Hokkaido Mathematical Journal Vol. 19 (1990) p. 281-287

On strongly separable extensions

Yasukazu YAMASHIRO

Dedicated to Professor Kazuhiko HIRATA on his 60th birthday

(Received April 4, 1989, Revised December 7, 1989)

E. McMahon and A. C. Mewborn introduced a type of separable extensions in [4], which is called strongly separable extension. In this paper, we shall study some properties of strongly separable extensions corresponding to H-separable extensions. In § 1, we give some equivalent conditions (1.4) and in § 2, we give the commutor theorem for strongly separable extensions (2.5).

The author expresses his gratitude to professor Kazuhiko Hirata for his advices during the preparation of this paper.

1. Strongly separable extensions

Let R be a ring and M and N left R-modules. We shall denote $M \rightsquigarrow N$ if M is a direct sum of submodules S and K such that ${}_RS < \oplus_R(N\oplus\cdots\oplus N)$ and $\mathrm{Hom}({}_RK, {}_RN)=0$. It is easy to see that K coincides with the reject of N in M (cf. [1]), which is defined by

$$\mathrm{Rej}_M(N)=\cap\{\ker f\,;\ f\in\mathrm{Hom}({}_RM, {}_RN)\}.$$

Using this notation, we can state that a ring Λ is a strongly separable extension of a subring Γ if and only if $\Lambda\otimes_\Gamma\Lambda \rightsquigarrow \Lambda$ as Λ-Λ-medules.

LEMMA 1. 1. *Let R be a ring and M and N left R-modules such that $M\rightsquigarrow N$. Then for every R-direct summand L_1 of M, $L_1\rightsquigarrow N$.*

PROOF. We can writ $M=L_1\oplus L_2$ and $M=S\oplus K$ with ${}_RS<\oplus_R(N\oplus\cdots\oplus N)$, $\mathrm{Hom}({}_RK, {}_RN)=0$.

Let π_1 and π_2 be projections of M to L_1 and L_2, respectively, and p_K the projection M to K. By (8.18) in [1], we have $K=\pi_1(K)\oplus\pi_2(K)$. Then the restriction of π_ip_K to L_i is the projection of L_i to $\pi_i(K)$ ($i=1, 2$). Hence we can write $L_1=S_1\oplus\pi_1(K)$ and $L_2=S_2\oplus\pi_2(K)$. Then we have $M=S\oplus K=S_1\oplus S_2\oplus K$ and $S\simeq M/K\simeq S_1\oplus S_2$. Hence $S_1<\oplus S<\oplus(N\oplus\cdots N)$. Since $\pi_1(K)<\oplus K$, $\mathrm{Hom}({}_R\pi_1(K), {}_RN)=0$. Then $L_1\rightsquigarrow N$.

Let $\Gamma\subset B\subset\Lambda$ be rings. In case the map $B\otimes_\Gamma\Lambda\longrightarrow\Lambda$ such that $b\otimes\lambda\longmapsto b\lambda$ for $b\in B$ and $\lambda\in\Lambda$ splits as a B-Λ-map, we shall call briefly that $B\otimes_\Gamma\Lambda\longrightarrow\Lambda$ splits. In this case, by tensoring on the left with Λ over B, $\Lambda\otimes_B\Lambda<\oplus\Lambda\otimes_\Gamma\Lambda$ as Λ-Λ-modules. So, from the above lemma, we obtain

PROPOSITION 1. 2. *Let Λ be a strongly separable extension of Γ. Then for every subring B of Λ such that $\Gamma\subset B$ and $B\otimes_{\Gamma}\Lambda\longrightarrow\Lambda$ splits, Λ is strongly separable over B.*

COROLLARY 1. 3. *Let Λ be a strongly separable extension of Γ. Then for every separable subextension B of Λ over Γ, Λ is strongly separable over B.*

For any Λ-Λ-module M, we denote by M^{Λ} the subset $\{m\in M\,;\,\lambda m=m\lambda$ for all $\lambda\in\Lambda\}$ of M, and for any subring A of Λ, we denote by $V_{\Lambda}(A)$ the commutor ring of A in Λ.

