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On the bisectable metrics on the 2-sphere S^2

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Introduction

A Riemannian manifold (M, g) is called an $SC_{2\pi}$-manifold, and its Riemannian metric g an $SC_{2\pi}$-metric on M, if all of its (maximal) geodesics are simply closed and have the same length 2π. For example, the standard 2-sphere (S^2, g_o) is an $SC_{2\pi}$-manifold. Moreover it is known that there are many $SC_{2\pi}$-metrics on S^2 which are essentially different from each other (see [1], Chapter 4, C, and [2] for details). Among $SC_{2\pi}$-metrics on S^2, L. W. Green characterized the standard 2-sphere (S^2, g_o) in terms of the Blaschke condition (see [1] p. 143). It seems interesting to study another sufficient condition for an $SC_{2\pi}$-manifold (S^2, g) to be isometric to (S^2, g_o).

An $SC_{2\pi}$-metric g on S^2 has the property that every geodesic γ divides S^2 into two domains. More precisely, there are two domains (connected open subsets) H_1 and H_2 in S^2 which satisfy $S^2=\gamma\cup H_1\cup H_2$ (disjoint union) and $\gamma=\partial H_1=\partial H_2$, where γ should be confounded with a subset of S^2. Either of the two domains is called a hemisphere with respect to g. In the case of the standard metric g_o all of its hemispheres have the same area 2π. This suggests the following definition:

An $SC_{2\pi}$-manifold (S^2, g) is a (geodesically) bisectable manifold if each geodesic divides S^2 into two hemispheres having the same area. The $SC_{2\pi}$-metric g on S^2 is then called a (geodesically) bisectable metric.

Here, we are led to the following conjecture:

If an $SC_{2\pi}$-metric g on S^2 satisfies the bisectability condition, then (S^2, g) is isometric to (S^2, g_o).

The main purpose of this paper is to give a partial affirmative answer to this conjecture.

Let us consider a one-parameter deformation $\{g_t\}$ of the standard metric g_o on S^2 (such that $g_t|_{t=0}=g_o$). Then we define a symmetric 2-form h on S^2 by $h=\frac{\partial}{\partial t}g_t|_{t=0}$, which is called the linearization of $\{g_t\}$ (at $t=0$).

The one-parameter deformation $\{g_t\}$ is said to be a one-parameter bisectable deformation of g_o on S^2 if each g_t is a bisectable metric. Furthermore a symmetric 2-form h on S^2 is said to be an infinitesimal bisectable deformation of g_o on S^2 if it is the linearization of a certain one-parameter bisectable deformation of g_o on S^2.

Here, our main results may be stated as follows:

THEOREM A. *Any infinitesimal bisectable deformation h of g_o on S^2 is trivial, that is, there exists a vector field X on S^2 such that $h=\mathscr{L}_X g_o$, $\mathscr{L}_X$ being the Lie derivation with respect to X.*

THEOREM B. *Let $\{g_t\}_{t\in I}$ be a one-parameter bisectable deformation of g_o on S^2, where I is an open interval containing 0. If g_t depends real analytically on the parameter t, then there exists a one-parameter family $\{\Psi_t\}_{t\in I}$ of transformations of S^2, defined on the same interval I, such that $\Psi_0=$ identity and $g_t=\Psi_t{}^* g_o$.*

In § 1 we study even functions and hemispherically even functions on (S^2, g_o). As usual an even function means a function on (S^2, g_o) which is invariant under the antipodal transformation of (S^2, g_o). A hemispherically even function is defined as a function f on (S^2, g_o) which satisfies

$$\int_{H_1} f dA = \int_{H_2} f dA ,$$

where (H_1, H_2) is any pair of standard hemispheres determined by a geodesic or a great circle of (S^2, g_o), and dA is the standard area element of (S^2, g_o). We show there the following

THEOREM C. *A function on (S^2, g_o) is an even function if and only if it is a hemispherically even function.*

In § 2 we show a necessary and sufficient condition for an $SC_{2\pi}$-metric on S^2 to be bisectable. In § 3 we prove Theorem A by using a corollary to Theorem C. In § 4 we prove Theorem B, following the argument due to N. Tanaka in [3].

Throughout this paper, we assume the differentiability of class C^∞ unless otherwise stated.

