Hokkaido Mathematical Journal Vol. 19 (1990) p. 241-248

Relative bounds of closable operators in non-reflexive Banach spaces

Hideo KOZONO and Tohru OZAWA
(Received February 1, 1989, Revised February 21, 1989)

Introduction

In this paper we discuss some perturbation problems related to the relative compactness and boundedness of closable operators in complex Banach spaces which are *not necessarily reflexive*.

Let X, Y and Z be Banach spaces, let A be an operator from X into Z and let B be an operator from X into Y with $D(A)\subset D(B)$, where $D(T)$ denotes the domain of an operator T. We consider the following three conditions (see T. Kato [3] and S. G. Krein [4]):

(I) B is A-compact, i. e., for any sequence $\{u_n\}$ in $D(A)$ with $\sup_{n\in N}(\|u_n\|_X+\|Au_n\|_Z)<\infty$, $\{Bu_n\}$ has a convergent subsequence $\{Bu_{n_j}\}$ in Y.

(II) B is subordinate to A with exponent $\alpha\in(0,1)$, i. e., there is a constant C_α such that for all $u\in D(A)$

$$\|Bu\|_Y\leq C_\alpha\|Au\|_Z^\alpha\|u\|_X^{1-\alpha}.$$

(III) B is A-bounded with A-bound zero, i. e., for any $\varepsilon>0$ there is a constant C_ε such that for all $u\in D(A)$

$$\|Bu\|_Y\leq\varepsilon\|Au\|_Z+C_\varepsilon\|u\|_X.$$

It is clear that (II) implies (III). P. Hess [1][2] has proved that (I) implies (III) in the case $X=Y=Z$, where X is *reflexive* and A is *closed*. He has also observed that both reflexivity of X and closedness of A are necessary. M. Schechter [6] has proved that (I) implies (III) in the case $X=Y=Z$, where X is *not necessarily reflexive*, A is *closed*, and B is *closable*.

In § 1 we prove that exen when X, Y, Z are not reflexive and A is not closed, (I) implies (III) under the condition that B is closable, which is also shown not removable. Moreover, we prove that there exist a Banach space X, a closed operator A and a non-closable operator B in X satisfying (I) and (II). Furthermore, we prove that there exist a Banach space X and closed operators A, B in X such that (II) does not hold for any $\alpha\in(0,1)$ but (I) holds. Let $X=Y=Z=L^2(\boldsymbol{R}^n)$ and let

$\Lambda^s=(1-\Delta)^{s/2}$, $s\in\boldsymbol{R}$, with $D(\Lambda^s)=H_2^s(\boldsymbol{R}^n)$, the Sobolev space of order s. Then $A:=\Lambda^s$ and $B:=\Lambda^t$ with $0<t<s$ does not satisfy (I) but satisfies (II) for $\alpha=t/s$.

Therefore in general, (I) and (II) are irrelevant to each other.

In § 2 we investigate the conditions (II) and (III) in view of the spectral properties of A and B in the case where $X=Y=Z$ and A is a non-negative operator in X. We show that the decay property of $B(A+\lambda)^{-1}$ in the operator norm on X as $\lambda\to\infty$ is closely related to the properties (II) and (III). We remark that the information about the decay rate for $B(A+\lambda)^{-1}$ with respect to λ also plays an important role in determining the domains of the fractional powers $(A+B)^{\alpha}$, $\alpha\in\boldsymbol{R}$, of the perturbed operator $A+B$ (see H. Kozono & T. Ozawa [4]).

§ 1. Results on relatively compact perturbations

Our first result is:

THEOREM 1.1 *Let X, Y and Z be Banach spaces. Let A be an operator from X into Z and let B be an A-compact operator from X into Y. If B is closable, then B is A-bounded with A-bound zero.*

The converse of Theorem 1.1 does not hold:

THEOREM 1.2 *Let X be the Banach space $C(\mathrm{I})$, $I=[0,1]$, of continuous functions on I with the uniform norm. Let A and B be the operators in X given respectively by*

$$D(A)=\{u\in X\ ;\ u''=\left(\frac{d}{dx}\right)^2u\in X,\ u(0)=u(1)=0\},\ Au=-u'',\ u\in D(A),$$

$$D(B)=\{u\in X\ ;\ u'\in X\},\ (Bu)(x)=u'(0),\ u\in D(B),\ x\in I.$$ *Then:*

(1) *B is A-compact.*

(2) *B it subordinate to A with exponent* 1/2.

(3) *B is not closable.*

The following theorem shows that (I) does not imply (II).