Let $\Gamma\subset\Lambda$ be arbitrary rings C the center of Λ and $\Delta=V_{\Lambda}(\Gamma)$. Then we always have a Λ-Λ-map $\varphi:\Lambda\otimes_{\Gamma}\Lambda\longrightarrow\mathrm{Hom}_{C}(\Delta,\Lambda)$ defined by $\varphi(\lambda\otimes\lambda')(\delta)=\lambda\delta\lambda'$ for $\lambda,\lambda'\in\Lambda$ and $\delta\in\Delta$. We shall denote its kernel by $R_{\Gamma}(\Lambda)$. Since $\mathrm{Hom}({}_{\Lambda}\Lambda\otimes_{\Gamma}\Lambda_{\Lambda},{}_{\Lambda}\Lambda_{\Lambda})\simeq\Delta$ by the map $f\longmapsto f(1\otimes1)$ for $f\in\mathrm{Hom}({}_{\Lambda}\Lambda\otimes_{\Gamma}\Lambda_{\Lambda},{}_{\Lambda}\Lambda_{\Lambda})$, $R_{\Gamma}(\Lambda)$ coincides with the reject of Λ in $\Lambda\otimes_{\Gamma}\Lambda$ as a Λ-Λ-module. In particular, if Λ is strongly separable over Γ then we can write

$$\Lambda\otimes_{\Gamma}\Lambda\simeq\mathrm{Hom}_{C}(\Delta,\Lambda)\oplus R_{\Gamma}(\Lambda)$$

as Λ-Λ-modules.

The next theorem is a generalization of Theorem 1. 2 in [6].

THEOREM 1. 4. *Let $\Gamma\subset\Lambda$ be rings, C the center of Λ and $\Delta=V_{\Lambda}(\Gamma)$. Then the following statements are equivalent.*

(1) *Λ is a strongly separable extension of Γ.*

(2) *For every Λ-Λ-module M,*

$$M^{\Gamma}=\Delta M^{\Lambda}\otimes X$$

such that the map $g:\Delta\otimes_{C}M^{\Lambda}\longrightarrow\Delta M^{\Lambda}$ defined by $g(\delta\otimes m)=\delta m$ for $\delta\in\Delta$ and $m\in M^{\Lambda}$ is an isomorphism and $X\in\mathrm{Rej}_{M}(\Lambda)$.

(3) $(\Lambda\otimes_{\Gamma}\Lambda)^{\Gamma}=\Delta(\Lambda\otimes_{\Gamma}\Lambda)^{\Lambda}\otimes X$

such that the map g for $M=\Lambda\otimes_{\Gamma}\Lambda$ is an isomorphism and $X\subset R_{\Gamma}(\Lambda)$.

PROOF. Assum (1). By (3. 10) in [4],

$$M^{\Gamma}\simeq(\Delta\otimes_{C}M^{\Lambda})\otimes\mathrm{Hom}({}_{\Lambda}R_{\Gamma}(\Lambda)_{\Lambda},{}_{\Lambda}M_{\Lambda}).$$

In this case, the injection $\mathrm{Hom}({}_{\Lambda}R_{\Gamma}(\Lambda)_{\Lambda},{}_{\Lambda}M_{\Lambda})\longrightarrow M^{\Gamma}$ is given by $f\longmapsto f(k)$ for $f\in\mathrm{Hon}({}_{\Lambda}R_{\Gamma}(\Lambda)_{\Lambda},{}_{\Lambda}M_{\Lambda})$, where k is the image of $1\otimes1$ in $R_{\Gamma}(\Lambda)$ by the projection $p:\Lambda\otimes_{\Gamma}\Lambda\longrightarrow R_{\Gamma}(\Lambda)$. For any $g\in\mathrm{Hom}({}_{\Lambda}M_{\Lambda},{}_{\Lambda}\Lambda_{\Lambda})$, $g\circ f\circ$