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§ 1. Hemispherically even functions

In this section every function on (S^2, g_o) is understood to be continuous. As usual a function f on (S^2, g_o) is called an odd function if $\tau^* f = -f$, where τ is the antipodal transformation of (S^2, g_o). We note that every function is uniquely written as a sum of an even function and an odd function. To prove Theorem C, it is therefore sufficient to show the following

THEOREM C′. *Let f be a hemispherically even function on (S^2, g_o). If f is an odd function, then f vanishes identically.*

PROOF. We utilize the canonical polar coordinate (θ, φ) on (S^2, g_o) which is defined for $0 \leq \theta \leq \pi$, $0 \leq \varphi < 2\pi$. In this coordinate the north pole N corresponds to $\theta = 0$. We denote by γ_λ the meridian great circle which is represented by $\varphi = \lambda$, $\pi + \lambda$. If $0 < \lambda < \pi$, it can be immediately seen that γ_0 and γ_λ split (S^2, g_o) into four pieces of domains $D(0 < \varphi < \lambda)$, $D(\lambda < \varphi < \pi)$, $D(\pi < \varphi < \pi + \lambda)$ and $D(\pi + \lambda < \varphi < 2\pi)$, where $D(a < \varphi < b)$ means the domain represented by $0 < \theta < \pi$, $a < \varphi < b$. Since f is a hemispherically even function on (S^2, g_o), we have

$$\int_{D(0<\varphi<\lambda)} f dA = \int_{D(\pi<\varphi<\pi+\lambda)} f dA \qquad \text{for } \lambda \in]0, \pi[.$$

This can be written in the form

$$\int_0^\lambda \int_0^\pi f(\theta, \varphi) \sin\theta \, d\theta d\varphi = \int_0^{\pi+\lambda} \int_0^\pi f(\theta, \varphi) \sin\theta \, d\theta d\varphi \quad \text{for } \lambda \in]0, \pi[.$$

Since f is an odd function on (S^2, g_o), it follows that

$$\int_0^\lambda \int_0^\pi f(\theta, \varphi) \sin\theta \, d\theta d\varphi = 0 \qquad \text{for } \lambda \in]0, \pi[.$$

Differentiating this equation with respect to λ, we find that

$$\int_0^\pi f(\theta, \lambda) \sin\theta \, d\theta = 0 \qquad \text{for } \lambda \in]0, \pi[.$$

This implies that

$$\int_0^\pi f(c(s)) \sin s \, ds = 0$$

for any great hemicircle $c : [0, \pi] \longrightarrow (S^2, g_o)$ parametrized by arc-length s.

Now, we take an arbitrary point q of (S^2, g_o), and show that $f(q) = 0$. Clearly we may assume that q is the north pole N. Let

$c: [0, \pi] \longrightarrow (S^2, g_o)$ be a great hemicircle parametrized by arc-length s which satisfies the following conditions:

1) It issues from the point represented by $(\pi/2, 0)$ in the canonical polar coordinate.

2) It does not pass through the north pole N.

3) The upper standard hemisphere contains $c(]0, \pi[)$.

Let S be the antipodal point of N, that is, the south pole. Let r be the angle between the NS-axis and the plane which contains the great hemicircle c. Then, in the canonical polar coordinate, c can be represented in the form

$$c(s) = (\theta(s), \ \varphi(s)),$$

and $(\theta, \varphi) = (\theta(s), \varphi(s))$ is related to s by the following rule:

$$\begin{cases} \sin\theta\cos\varphi = \cos s \\ \sin\theta\sin\varphi = \sin s \sin r, \\ \cos\theta \qquad\quad = \sin s \cos r \end{cases}$$

Recalling the argument above, we have

$$\int_0^\pi f(\theta(s), \varphi(s)) \sin s \, ds = 0.$$

Performing the change of the variables, $s \longrightarrow \theta$, in this integral equation, we get

$$\int_r^{\frac{\pi}{2}} \{f(\theta, \varphi(\theta)) + f(\theta, \pi - \varphi(\theta))\} \frac{\sin\theta\cos\theta}{\sqrt{\sin^2\theta - \sin^2 r}} d\theta = 0.$$

Since this equation is invariant under the S^1-action of revolution whose rotation axis is the NS-axis, it follows that

$$\int_r^{\frac{\pi}{2}} \{f(\theta, \varphi(\theta) + u) + f(\theta, \pi - \varphi(\theta) + u)\} \frac{\sin\theta\cos\theta}{\sqrt{\sin^2\theta - \sin^2 r}} d\theta = 0.$$

for any $u \in [0, 2\pi[$. Thus this gives

$$\int_0^{2\pi} \int_r^{\frac{\pi}{2}} \{f(\theta, \varphi(\theta) + u) + f(\theta, \pi - \varphi(\theta) + u)\} \frac{\sin\theta\cos\theta}{\sqrt{\sin^2\theta - \sin^2 r}} d\theta du = 0.$$

