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On Hardy's Inequality and Paley's Gap Theorem

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Dedicated to Professor Shozo Koshi on his sixtieth birthday
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Let $\boldsymbol{T}=\{z\in\boldsymbol{C}:|z|=1\}$ be the circle group, and let λ be the Lebesgue measure on $\boldsymbol{T}$ normalized so that $\lambda(\boldsymbol{T})=1$. Thus the Fourier coefficients of $f\in L^{1}(\boldsymbol{T})$ are defined by

$$\hat{f}(n)=\int_{\boldsymbol{T}}z^{-n}f(z)d\lambda(z)\qquad\forall n\in\boldsymbol{Z}.$$

The Hardy class $H^{1}(\boldsymbol{T})$ consists of all $f\in L^{1}(\boldsymbol{T})$ such that $\hat{f}(n)=0$ for all $n<0$. The classical inequality of Hardy states that

$$(1)\qquad \sum_{n=1}^{\infty}\frac{1}{n}|\hat{f}(n)|\leq C_{1}\|f\|_{1}\qquad\forall f\in H^{1}(\boldsymbol{T}),$$

where C_{1} is a positive constant $\leq\pi$; see, e.g., K. Hoffman [2; p. 70] or A. Zygmund [5; p. 286]. On the other hand, Paley's Gap Theorem [3] asserts that given a sequence $(n_{k})_{1}^{\infty}$ of natural numbers with $\inf\{n_{k+1}/n_{k}:k\geq 1\}>1$, there exists a finite constant C_{2} such that

$$(2)\qquad \sum_{k=1}^{\infty}|\hat{f}(n_{k})|^{2}\leq C_{2}^{2}\|f\|_{1}^{2}\qquad\forall f\in H^{1}(\boldsymbol{T}).$$

For a generalization of (2) to connected compact abelian groups, we refer to W. Rudin [4; p. 213]. In the present paper, we shall give some generalizations of these well known results both in the classical setting and the abstract setting.

Let α be a Borel measurable function on $\boldsymbol{T}$ such that $|\alpha|=1$ almost everywhere. Given $f\in L^{1}(\boldsymbol{T})$, let $\alpha^{*}f$ denote the complex measure on $\boldsymbol{T}$ defined by

$$(3)\qquad \int hd(\alpha^{*}f)=\int(h\circ\alpha)fd\lambda$$

for all bounded Borel functions h on $\boldsymbol{T}$. In other words, $\alpha^{*}f$ is the image measure of $f\lambda$ by α. Let $H_{0}^{1}(\boldsymbol{T})=\{f\in H^{1}(\boldsymbol{T}):\hat{f}(0)=0\}$. Finally recall that an inner function is an element α of $H^{1}(\boldsymbol{T})$ such that $|\alpha|=1$ almost everywhere.

THEOREM 1. *Let α,β be two functions in* $H^{1}(\boldsymbol{T})$ such that $|\alpha|=1\geq|\beta|$ *a.e. and $\hat{\alpha}(0)\hat{\beta}(0)=0$. Then*

$$\sum_{n=1}^{\infty}\frac{1}{n}\|(\alpha^*\beta^n)*f\|_1\leq C_1\|f\|_1 \qquad \forall f\in H_0^1(\boldsymbol{T}),$$

where C_1 *is any finite constant satisfying*(1).

THEOREM 2. *Let* $\alpha, \beta\in H^1(\boldsymbol{T})$ *be as in Theorem* 1, *and let* $(n_k)_1^{\infty}$ *be a sequence of natural numbers that satisfies* (2) *for some* $C_2<\infty$. *Then*

$$\sum_{k=1}^{\infty}\|(\alpha^*\beta^{n_k})*f\|_1^2\leq C_2^2\|f\|_1^2 \qquad \forall f\in H_0^1(\boldsymbol{T}).$$

Example 5 (iii) given below includes a precise calculation of the measure α^*g for Möbius transformations α and $g\in L^1(\boldsymbol{T})$. In order to prove the above two results, let G be a locally compact (Hausdorff) space, and let $M(G)$ be the Banach space of all bounded regular Borel measures on G. Given a bounded Borel function f on G and $\mu\in M(G)$, we shall often write $\langle f, \mu\rangle$ for $\int f d\mu$. For a linear subspace $\mathscr{A}$ of $C_0(G)$, define

$$\mathscr{A}^{\perp}=\{\mu\in M(G):\langle\phi, \mu\rangle=0 \quad \forall\phi\in\mathscr{A}\}.$$

For $1<p<\infty$, let $p'=p/(p-1)$.