p is a map of $\Lambda\otimes_{\Gamma}\Lambda$ to Λ. Since $k\in R_{\Gamma}(\Lambda)$, the reject of Λ in $\Lambda\otimes_{\Gamma}\Lambda$, $g(f(k))=g\circ f\circ p(k)=0$. Then $g(X)=0$ and $X\subset \mathrm{Rej}_{M}(\Lambda)$. Hence (2) holds. If we put $M=\Lambda\otimes_{\Gamma}\Lambda$ then (2) implies (3). Assume (3). We can write

$$1\otimes 1=\Sigma_{ij}\delta_{i}x_{ij}\otimes y_{ij}+k$$

for some $\delta_{i}\in\Delta$, $\sum_{j}x_{ji}\otimes y_{ij}\in(\Lambda\otimes_{\Gamma}\Lambda)^{\Delta}$ and $k\in X$. By definition of $R_{\Gamma}(\Lambda)$,

$$\delta=\varphi(1\otimes 1)(\delta)=\textstyle\sum_{ij}\delta_{i}x_{ij}\delta y_{ij} \qquad \text{for all } \delta\in\Delta.$$

Hence Λ is strongly separable over Γ by (3.5) (2) in [4]. This completes the proof.

As a generalization of (3.4) in [2], we have

PROPOSITION 1. 5. *Let Γ be a ring R the center of Γ and Δ a separable R-algebra such that Δ is R-f. g. projective. Then $\Lambda=\Delta\otimes_{R}\Gamma$ is a strongly separable extension of Γ' where Γ' is a natural homomorphic image of $R\otimes_{R}\Gamma$ in $\Delta\otimes_{R}\Gamma$.*

PROOF. By Lemma 3 in [7], $\Lambda^{\Gamma}=\Delta$. By (3.3) in [4], Δ is strongly separable over R. Then we have $\Delta\otimes_{R}\Delta=S\oplus K$ with

$${}_{\Delta}S_{\Delta}<\oplus_{\Delta}(\Delta\oplus\cdots\oplus\Delta)_{\Delta},\ \mathrm{Hom}({}_{\Delta}K_{\Delta},{}_{\Delta}\Delta_{\Delta})=0.$$

Hence

$$\Lambda\otimes_{\Gamma}\Lambda=(S\otimes_{R}\Gamma)\oplus(K\otimes_{R}\Gamma)$$

with

$${}_{\Lambda}(S\otimes_{R}\Gamma)_{\Lambda}<\oplus_{\Lambda}(\Lambda\oplus\cdots\oplus\Lambda)_{\Lambda}.$$

On the orther hand,

$$\begin{aligned}\mathrm{Hom}({}_{\Lambda}K\otimes_{R}\Gamma_{\Lambda},{}_{\Lambda}\Lambda_{\Lambda})&=\mathrm{Hom}({}_{\Gamma-\Delta}K\otimes\Gamma_{\Gamma-\Delta},{}_{\Gamma-\Delta}\Lambda_{\Gamma-\Delta})\\ &=\mathrm{Hom}({}_{\Delta}K_{\Delta},{}_{\Delta}\mathrm{Hom}({}_{\Gamma}\Gamma_{\Gamma},{}_{\Gamma}\Lambda_{\Gamma})_{\Delta})=\mathrm{Hom}({}_{\Delta}K_{\Delta},{}_{\Delta}\Delta_{\Delta})=0\end{aligned}$$

Thus $\Lambda\otimes_{\Gamma}\Lambda\rightsquigarrow\Lambda$ as Λ-Λ-modules and Λ is strongly separable over Γ.

For any Λ-Λ-module M, we call that M is centrally projective over Λ if M is a direct summand of a finite direct sum of copies of Λ as a Λ-Λ-module.

COROLLARY 1. 6. *Let Λ be a separable extension of Γ such that Λ is Γ-centrally projective. Then Λ is strongly separable over Γ.*

PROOF. Let R be the center of Γ and $\Delta=V_{\Lambda}(\Gamma)$. By (5.6) in [3], $\Lambda=\Gamma\otimes_{R}\Delta$ and Δ is R-f. g. projective. By Theorem 2 in [7], Δ is a separable R-algebra. Then by (1.5), Λ is strongly separable over Γ.

