h_\varphi^{1/q}xx^*)\geq 0$, $x\in L^{2p}(M)$. Thus we get the desired isometric bijective correspondence which implies that $\gamma=\delta$. This completes the proof. □

DEFINITION 4. 3. *A positive element a in $L^p(M)$ is said to be finite if for each $a'\in L^p(M)_+$ such that $a'\leq a$ and $a'\sim a$ implies that $a'=a$. We say that $L^p(M)$ is finite if every element in $L^p(M)_+$ is finite.*

REMARK 4. 4. For the case of $p=1$, the above definition is vacuous. Let a, b be elements in $L^1(M)_+$. Suppose that $a\sim b\leq a$. Then we have $tr(a)=tr(b)$. It follows that $\|a-b\|_1=tr(a-b)=0$, i. e. $a=b$. Therefore, the space $L^1(M)$ is always finite in the sense defined above for an arbitrary von Neumann algebra.

It is easy to verify the following lemmas.

LEMMA 4. 5 (cf. [2; Lemma 3. 3]). *$L^p(M)$ is finite if and only if $L^p(M)_+\cap L_0^p=\{0\}$.*

LEMMA 4. 6. *Suppose that $\{\varphi_\lambda\}_{\lambda\in\Lambda}$ is a family of positive normal functionals on M. Then the following conditions are equivalent.*
(1) *The supremum of the support projections of φ_λ equals to* 1.
(2) *$\{tr(h_{\varphi_\lambda}\cdot);\lambda\in\Lambda\}$ is separating for M_+.*
(3) *$\{tr(h_{\varphi_\lambda}{}^{1/q}\cdot);\lambda\in\Lambda\}$ is separating for $L^p(M)_+$.*

The following theorem shows that our notion of finiteness of non-commutative L^p-spaces for $1<p<\infty$ precisely agrees with that of von Neumann algebras.

THEOREM 4. 7. *Let $1<p<\infty$. The $L^p(M)$ is finite if and only if M is a finite von Neumann algebra.*

PROOF. Suppose that $L^p(M)$ is finite. Let a be an element in $L^p(M)_+$. If $tr(h_\varphi^{1/q}a)=0$ for any normal finite trace φ on M, then $Q(a)=0$ by Theorem 4. 2, where Q denotes the quotient map. Since Q is faithful on $L^p(M)_+$ by Lemma 4. 5, we have $a=0$. Thus the set $\{tr(h_\varphi^{1/q}\cdot)$; *$\varphi$ is a normal finite trace on M*$\}$ is separating for $L^p(M)_+$. It follows from the previous lemma that M has a sufficient family of normal finite traces. Conversely, if M has a sufficient family $\{\varphi_\lambda\}_{\lambda\in\Lambda}$ of normal tracial states, then $\{tr(h_{\varphi_\lambda}{}^{1/q}\cdot);\lambda\in\Lambda\}$ is separating for $L^p(M)_+$ by Lemma 4. 6. For $a\in$

$L^p(M)_+ \cap L_0^p$, we have by Theorem 4.2,

$$0=\|Q(a)\|\geq \sup\{tr(h_{\varphi_\lambda}^{1/q}a)\ ;\ \lambda\in\Lambda\}0.$$

Thus $a=0$, hence the result follows from Lemma 4.5. □

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