Putting $F(\theta) = \int_0^{2\pi} f(\theta, u) du$, we see from the Fubini theorem that

$$\int_r^{\frac{\pi}{2}} \frac{F(\theta) \sin\theta\cos\theta}{\sqrt{\sin^2\theta - \sin^2 r}} d\theta = 0.$$

Here we remark that this equation holds for any $r \in]0, \pi/2[$, and $F(\theta)$ is independent of r. Thus by the same way as in the proof of Theorem 4.13 in [1], pp. 102-104, we obtain

$$F(\theta)=\int_0^{2\pi} f(\theta, u)\,du=0 \qquad \text{for any } \theta \in]0, \frac{\pi}{2}[.$$

Letting θ tend to 0, we get

$$f(N)=0. \qquad \text{Q. E. D.}$$

A function f on (S^2, g_o) is called a hemispherically zero function if it satisfies

$$\int_H f dA=0$$

for any standard hemisphere H of (S^2, g_o).

COROLLARY. *Every hemispherically zero function on (S^2, g_o) is an even function on (S^2, g_o).*

REMARK. R. Michel has obtained the following result which is equivalent to this corollary:

THEOREM (R. Michel [4]). *Let ω be a 1-form on the 2-dimensional standard real projective space $(\boldsymbol{RP}^2, \hat{g}_o)$. If the integral of ω on any closed geodesic of $(\boldsymbol{RP}^2, \hat{g}_o)$ vanishes, then ω is exact.*

In fact, it can be easily seen that a function on (S^2, g_o) is an odd function if and only if there exists a 1-form ω on $(\boldsymbol{RP}^2, \hat{g}_o)$ such that

$$f dA=d(p^*\omega),$$

where p is the canonical projection of (S^2, g_o) onto $(\boldsymbol{RP}^2, \hat{g}_o)$.
Furthermore we have

$$\int_H f dA=\int_H d(p^*\omega)=\int_{\partial H} p^*\omega=2\int_{p(\partial H)}\omega$$

for any standard hemisphere H of (S^2, g_o). From these facts follows immediately the equivalence.

§ 2. The bisectability condition

In this section we discuss the bisectability condition for an $SC_{2\pi}$-metric on S^2. For this purpose we may only deal with an $SC_{2\pi}$-metric g of the following form:

$$g=e^{2\rho}g_o\,,$$

where ρ is a certain function on S^2.

PROPOSITION. *Let $g=e^{2\rho}g_o$ be an $SC_{2\pi}$-metric on S^2. Then, g is a bisectable metric if and only if the following equality holds for each hemisphere H with respect to g:*

$$\int_H (1-e^{2\rho}+\Delta\rho)\,dA=0\,,$$

where Δ and dA are the Laplacian and the area element of g_o respectively.

PROOF. Let K and dA_ρ be the Gaussian curvature and the area element with respect to $g=e^{2\rho}g_o$ respectively. A short calculation gives

$$K-e^{-2\rho}=e^{-2\rho}\Delta\rho\,.$$

Since $\int_H K dA_\rho=2\pi$ by the Gauss-Bonnet theorem, it follows that

$$2\pi-\int_H dA_\rho=\int_H (1-e^{2\rho}+\Delta\rho)\,dA\,.$$

By the Weinstein's theorem ([1] p. 59 and [5]) we know that the area of (S^2, g) is equal to 4π. Hence we obtain the proposition. Q. E. D.

§ 3. Proof of Theorem A

Now, we will prove Theorem A. Let h be an infinitesimal bisectable deformation of g_o on S^2, that is, it is the linearization of a certain one-parameter bisectable deformation $\{g_t\}$ of g_o on S^2. For the one-parameter deformation, we can find a smooth one-parameter family $\{\varphi_t\}$ of transformation of S^2 such that $\varphi_0=$identity and $\varphi_t{}^*g_t$ is of the form:

$$\varphi_t{}^*g_t=\exp(2\rho_t)g_o\,,$$

where $\{\rho_t\}$ is a certain one-parameter family of functions on S^2 which satisfies $\rho_0\equiv 0$ (see [6], the proof of lemma 1, and [7]). Then h can be written as

$$h=2\dot{\rho}g_o-\mathscr{L}_Y g_o\,,$$

where $\dot{\rho}$ is the derivation of ρ_t with respect to t at $t=0$, and Y is the infinitesimal transformation of the one-parameter family $\{\varphi_t\}$ of transformations of S^2. Since $\varphi_t{}^*g_t=\exp(2\rho_t)g_o$ is also a one-parameter bisectable deformation of g_o, it follows from Proposition in § 2 that