LEMMA 3. *Let* $\mathscr{A}$ *be a linear subspace of* $C_0(G)$, *let* $(\gamma_k)_1^{\infty}$ *be a sequence in* $C_0(G)$, *let* $1\leq p<\infty$, *and let* $(a_k)_0^{\infty}$ *be a sequence of real positive numbers. Then the following conditions are equivalent*:

(a) *For each* $\mu\in\mathscr{A}^{\perp}$, $\sum_{k=1}^{\infty} a_k|\langle\gamma_k, \mu\rangle|^p\leq a_0^p\|\mu\|^p$.

(b) *Whenever* $c_1, \ldots, c_n$ *are finitely many complex numbers such that* $\sup\{|c_k|:1\leq k\leq n\}\leq 1$ *if* $p=1$ *or* $\sum_{k=1}^{n} a_k|c_k|^{p'}\leq 1$ *if* $p>1$, *then*

$$\inf\left\{\left\|\sum_{k=1}^{n}a_kc_k\gamma_k+\phi\right\|_{\infty}:\phi\in\mathscr{A}\right\}\leq a_0.$$

PROOF: That (a) implies (b) follows from the Hahn-Banach Theorem combined with the Riesz Representation Theorem. The converse is an easy exercise.

PROOF OF THEOREM 1: Choose and fix any $\alpha, \beta\in H^1(\boldsymbol{T})$ such that $|\alpha|=1\geq|\beta|$ almost everywhere and $\hat{\alpha}(0)\hat{\beta}(0)=0$. Then

$$\left(\alpha^*\overline{\beta}^k\right)*f=0 \qquad \forall k\in\boldsymbol{N} \textit{ and } f\in H_0^1(\boldsymbol{T}). \tag{4}$$

In fact, $n\in\boldsymbol{N}$ implies $\alpha^n\beta\in H_0^1(\boldsymbol{T})$, so

$$(\alpha^*\overline{\beta})^{\wedge}(n)=\int\bar{z}^n d(\alpha^*\overline{\beta})=\int\bar{\alpha}^n\overline{\beta}d\lambda$$

$$=\left(\int \alpha^n \beta d\lambda\right)^{-}=0.$$

Therefore $(\alpha^* \bar{\beta})*f=0$ for each $f\in H_0^1(\boldsymbol{T})$. Applying this result to β^k, we obtain (4).

Now let $a_k=1/k$ for $k\geq 1$, and let $c_1,\ldots,c_n$ be finitely many complex numbers with sup $\{|c_k|:1\leq k\leq n\}\leq 1$. Notice that $\{\mu\in M(\boldsymbol{T}):\hat{\mu}(n)=0\ \forall n\in\boldsymbol{N}\}=\{\bar{f}\lambda:f\in H^1(\boldsymbol{T})\}$ by the F. and M. Riesz Theorem. Therefore Lemma 3 (with $p=1$) combined with (1) yields complex numbers $b_1,\ldots,b_m$ such that

$$\left|\sum_{k=1}^{n} a_k c_k z^k+\sum_{j=1}^{m} b_j \bar{z}^j\right|<C \qquad \forall z\in\boldsymbol{T}, \tag{5}$$

where C is any preassigned finite constant $>C_1$. Define $g\in L^\infty(\boldsymbol{T})$ by setting

$$g=\sum_{k=1}^{n} a_k c_k \beta^k+\sum_{j=1}^{m} b_j \bar{\beta}^j. \tag{6}$$

Notice that (5) holds for all $z\in\boldsymbol{C}$ with $|z|\leq 1$ by the maximum modulus principle for harmonic functions. Since $|\beta|\leq 1$ ($a, e.$), it follows from (6) that $|g|\leq C$. Therefore $f\in H_0^1(\boldsymbol{T})$ implies

$$\begin{aligned}\left|\sum_{k=1}^{n} a_k c_k(\alpha^*\beta^k)*f\right| &=\left|\sum_{k=1}^{n} a_k c_k(\alpha^*\beta^k)*f+\sum_{k=1}^{m} b_k(\alpha^*\bar{\beta}^k)*f\right| \qquad \text{by (4)}\\ &=|(\alpha^* g)*f| \qquad \text{by (6)}\\ &\leq(\alpha^*|g|)*|f|\leq C(\alpha^* 1)*|f|.\end{aligned}$$

Since this holds for all $c_1,\ldots,c_n\in\boldsymbol{C}$ with sup $\{|c_k|:1\leq k\leq n\}\leq 1$, it follows that

$$\sum_{k=1}^{n} a_k|(\alpha^*\beta^k)*f|\leq C(\alpha^* 1)*|f| \qquad \forall f\in H_0^1(\boldsymbol{T}). \tag{7}$$

Upon integrating both sides of (7) over $\boldsymbol{T}$ and noting that $\alpha^* 1$ is a probability measure, we obtain $\sum_{k=1}^{n} a_k\|(\alpha^*\beta^k)*f\|_1\leq C\|f\|_1$ for each $f\in H_0^1(\boldsymbol{T})$. Since $n\in\boldsymbol{N}$ and $C>C_1$ were arbitrary, this completes the proof of Theorem 1.