THEOREM 1.3 *Let X be as in Theorem* 1.2. *Let A and B be the operators given respectively by*

$$D(A)=X,\ (Au)(x)=u(0),\ u\in X,\ x\in I,$$

$$D(B)=X,\ (Bu)(x)=\int_0^x u(y)\,dy,\ u\in X,\ x\in I.$$ *Then:*

(1) *B is A-compact.*

(2) *For any $\alpha\in(0,1)$, B is not subordinate to A with exponent α.*

PROOF OF THEOREM 1.1 We prove the theorem by contradiction. Suppose that there exist $\varepsilon_0>0$ and a sequence $\{u_n\}$ in $D(A)$ satisfying $u_n\neq 0$ and

(1.1) $\quad \|Bu_n\|_Y>\varepsilon_0\|Au_n\|_Z+n\|u_n\|_X$

For all $n\in \boldsymbol{N}$. We set $v_n=u_n/\|Bu_n\|_Y$. It follows from (1.1) that $\sup_{n\in N}(\|v_n\|_X+\|Av_n\|_Z)<\infty$ and therefore $\{Bv_n\}$ has a subsequence $\{Bv_{n_j}\}$ such that for some $w\in Y$, $Bv_{n_j}\to w$ in Y as $j\to\infty$. On the other hand we see from (1.1) that $v_{n_j}\to 0$ in X as $j\to\infty$. Since B is closable, we have $w=0$. This contradicts the fact that $\|Bv_n\|_Y=1$ for all $n\in \boldsymbol{N}$.

PROOF OF THEOREM 1.2 (1) For any $\lambda\notin\{n^2\pi^2;\ n\in \boldsymbol{N}\cup\{0\}\}$, we have $\lambda\in\rho(A)$, the resolvent set of A, and

$$((\lambda-A)^{-1}u)(x)$$
$$+(\lambda^{1/2}\sin\lambda^{1/2})^{-1}\left(\sin(\lambda^{1/2}(x-1))\int_0^x\sin(\lambda^{1/2}y)u(y)\,dy\right.$$
$$\left.+\sin(\lambda^{1/2}x)\int_x^1\sin(\lambda^{1/2}(y-1))u(y)\,dy\right),\quad u\in X,\ x\in I.$$

Thus

$$(B(\lambda-A)^{-1}u)(x)$$
$$=\left(\frac{d}{dx}(\lambda-A)^{-1}u\right)(0)=(\sin\lambda^{1/2})^{-1}\int_0^1\sin(\lambda^{1/2}(y-1))u(y)\,dy.$$

It therefore follows from the Ascoli-Arzelã theorem that $B(\lambda-A)^{-1}$ is a compact operator. This proves part (1).

(2) We prove that $\|Bu\|\le 2^{1/2}\|Au\|^{1/2}\|u\|^{1/2}$, $u\in D(A)$. Let

(1.2) $\quad u(x)=xu'(0)+\int_0^x(x-y)u''(y)\,dy,\ x\in I,$

(1.3) $\quad u'(0)=-\int_0^1(1-y)u''(y)\,dy.$

If $\|Au\|/\|u\|\ge 2$, then by (1.2) we obtain for $x\in(0,1]$

(1.4) $\quad |u'(0)|\le x^{-1}\|u\|+x^{-1}\int_0^x(x-y)\|Au\|\,dy=x^{-1}\|u\|+2^{-1}x\|Au\|.$

We set $x_0=2^{1/2}\|u\|^{1/2}\|Au\|^{-1/2}$. Since $x_0\in(0,1]$, we obtain the desired inequality by replacing x by x_0 in (1.4).
If $\|Au\|/\|u\|<2$, then by (1.3)

$$|u'(0)|\le\int_0^1(1-y)\|Au\|\,dy=2^{-1/2}\|Au\|\le 2^{-1/2}\|Au\|^{1/2}\|u\|^{1/2}.$$

This proves part (2).

(3) Let $\{u_n\}$ be the sequence in $D(B)$ defined by $u_n(x)=n^{-1/2}\sin(n^{1/2}\pi x)$, $x\in I$. Then, $\|u_n\|=n^{-1/2}\to 0$ as $n\to\infty$. On the other hand $(Bu_n)(x)=\pi$ for all $x\in I$ and $n\in \mathbf{N}$, so that B is not closable.

PROOF OF THEOREM 1.3 Part (1) follows from the Ascoli-Arzelâ theorem. We prove part (2) by contradiction. Suppose that there exist $\alpha\in(0,1]$ and $C>0$ such that $\|Bu\|\le C\|Au\|^{\alpha}\|u\|^{1-\alpha}$, $u\in X$. But this does not hold for $u(x)=x\in X$.

§ 2. Results on relatively bounded perturbations with relative bound zero

For a Banach space X, $\boldsymbol{B}(X)$ denotes the space of all bounded linear operators in X with norm $\|\cdot\|_{B(X)}$.