$$\int_{H_t}\{1-\exp(2\rho_t)+\Delta\rho_t\}dA=0$$

for each hemisphere H_t with respect to $\varphi_t{}^*g_t$. Notice that $1-\exp(2\rho_0)+\Delta\rho_0\equiv 0$. Differentiating this equation with respect to t at $t=0$, we have

$$\int_H(\Delta\dot{\rho}-2\dot{\rho})dA=0$$

for each standard hemisphere H of (S^2, g_o), or, in other words, $\Delta\dot{\rho}-2\dot{\rho}$ is a hemispherically zero function on (S^2, g_o). By Corollary in § 1 this equation implies that $\Delta\dot{\rho}-2\dot{\rho}$ is an even function on (S^2, g_o). On the other hand we assert that $\Delta\dot{\rho}-2\dot{\rho}$ is an odd function on (S^2, g_o). Indeed, $h=2\dot{\rho}g_o$ satisfies the so-called zero energy condition (cf. [1] p. 151), because $\{g_t\}$ is a one-parameter $SC_{2\pi}$-deformation of g_o on S^2, that is, each g_t is an $SC_{2\pi}$-metric on S^2. It follows that $\dot{\rho}$ and hence $\Delta\dot{\rho}-2\dot{\rho}$ are odd functions on (S^2, g_o), which proves our assertion (see [1] p. 123). We have therefore shown that

$$\Delta\dot{\rho}=2\dot{\rho}\ .$$

Hence we see that $\dot{\rho}$ is the restriction of a certain linear function on $\boldsymbol{R}^3$ to (S^2, g_o) (see [8] p. 160). As is well known, this implies that there is an infinitesimal conformal transformation X of (S^2, g_o) such that $\mathscr{L}_X g_o=2\dot{\rho}g_o$, which proves Theorem A.

§ 4. Proof of Theorem B

In this section, we will give the proof of Theorem B, following the same reasoning as in Appendix in [3].

Let $\{h_t\}_{t\in I}$ and $\{\tilde{h}_t\}_{t\in I}$ be two one-parameter families of symmetric 2-forms on S^2, I being an open interval containing 0. By the notation $h_t\equiv\tilde{h}_t \pmod{t^m}$ we mean that there is a one-parameter family $\{k_t\}_{t\in I}$ of symmetric 2-forms on S^2 such that

$$h_t=\tilde{h}_t+t^m k_t\ , \qquad t\in I.$$

The same notation will be also used for one-parameter families of functions on S^2.

LEMMA. *Let $\{\bar{g}_t\}$ be a one-parameter bisectable deformation of g_o on*

S^2. *Assume that* $\bar{g}_t$ *is of the following form*:

$$\bar{g}_t = \exp(2\rho_t) g_o ,$$

where ρ_t *is a certain function on* S^2. *Then, there exist a series of infinitesimal conformal transformations* $\{X^{(i)}\}_{i=0}^{\infty}$ *of* (S^2, g_o) *such that for each integer* $m \geq 0$

$$(*)_m \qquad \mathscr{L}_{X_t} \bar{g}_t \equiv \frac{\partial}{\partial t} \bar{g}_t \pmod{t^{m+1}} ,$$

where $X_t = \sum_{i=0}^{m} t^i X^{(i)}$.

PROOF. We will define $\{X^{(i)}\}_{i=0}^{\infty}$ inductively as follows. By the proof of Theorem A we can find an infinitesimal conformal transformation $X^{(0)}$ such that

$$\mathscr{L}_{X^{(0)}} g_o = 2\dot{\rho} g_o .$$

This implies $(*)_0$. Now, we assume that there are infinitesimal conformal transformations $\{X^{(i)}\}_{i=0}^{m}$ which satisfy $(*)_m$. Let $\{\Phi_t^{(m)}\}$ be the (smooth) one-parameter family of conformal transformations of (S^2, g_o) generated by $X_t = \sum_{i=0}^{m} t^i X^{(i)}$, that is, $\Phi_0^{(m)} =$ identity and

$$\frac{\partial}{\partial t} \Phi_t^{(m)}(q) = X_t(\Phi_t^{(m)}(q)) \qquad \text{for } q \in S^2.$$

We define a Riemannian metric $\bar{g}_t^{(m)}$ on S^2 by $\bar{g}_t = (\Phi_t^{(m)})^* \bar{g}_t^{(m)}$. Since $\Phi_t^{(m)}$ is a conformal transformation, $\bar{g}_t^{(m)}$ can be also written as

$$\bar{g}_t^{(m)} = \exp(2\rho_t^{(m)}) g_o ,$$

where $\rho_t^{(m)}$ is a certain function on S^2. Now, we have

$$\mathscr{L}_{X_t} \bar{g}_t + (\Phi_t^{(m)})^* \left(\frac{\partial}{\partial t} \bar{g}_t^{(m)} \right) = \frac{\partial}{\partial t} \bar{g}_t .$$