The proof of Theorem 2 is quite similar to the above proof. We leave the details to the reader.

REMARK 4: Suppose that $\hat{\beta}(0)=0$. Then (4) holds for all $f\in H^1(\boldsymbol{T})$. Consequently $H_0^1(\boldsymbol{T})$ in Theorems 1 and 2 may be replaced by $H^1(\boldsymbol{T})$ in this case.

EXAMPLES 5. (i) Let α be a nonconstant inner function on $\boldsymbol{T}$ with $c=\hat{\alpha}(0)$, so that $|c|<1$. Let P_c denote the Poisson kernel at c:

$$(8)\qquad P_c(z)=Re\frac{1+cz}{1-cz}=1+2Re\sum_{k=1}^{\infty}c^k z^k \qquad \forall z\in\boldsymbol{T}.$$

Then we have

$$(9)\qquad \int(h\circ\alpha)\,d\lambda=\int hP_c^{\#}\,d\lambda \qquad \forall h\in L^1(\boldsymbol{T}),$$

where $P_c^{\#}(z)=P_c(z^{-1})$ for $z\in\boldsymbol{T}$ (cf. R. B. Burckel [1 ; p. 134]). Consequntly $\alpha^*1=P_c^{\#}\bar{\lambda}$

To prove this, first suppose that h is a trigonometric polynomial on $\boldsymbol{T}$: $h(z)=\sum_{k=-n}^{n} a_k z^k$. Then

$$\int(h\circ\alpha)\,d\lambda=\sum_{k=-n}^{n}a_k\int\alpha^k d\lambda$$
$$=a_0+\sum_{k=1}^{n}(a_k c^k+a_{-k}\bar{c}^k)=\int h\,P_c^{\#}\,d\lambda.$$

Thus (9) holds for all trigonometric polynomials h and hence for all $h\in C(\boldsymbol{T})$. Therefore it is an easy exercise to show that (9) holds for all $[0,\infty]$-valued Borel functions h and hence for all $h\in L^1(\boldsymbol{T})$.

(ii) Let α, c be as in Part (i), and let $\beta\in H^1(\boldsymbol{T})$ be such that $|\beta|\leq 1$ a. e. and $c\hat{\beta}(0)=0$. Then our proof of Theorem 1 combined with (9) shows that

$$(10)\qquad \sum_{k=1}^{\infty}\frac{1}{k}|(\alpha^*\beta^k)*f|\leq C_1P_c^{\#}*|f| \qquad \forall f\in H_0^1(\boldsymbol{T}).$$

Similarly we have

$$(11)\qquad \sum_{k=1}^{\infty}|(\alpha^*\beta^{nk})*f|^2\leq C_2^2(P_c^{\#}*|f|)^2 \qquad \forall f\in H_0^1(\boldsymbol{T})$$

under the hypotheses of Theorem 2. In case $\hat{\beta}(0)=0$, both (10) and (11) hold for all $f\in H^1(\boldsymbol{T})$.

(iii) Now we consider a special case. Fix any $c\in\boldsymbol{C}$ with $|c|<1$, and let $\alpha=\alpha_c$ denote the Möbius transformation defined by

$$\alpha(z)=\frac{c-z}{1-\bar{c}z} \qquad \forall z\in\boldsymbol{T}.$$

Thus α is an inner function with $\hat{\alpha}(0)=c$ and $\alpha\circ\alpha$ is nothing but the identity mapping on $\boldsymbol{T}$.