Let Λ be a strongly separable extension of Γ, C the center of Λ, $\Delta = V_\Lambda(\Gamma)$ and $\Gamma' = V_\Lambda(\Delta)$. By (3.6) in [4], Λ is strongly separable over Γ', and by (3.9) in [4], the map $\Delta \otimes_C \Lambda \longrightarrow \mathrm{Hom}({}_\Gamma\Lambda, {}_\Gamma\Lambda)$ defined by $\delta \otimes \lambda \longmapsto [\lambda' \longmapsto \delta\lambda'\lambda]$ for $\delta \in \Delta$ and $\lambda, \lambda' \in \Lambda$ is a (split) monomorphism. Then a generalization of (2.2) in [4] can be obtained in the same way.

PROPOSION 1.8. *Let Λ be a strongly separable extension of Γ such that $\Lambda = \Delta\Gamma$. Then $\Lambda \simeq \Delta \otimes_C \Gamma'$, Δ is a central C-separable algebra and Λ is an H-separable extension of Γ', where $\Gamma' = V_\Lambda(\Delta)$.*

PROOF. The latter assertion follows from (3.4) in [2].

2. Commutor theorem

Throughout this section, whenever we denote a ring and its subring by Λ and Γ, respectively, we denote the center of Λ by C and $V_\Lambda(\Gamma) = \Delta$.

Let $\mathscr{B}_l$ be the set of subrings B of Λ such that $\Gamma \subset B$, $B \otimes_\Gamma \Lambda \longrightarrow \Lambda$ splits and there exists a B-Γ-projection $p_B : \Lambda \longrightarrow B$ such that $(1_\Lambda \otimes p_B)(R_B(\Lambda)) = 0$, where 1_Λ is the identity map of Λ and $1_\Lambda \otimes p_B$ is the map of $\Lambda \otimes_B \Lambda$ to Λ given by $(1_\Lambda \otimes p_B)(\lambda \otimes \lambda') = \lambda p_B(\lambda')$ for $\lambda, \lambda' \in \Lambda$, and $\mathscr{D}_l$ the set of C-subalgebras D of Δ such that ${}_D D < \oplus_D \Delta$ and $D \otimes_C \Delta \longrightarrow \Delta$ splits. $\mathscr{B}_r$ and $\mathscr{D}_r$ are defined similarly. Furthermore, let $\mathscr{B}$ be the set of subrings B of Λ such that B is a separable extension of Γ and there exists a B-B-projection $p_B : \Lambda \longrightarrow B$ such that $(1_\Lambda \otimes p_B)(R_B(\Lambda)) = 0$ and $\mathscr{D}$ the set of separable C-subalgebras of Δ.

Firstly, we prove

PROPOSION 2.1. *Let Λ be a strongly separable extension of Γ, D a C-subalgebra of Δ such that $D \otimes_C \Delta \longrightarrow \Delta$ splits, and $B = V_\Lambda(D)$. Then there exists a B-Γ-projection $p_B : \Lambda \longrightarrow B$ such that $(1_\Lambda \otimes p_B)(R_B(\Lambda)) = 0$ and the map $\psi_B : B \otimes_\Gamma \Lambda \longrightarrow \mathrm{Hom}({}_D\Delta, {}_D\Lambda)$ defined by $\psi_B(b \otimes \lambda)(\delta) = b\delta\lambda$ for $b \in B$, $\lambda \in \Lambda$ and $\delta \in \Delta$ is a split epimorphism as a B-Λ-map. If furthermore ${}_D D < \otimes_D \Delta$, then $B \otimes_\Gamma \Lambda \longrightarrow \Lambda$ splits.*