THEOREM 2.1 *Let X be a Banach space. Let A be a closed operator in X such that the resolvent set $\rho(A)$ of A contains the negative real axis $(-\infty, 0)$ and $\sup_{\lambda>0}\lambda\|(A+\lambda)^{-1}\|_{B(X)}<\infty$. Let B be a closable operator in X with $D(B)\supset D(A)$. Let $\alpha\in[0,1]$. Then, B is subordinate to A with exponent α if and only if $\sup_{\lambda>0}\lambda^{1-\alpha}\|B(A+\lambda)^{-1}\|_{B(X)}<\infty$.*

THEOREM 2.2 *Let X, A and B be as in Theorem 2.1. Suppose that there is $\lambda_0\ge 0$ such that $\int_{\lambda_0}^{\infty}\|B(A+\lambda)^{-2}\|_{B(X)}d\lambda<\infty$. Then, B is A-bounded with A-bound zero.*

COROLLARY. *Let X, A and B be as in Theorem 2.1. Suppose that there is $\lambda_0\ge 0$ such that $\int_{\lambda_0}^{\infty}\lambda^{-1}\|B(A+\lambda)^{-1}\|_{B(X)}d\lambda<\infty$. Then, B is A-bounded with A-bound zero.*

The converse of Theorem 2.2 does not hold:

THEOREM 2.3 *Let X be a Hilbert space and let H be a self-adjoint operator. Let A and B be the operators in X given respectively by $A=|H|$, $B=|H|/\log(1+|H|)$. Then:*

(1) *B is A-bounded with A-bound zero.*

(2) *For any $N\ge 0$, the map $(N,\infty)\ni\lambda\mapsto\|B(A+\lambda)^{-2}\|_{B(X)}\in\boldsymbol{R}$ is not integrable.*

PROOF OF THEOREM 2.1 Since A is closed and B is closable with $D(B)\supset D(A)$, we have $B(A+\lambda)^{-1}\in\boldsymbol{B}(X)$ for all $\lambda>0$. We set $M=$

$\sup_{\lambda>0}\lambda\|(A+\lambda)^{-1}\|_{B(X)}$. If B is subordinate to A with exponent α, then there is a constant C such that $\|Bu\|\leq C\|Au\|^{\alpha}\|u\|^{1-\alpha}$ for all $u\in D(A)$. Therefore for any $v\in X$ we have

$$\begin{aligned}\|B(A+\lambda)^{-1}v\| &\leq C\|A(A+\lambda)^{-1}v\|^{\alpha}\|(A+\lambda)^{-1}v\|^{1-\alpha}\\ &\leq C(\|v\|+\lambda\|(A+\lambda)^{-1}\|_{B(X)}\|v\|)^{\alpha}(\lambda^{-1}M\|v\|)^{1-\alpha}\\ &\leq C(1+M)^{\alpha}M^{1-\alpha}\lambda^{\alpha-1}\|v\|.\end{aligned}$$

Hence,

$$\sup_{\lambda>0}\lambda^{1-\alpha}\|B(A+\lambda)^{-1}\|_{B(X)}\leq C(1+M)^{\alpha}M^{1-\alpha}.$$

Conversely, suppose that $\tilde{M}:=\sup_{\lambda>0}\lambda^{1-\alpha}\|B(A+\lambda)^{-1}\|_{B(X)}<\infty$. Let $u\in D(A)$, $u\neq 0$. We obtain for any $\lambda>0$

$$\|Bu\|\leq\|B(A+\lambda)^{-1}\|_{B(X)}(\|Au\|+\lambda\|u\|)\leq\tilde{M}\lambda^{\alpha-1}(\|Au\|+\lambda\|u\|).$$

Setting $\lambda=\|Au\|/\|u\|$, we have the desired estimate.

PROOF OF THEOREM 2.2 It follows from the resolvent equation that for any $j\in \mathbf{N}$ the map $(0,\infty)\ni\lambda\mapsto B(A+\lambda)^{-j}\in\mathbf{B}(X)$ is continuous. Since we have for any $h\neq 0$,

$$\begin{aligned}&\|h^{-1}(B(A+\lambda+h)^{-1}-B(A+\lambda)^{-1})+B(A+\lambda)^{-2}\|_{B(X)}\\ &=|h|\|B(A+\lambda)^{-2}(A+\lambda+h)^{-1}\|_{B(X)},\end{aligned}$$

the map $(0,\infty)\ni\lambda\mapsto B(A+\lambda)^{-1}\in\mathbf{B}(X)$ is continuously differentiable and $\frac{d}{d\lambda}B(A+\lambda)^{-1}=-B(A+\lambda)^{-2}$. Therefore