If $g\in L^1(\boldsymbol{T})$, then we have $\alpha^*g=(g\circ\alpha)P_c^{\#}\,\lambda$; in fact, $h\in C(\boldsymbol{T})$ implies

$$\int hd(\alpha^* g)=\int(h\circ\alpha)gd\lambda$$
$$=\int(h\circ\alpha)(g\circ\alpha\circ\alpha)d\lambda=\int h(g\circ\alpha)P_c^\# \ d\lambda$$

by (9). Therefore (10) and (11) become respectively

$$(10)' \qquad \sum_{k=1}^{\infty}\frac{1}{k}|((\beta\circ\alpha)^k P_c^\#)*f|\leq C_1 P_c^\# *|f| \qquad \forall f\in H_0^1(\boldsymbol{T}),$$

and

$$(11)' \qquad \sum_{k=1}^{\infty}|((\beta\circ\alpha)^{nk}P_c^\#)*f|^2\leq C_2^2(P_c^\# *|f|)^2 \qquad \forall f\in \mathrm{H}_0^1(\boldsymbol{T}).$$

Now let G, X, Y be three locally compact spaces, and let $u: G\times X\rightarrow Y$ be a Borel measurable mapping. Given $\nu\in M(G)$ and $\mu\in M(X)$, let $\nu *_u \mu$ denote the complex Borel measure on Y defined by

$$(12) \qquad \int_Y hd(\nu *_u \mu)=\iint h(u(t, x))d\nu(t)d\mu(x)$$

for all bounded Borel functions h on Y. It is readily seen that if u is continuous, then $\nu *_u \mu$ is a regular measure.

To give an example, let $\alpha: \boldsymbol{C}\rightarrow\boldsymbol{C}$ be a Borel function. Define $u: \boldsymbol{C}^2\rightarrow\boldsymbol{C}$ by setting $u(z, w)=\alpha(z)w$ *for* $z, w\in\boldsymbol{C}$. Regard $\boldsymbol{C}$ as a topological semigroup with respect to the usual multiplication of complex numbers. If $\nu, \mu\in M(\boldsymbol{C})$ and h is a bounded Borel function on $\boldsymbol{C}$, then

$$\int hd(\nu *_u \mu)=\iint h(\alpha(z)w)d\nu(z)d\mu(w)$$
$$=\iint h(zw)d(\alpha^*\nu)(z)d\mu(w)$$
$$=\int hd[(\alpha^*\nu)*\mu],$$

where $(\alpha^*\nu)*\mu$ denotes the convolution product of $\alpha^*\nu$ and μ on the topological semigroup $\boldsymbol{C}$. Consequently we have $\nu *_u \mu=(\alpha^*\nu)*\mu$.

THEOREM 6 (NOTATION AS BEFORE). *Suppose that one of the conditions in Lemma* 3 *obtains,* $\nu\in M(G)$ *and* $\mu\in M(X)$. *If* $(\phi\nu)*_u\mu=0$ *for all* $\phi\in\mathscr{A}$, *then*

$$\sum_{k=1}^{\infty}a_k\|(\gamma_k\nu)*_u\mu\|^p\leq a_0^p\|\nu\|^p\|\mu\|^p.$$

PROOF: Given $t\in G$, let μ_t denote complex Borel measure on Y defined by

(13) $$\langle h, \mu_t\rangle=\int_Y h d\mu_t=\int_X h(u(t, x))\, d\mu(x)$$

for all bounded Borel functions h on Y. Then we have

(14) $$\|\mu_t\|\leq\|\mu\| \qquad \forall t\in G$$

and

(15) $$\int\langle h, \mu_t\rangle\phi(t)\, d\nu(t)=\int hd(\phi\nu *_u\mu)$$

for all h as above and all $\phi\in L^1(\nu)$. Thus the assumption that $(\phi\nu)*_u\mu=0$ for all $\phi\in\mathscr{A}$ can be expressed as

(16) $$\int\langle h, \mu_t\rangle\phi(t)\, d\nu(t)=0 \qquad \forall\phi\in\mathscr{A}$$

whenever h is a bounded Borel function on Y.

Now choose and fix any finite constant $C>a_0$ and any natural number n. Suppose $z=(z_k)_1^n\in \boldsymbol{C}^n$ and sup $\{|z_k| : 1\leq k\leq n\}\leq 1$ (if $p=1$) or $\sum_{k=1}^n a_k|z_k|^{p'}\leq 1$ (if $p>1$). Then condition (b) of Lemma 3 yields $\phi\in\mathscr{A}$ such that

(17) $$\left\|\sum_{k=1}^n a_k z_k\gamma_k+\phi\right\|_\infty<C.$$

Notice that this inequality is valid for all z' in a sufficiently small neighborhood of z in $\boldsymbol{C}^n$. Therefore we can find finitely many simple Borel functions $g_1, \dots, g_m$ on $\boldsymbol{C}^n$ and $\phi_1, \dots, \phi_m\in\mathscr{A}$ such that

(18) $$\left\|\sum_{k=1}^n a_k z_k\gamma_k+\sum_{j=1}^m g_i(z)\phi_j\right\|_\infty<C.$$

for all $z\in \boldsymbol{C}^n$ as above.