It follows from $(*)_m$ that

$$\frac{\partial}{\partial t} \bar{g}_t^{(m)} \equiv 0 \pmod{t^{m+1}} ,$$

which implies

$$\partial^j \rho^{(m)} = \frac{\partial^j}{\partial t^j} \rho_t^{(m)} \Big|_{t=0} = 0 \qquad \text{for } j = 1, \cdots, m+1.$$

Since $\{\bar{g}_t^{(m)}\}$ is a one-parameter bisectable deformation of g_o on S^2, it follows from Proposition in § 2 that

$$\int_H \{\Delta(\partial^{m+2}\rho^{(m)}) - 2\partial^{m+2}\rho^{(m)}\} dA = 0$$

for each standard hemisphere H of (S^2, g_o). Furthermore we can easily verify that $\partial^{m+2}\rho^{(m)}$ is an odd function on (S^2, g_o). Hence, in the same way as in § 3, we know that

$$\Delta(\partial^{m+2}\rho^{(m)}) = 2\partial^{m+2}\rho^{(m)} ,$$

which implies that there exists an infinitesimal conformal transformation $X^{(m+1)}$ such that

$$\mathscr{L}_{X^{(m+1)}} g_o = \frac{2\partial^{m+2}\rho^{(m)}}{(m+1)\,!} g_o .$$

Therefore, putting $Y_t = X_t + t^{m+1} X^{(m+1)}$, we have

$$\mathscr{L}_{Y_t} \bar{g}_t \equiv \frac{\partial}{\partial t} \bar{g}_t \pmod{t^{m+2}} . \qquad \text{Q. E. D.}$$

We are now in a position to prove Theorem B. We know that there is a smooth one-parameter family $\{\varphi_t\}_{t\in I}$ of transformations of S^2 such that φ_0 = identity and $\varphi_t{}^* g_t$ is of the form:

$$\varphi_t{}^* g_t = \exp(2\rho_t) g_o ,$$

where ρ_t is a certain function on S^2. We put $\bar{g}_t = \varphi_t{}^* g_t$, and apply Lemma together with its proof to the one-parameter bisectable deformation $\{\bar{g}_t\}$ of g_o on S^2. For each m let $\Phi_t^{(m)}$ and $\bar{g}_t^{(m)}$ be as in the proof of Lemma, and let $\bar{K}_t^{(m)}$ and K_t be the Gaussian curvature of $\bar{g}_t^{(m)}$ and g_t respectively. Now, we have

$$\frac{\partial}{\partial t} \bar{g}_t^{(m)} \equiv 0 \pmod{t^{m+1}} \qquad \text{for each } m,$$

which means that

$$\bar{g}_t^{(m)} \equiv g_o \pmod{t^{m+2}} \qquad \text{for each } m.$$

Therefore we obtain

$$\bar{K}_t^{(m)} \equiv K_0 (=1) \pmod{t^{m+2}} \qquad \text{for each } m.$$

Since $K_t = (\Phi_t^{(m)} \circ \varphi_t^{-1})^* \bar{K}_t^{(m)}$, it follows that

$$K_t \equiv K_0 \pmod{t^{m+2}} \qquad \text{for each } m.$$

Since g_t and hence K_t depend real analytically on the parameter t, we have thus seen that

$$K_t = K_0 = 1 \qquad \text{for any } t \in I.$$

By a standard method we can therefore construct a one-parameter family $\{\Psi_t\}_{t\in I}$ of transformations of S^2 such that Ψ_0=identity and $g_t = \Psi_t{}^* g_o$, which completes the proof of Theorem B.

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Added in Proof

For the latter half of the proof of Theorem C′ in § 1, more precisely, the part which followed after the sentence "Now, we take……", a simpler and more elegant method was suggested by the referee of this paper. He proposed to use a linear function $x \longrightarrow \langle v, x\rangle$ on $\boldsymbol{R}^3$, where $\langle \cdot, \cdot \rangle$ is the canonical inner product in $\boldsymbol{R}^3$ and v is an arbitrary element of $\boldsymbol{R}^3$. Since $x \longrightarrow \langle v, x\rangle f(x)$ is an even function on (S^2, g_o), $\int_0^{2\pi} f(c(s))\langle v, c(s)\rangle ds = 0$ *implies* $f(x) \equiv 0$ from the Theorem 4.53 in [1], where $c(s)$ is an arbitrary great circle parametrized by the arc-length s. We are grateful to the referee for this comment.