$$B(A+\lambda)^{-1}-B(A+\mu)^{-1}=-\int_{\mu}^{\lambda}B(A+\nu)^{-2}d\nu,\ \lambda>\mu>0,$$

and by our assumption we see that $\{B(A+\lambda)^{-1};\ \lambda\geq\lambda_0\}$ is convergent in $\mathbf{B}(X)$. Hence there is an operator $T\in\mathbf{B}(X)$ such that $B(A+\lambda)^{-1}\to T$ in $\mathbf{B}(X)$ as $\lambda\to\infty$. Let $u\in X$. We have $(A+\lambda)^{-1}u\to 0$, $B(A+\lambda)^{-1}u\to Tu$ *in* X as $\lambda\to\infty$. Since B is closable, we conclude that $Tu=0$ for all $u\in X$. This implies that $B(A+\lambda)^{-1}\to 0$ in $\mathbf{B}(X)$ as $\lambda\to\infty$. The result now follows from the inequality

$$\|Bu\|\leq\|B(A+\lambda)^{-1}\|_{B(X)}(\|Au\|+\lambda\|u\|),\ u\in D(A),\ \lambda>0.$$

PROOF OF THEOREM 2.3 (1) Since $(-\infty,0)\subset\rho(A)$, it suffices to prove that $B(A+\lambda)^{-1}\to 0$ in $\mathbf{B}(X)$ as $\lambda\to\infty$. We estimate $B(A+\lambda)^{-1}$ in

$\boldsymbol{B}(X)$ for $\lambda \geq 10$ as

$$\begin{aligned}\|B(A+\lambda)^{-1}\|_{B(X)} &= \sup_{\mu\geq 0} \mu(\log(1+\mu))^{-1}(\mu+\lambda)^{-1} \\ &\leq \sup_{j\in N} \sup_{2^{j-1}\lambda\leq\mu<2^j\lambda} \mu(\log(1+\mu))^{-1}(\mu+\lambda)^{-1} \\ &+ \sup_{0\leq\mu<\lambda} \mu(\log(1+\mu))^{-1}(\mu+\lambda)^{-1} \\ &\leq \sup_{j\in N} 2^j\lambda(\log(1+2^{j-1}\lambda))^{-1}(2^{j-1}\lambda+\lambda)^{-1} \\ &+ \lambda(\log(1+\lambda))^{-1}(2\lambda)^{-1} \\ &\leq \sup_{j\in N} 2^j((j-1)\log 2+\log\lambda)^{-1}(2^{j-1}+1)^{-1} \\ &+ (2\log\lambda)^{-1} \leq 3(\log\lambda)^{-1}.\end{aligned}$$

This proves part (1).

(2) For any $\lambda>0$. we have

$$\begin{aligned}\|B(A+\lambda)^{-2}\|_{B(X)} &= \sup_{\mu\geq 0} \mu(\log(1+\mu))^{-1}(\mu+\lambda)^{-2} \\ &\geq \lambda(\log(1+\lambda))^{-1}(2\lambda)^{-2} \geq (4(1+\lambda)\log(1+\lambda))^{-1}.\end{aligned}$$

The R. H. S. of the last inequality is not integrable. This proves part (2).

Acknowledgments. The present work was done while the first author stayed at the University of Paderborn as a research fellow of the Alexander von Humboldt Foundation. Their kind support and hospitality are gratefully acknowledged.

References

[1] HESS, P.: Zur Strörungtheorie linearer Operatoren: Relative Beschränktheit und relative Kompaktheit von Operatoren in Banachräumen, Comment. Math. Helv. 44, 245-248 (1969).

[2] HESS, P.: Zur Strörungtheorie linearer Operatoren in Banachräumen, Comment. Math. Helv. 45, 229-235 (1970).

[3] KATO, T.: Perturbation Theory for Linear Operators. 2nd ed. Berlin-Heiderberg-New York: Springer-Verlag 1980.

[4] KOZONO, H., OZAWA, T.: Stability in L^r for the Navier-Stokes Flow in a n-dimensional bounded domain, to appear in J. Math. Anal. Appl.

[5] KREIN, S. G.: Linear Differential Equations in Banach Space. Providence, R. I.: Amer. Math. Soc. Translations of Mathematical Monographs 29, 1971.

[6] SCHECHTER, M.: Spectra of Partial Differential Operators. 2nd ed. Amsterdam-New York-Oxford: North-Holland 1986.

Fachbereich Mathematik-Informatik
der Universität-Gesamthochschule
Paderborn, D-4790 Paderborn
Federal Republic of Germany

Department of Mathematics
Nagoya University
Nagoya 464, Japan