In order to confirm the desired inequality, let $h_1, \dots, h_n$ be any bounded Borel functions on Y such that $\sup\left\{\|h_k\|_\infty : 1\leq k\leq n\right\}\leq 1$ (if $p=1$) or $\sum_{k=1}^n a_k\|h_k\|_\infty^{p'}\leq 1$ (if $p>1$). Define

$$H_i(y)=g_i(h_1(y), \dots, h_n(y)) \qquad \forall y\in Y \text{ and } j=1, 2, \dots, m.$$

Then each H_j is a simple Borel function on Y and (18) ensures that

(19) $$\left\|\sum_{k=1}^n a_k\gamma_k(t)h_k+\sum_{k=1}^m\phi_j(t)H_i\right\|_\infty\leq C \qquad \forall t\in G.$$

It follows that

$$\left|\sum_{k=1}^{n} a_k \int_Y (h_k d[(\gamma_k \nu) *_u \mu]\right| = \left|\sum_{k=1}^{n} a_k \int_G \langle h_k, \mu_t\rangle \gamma_k(t)\, d\nu(t)\right| \qquad \text{by (15)}$$

$$= \left|\int_G \langle \sum_{k=1}^{n} a_k \gamma_k(t) h_k + \sum_{j=1}^{m} \phi_j(t) H_j, \mu_t\rangle d\nu(t)\right| \qquad \text{by (16)}$$

$$\leq C\|\mu\|\cdot\|\nu\| \qquad \text{by (19) and (14).}$$

Since this holds for all $h_1, \ldots, h_n$ as above, we obtain $\sum_{k=1}^{n} a_k \|(\gamma_k \nu) *_u \mu\|^p \leq C^p \|\nu\|^p \cdot \|\mu\|^p$. As $n \in \boldsymbol{N}$ and $C > a_0$ are arbitrary, this completes the proof.

REMARK 7: Theorem 6 has a purely measure-theoretical version. Since this version is somewhat complicated, we shall merely give an example instead of stating it.

Let X be a measurable space, and let μ be a complex measure on $\boldsymbol{R}\times X$, where $\boldsymbol{R}$ is equipped with its Borel field. Define the "maximal" function M of μ by setting

$$M(s) = \sup\left|\int_{\boldsymbol{R}\times X} e^{-ist} h(x)\, d\mu(t, x)\right| \qquad \forall s \in \boldsymbol{R},$$

where the supremum is taken over all measurable functions h on X with $\|h\|_\infty \leq 1$. If $M(s) = 0$ for all $s < 0$, then

$$\text{(20)} \qquad \int_0^\infty s^{-1} M(s)\, ds \leq C_1 \|\mu\|,$$

where C_1 is any finite constant satisfying (1).

First of all, note that M is a continuous function on $\boldsymbol{R}$. To prove (20), pick any $\varepsilon > 0$ and any finitely many measurable functions $h_1, \ldots, h_n$ on X such that $\|h_k\|_\infty \leq 1$ for each k. Then the proof of Theorem 6 combined with (5) shows that there exist finitely many simple functions $H_1, \ldots, H_m$ on X such that

$$\text{(21)} \qquad \left|\sum_{k=1}^{n} k^{-1} z^k h_k(x) + \sum_{j=1}^{m} z^{-j} H_j(x)\right| < C_1 + \varepsilon \qquad \forall (z, x) \in \boldsymbol{T}\times X.$$

Upon replacing z by $e^{-i\varepsilon t}$ in (21) and integrating both sides of the resulting inequality with respect to $d|\mu|$, we get

$$\left|\sum_{k=1}^{n} k^{-1} \int e^{-i\varepsilon kt} h_k(x)\, d\mu(t, x)\right| \leq (C_1 + \varepsilon)\|\mu\|$$

since $M(s) = 0$ for all $s < 0$. Therefore $\sum_{k=1}^{n} k^{-1} M(\varepsilon k) \leq (C_1 + \varepsilon)\|\mu\|$; hence

$$\text{(22)} \qquad \sum_{k=1}^{\infty} (\varepsilon k)^{-1} M(\varepsilon k)\varepsilon \leq (C_1 + \varepsilon)\|\mu\|.$$

Since M is continuous, (20) is obtained from (22) by applying Fatou's Lemma.

Finally observe that if X is a locally compact abelian group with dual $\hat{X}$, if $\mu\in M(\boldsymbol{R}\times\boldsymbol{X})$, and if

$$(23)\qquad \int e^{-ist}\gamma(x)\,d\mu(t,x)=0 \qquad \forall s<0 \text{ and } \gamma\in\hat{X},$$

then $M(s)=0$ for all $s<0$.

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