PROOF. Let $\sum_i d_i \otimes \delta_i \in (D \otimes_C \Delta)^D$ such that $\sum_i d_i \lambda \delta_i = 1$. If we put $p_B : \Lambda \longrightarrow B$ by $p_B(\lambda) = \sum_i d_i \lambda \delta_i$ for $\lambda \in \Lambda$, and $\pi_D : \mathrm{Hom}_C(\Delta, \Lambda) \longrightarrow \mathrm{Hom}({}_D\Delta, {}_D\Lambda)$ by $\pi_D(f)(\delta) = \sum_i d_i f(\delta_i \delta)$ for $\delta \in \Delta$ and $f \in \mathrm{Hom}_C(\Gamma, \Lambda)$ then these maps are split epimorphisms as a B-Γ-map and a B-Λ-map, respectively. Now, consider the commutative diagram

$$\begin{array}{ccc} \Lambda\otimes_\Gamma\Lambda & \xrightarrow{\varphi} & \mathrm{Hom}_C(\Delta, \Lambda) \\ {\scriptstyle p_B\otimes 1_\Lambda}\downarrow & & \downarrow{\scriptstyle \pi_D} \\ B\otimes_\Gamma\Lambda & \xrightarrow{\psi_B} & \mathrm{Hom}({}_D\Delta, {}_D\Lambda). \end{array}$$

Since φ is a split eprimorphism, ψ_B is a split epimorphism. If we put η: $\mathrm{Hom}_C(D', \Lambda)\longrightarrow\Lambda$ by $\eta(f)=\sum f(d_i)\delta_i$ for $f\in\mathrm{Hom}_C(D', \Lambda)$, where $D'=V_\Lambda(B)$, we have a commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow R_B(\Lambda) \longrightarrow & \Lambda\otimes_B\Lambda & & \xrightarrow{\psi_B} & & \mathrm{Hom}_C(D', \Lambda) \\ & {\scriptstyle 1_\Lambda\otimes p_B}\searrow & & & \swarrow{\scriptstyle \eta} & \\ & & & \Lambda & & \end{array}$$

where the row is exact. Then we have

$$(1_\Lambda\otimes p_B)(R_\Gamma(\Lambda))=\eta\circ\varphi_B(R_\Gamma(\Lambda))=0.$$

Consider the commutative diagram

$$\begin{array}{ccc} B\otimes_\Gamma\Lambda & \xrightarrow{\psi_B} & \mathrm{Hom}({}_D\Delta, {}_D\Lambda) \\ & \searrow \quad \swarrow{\scriptstyle \alpha} & \\ & \Lambda & \end{array}$$

where α is the map given by $\alpha(f)=f(1)$ for $f\in\mathrm{Hom}({}_D\Delta, {}_D\Lambda)$. If ${}_DD<\oplus{}_D\Delta$, then α is a split epimorphism and $B\otimes_\Gamma\Lambda\longrightarrow\Lambda$ splits.

PROPOSITION 2. 2. *Let Λ be a strongly separable extension of Γ. Then for every $B\in\mathscr{B}_l$, $V_\Lambda(B)\in\mathscr{D}_l$.*

PROOF. Since $B\otimes_\Gamma\Lambda\longrightarrow\Lambda$ splits, we have ${}_DD<\oplus{}_D\Delta$, where $D=V_\Lambda(B)$. By (1. 2), Λ is strongly separable over B. D is C-$f.g.$ projective since ${}_DD<\oplus{}_D\Delta$. Then we have the following isomorphisms

$$\begin{aligned} &\mathrm{Hom}({}_B\Lambda_\Gamma, {}_B\Lambda_\Gamma)\simeq\mathrm{Hom}({}_\Lambda\Lambda\otimes_B\Lambda_\Gamma, {}_\Lambda\Lambda_\Gamma) \\ &\simeq\mathrm{Hom}({}_\Lambda\mathrm{Hom}_C(D, \Lambda)_\Gamma, {}_\Lambda\Lambda_\Gamma)\oplus\mathrm{Hom}({}_\Lambda R_B(\Lambda)_\Gamma, {}_\Lambda\Lambda_\Gamma) \\ &\simeq(D\otimes_C\Delta)\oplus\mathrm{Hom}({}_\Lambda R_B(\Lambda)_\Gamma, {}_\Lambda\Lambda_\Gamma). \end{aligned}$$

In the above direct decomposition, the injection $\psi_D: D\otimes_C\Delta\longrightarrow\mathrm{Hom}({}_B\Lambda_\Gamma, {}_B\Lambda_\Gamma)$ is given by $\psi_D(d\otimes\delta)(\lambda)=d\lambda\delta$ for $d\in D$, $\delta\in\Delta$ and $\lambda\in\Lambda$. Clearly ψ_D is the D-Λ-homomorphism. In this case, the action of D and Δ to $\mathrm{Hom}({}_B\Lambda_\Gamma, {}_B\Lambda_\Gamma)$ is given by $(df)(\lambda)=df(\lambda)$ and $(f\delta)(\lambda)=f(\lambda)\delta$ for $d\in$

D, $\delta\in\Lambda$ and $f\in \mathrm{Hom}({}_B\Lambda_\Gamma, {}_B\Lambda_\Gamma)$. Let $\alpha: \mathrm{Hom}({}_B\Lambda_\Gamma, {}_B\Lambda_\Gamma)\longrightarrow \mathrm{Hom}({}_\Lambda R_B(\Lambda)_\Gamma, {}_\Lambda\Lambda_\Gamma)$ be the projection in the above decomposition, and M the map of $\Lambda\otimes_B\Lambda$ to Λ given by $M(\lambda\otimes\lambda')=\lambda\lambda'$ for $\lambda, \lambda'\in\Lambda$. Then $\alpha(f)(x)=M(1_\Lambda\otimes f)(x)$ for $f\in\mathrm{Hom}({}_B\Lambda_\Gamma, {}_B\Lambda_\Gamma)$ and $x\in R_\Gamma(\Lambda)$. Since $\alpha(p_B)=0$ by the definition of $\mathscr{B}_l$, we have $p_B\in\psi_D(D\otimes_C\Delta)$. Hence there exists $\sum d_i\otimes\delta_i\in D\otimes_C\Delta$ such that $p_B=\psi_D(\sum d_i\otimes\delta_i)$. Then we have

$$\sum d_i\delta_i=\psi_D(\sum d_i\otimes\delta_i)(1)=p_B(1)=1,$$

and for any $d\in D$,

$$\psi_D(\sum dd_i\otimes\delta_i)=d\psi_D(\sum d_i\otimes\delta_i)=dp_B=p_Bd=\psi_D(\sum d_i\otimes\delta_i)d= \psi_D(\sum d_i\otimes\delta_i d)$$

as the image of p_B is B. Since ψ_D is a monomorphism, $\sum dd_i\otimes\delta_i=\sum d_i\otimes\delta_i d$. Then $\sum d_i\otimes\delta_i\in(D\otimes_C\Delta)^D$ and this implies $D\otimes_C\Delta\longrightarrow\Delta$ splits.

As a generalization of Proposition 1.2 in [6], we have the next lemma.

LEMMA 2.3. *Let $\Gamma\subset\Lambda$ be rings and there exists a left Γ-projection $p:\Lambda\longrightarrow\Gamma$ such that $(1_\Lambda\otimes p)(R_\Gamma(\Lambda))=0$, the $V_\Lambda(V_\Lambda(\Gamma))=\Gamma$.*

PROOF. Let $x\in V_\Lambda(V_\Lambda(\Gamma))$. By definition of $R_\Gamma(\Lambda)$, $x\otimes1-1\otimes x\in R_\Gamma(\Lambda)$. By hypothesis, we have $x-p(x)=0$ and $x\in\Gamma$.

LEMMA 2.4. *Let Λ be a strongly separable extension of Γ. Then for every $D\in\mathscr{D}_l$, $V_\Lambda(V_\Lambda(D))=D$.*

PROOF. Since $D\otimes_C\Delta\longrightarrow\Delta$ splits and Δ is C-f.g. projective, Δ is left D-f.g. projective. Let $B=V_\Lambda(D)$ and $D'=V_\Lambda(B)$. By (2.1), ${}_B\mathrm{Hom}({}_D\Delta, {}_D\Lambda)_\Lambda<\otimes_B B\otimes_\Gamma\Lambda_\Lambda$. Then we have

$$D'\otimes_D\Delta\simeq\mathrm{Hom}({}_B\Lambda_\Lambda, {}_B\Lambda_\Lambda)\otimes_D\Delta\simeq\mathrm{Hom}({}_B\mathrm{Hom}({}_D\Delta, {}_D\Lambda)_\Lambda, {}_B\Lambda_\Lambda)$$
$$<\oplus\mathrm{Hom}({}_B B\otimes_\Gamma\Lambda_\Lambda, {}_B\Lambda_\Lambda)\simeq\mathrm{Hom}({}_B B_\Gamma, {}_B\Lambda_\Gamma)\simeq\Delta.$$

Hence the map $D'\otimes_D\Delta\longrightarrow\Delta$ ginen by $d'\otimes\delta\longrightarrow d'\delta$ is injective. Since this map is always surjective, $D'\otimes_D\Delta\simeq\Delta$. Then $D'=D$, since ${}_DD<\oplus_D\Delta$.

Now, we can obtain the commutor theorem for strongly separable extensions, which is a generalization of (1.3) in [9].

THEOREM 2.5. *Let Λ be a strongly separable extension of Γ, and consider the correspondence $V: A \rightsquigarrow V_\Lambda(A)$ for a subring A of Λ. Then we have*

(1) *V yields a one to one correspondence between $\mathscr{B}_l$ and $\mathscr{D}_l$ (resp. $\mathscr{B}_r$ and $\mathscr{D}_r$) such that $V^2=$identity.*

(2) *V yields a one to one correspondence between $\mathscr{B}$ and $\mathscr{D}$ such that $V^2=$identity.*

PROOF. (1) For any $B\in\mathscr{B}_l$, $V_A(B)\in\mathscr{D}_l$ by (2.2) and $V_A(V_A(B))=B$ by (2.3). For any $D\in\mathscr{D}_l$, $V_A(D)\in\mathscr{B}_l$ by (2.1) and $V_A(V_A(D))=D$ by (2.4).

(2) Since $\mathscr{B}\subset\mathscr{B}_l$, for any $B\in\mathscr{B}$, $V_A(V_A(B))=B$ and $V_A(B)=D\in\mathscr{D}_l$. Since $B\otimes_\Gamma B\longrightarrow B$ splits, ${}_DD_D<\oplus_D\Delta_D$. Hence D is a C-separable algebra by (1.4) in [9].

By (1.1) in [9], $\mathscr{D}\subset\mathscr{D}_l$. Then for any $D\in\mathscr{D}$, $V_A(V_A(D))=D$ and $V_A(D)=B\in\mathscr{B}_l$. Since $D\otimes_C D\longrightarrow D$ splits, ${}_BB_B<\oplus_B\Lambda_B$. Hence B is separable over Γ by (1.4) in [9].

References

[1] F. W. ANDERSON and K. R. FULLER: Rings and Categories of Modules, Springer G. T. M. **13**, 1974.

[2] K. HIRATA: Some types of separable extensions of rings, Nagoya Math. J. **33** (1968), 107-115.

[3] K. HIRATA: Separable extensions and centralizers of rings, Nagoya Math. J. **35** (1969), 31-45.

[4] E. MCMAHON and A. MEWBORN: Separable extensions of noncommutative rings, Hokkaido Math. J. **13** (1984), 74-88.

[5] T. NAKAMOTO and K. SUGANO: Note on H-separable extensions, Hokkaido Math. J. **4** (1975), 295-299.

[6] K. SUGANO: Note on semisimple extensions and separable extensions, Osaka J. Math. **4** (1967), 266-270.

[7] K. SUGANO: Separable extensions and Frobenius extensions, Osaka J. Math. **7** (1970) 291-299.

[8] K. SUGANO: On centralizers in separable extensions II, Osaka J. Math. **8** (1971), 465-469.

[9] K. SUGANO: On some commutor theorem of rings, Hokkaido Math. J. **1** (1972), 242-249.

[10] K. SUGANO: On projective H-separable extensions, Hokkaido Math. J. **5** (1976), 44-54.

Department of Mathematics
Faculty of Science
Chiba University
Ciba 260, Japan