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Holonomy groupoids of generalized foliations, I

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Introduction

We mean by a generalized foliation, a foliation with singular leaves in the sense of P. Stefan [St] and P. Dazord [D], and in the present paper we establish a notion of a holonomy groupoid of this generalized foliation. In order to keep similarities to the case of regular foliations, we set a limitation on singular leaves, however our result is applicable to some foliations of Poisson structures and to some foliations which are not locally simple (cf. [E]). In many cases, we call a generalized foliation simply a foliation.

According to [D], along a leaf F of a foliation $\mathscr{D}$ on a C^{∞}-manifold M there is a unique germ Δ_F of transverse structure. A singular leaf F is called *tractable* if F has a saturated neighborhood N in M with following properties:

(i) N is isomorphic to a fibre bundle over F, $\pi_F: N\to F$ having a fibre V with a foliation Δ_V which is a representative of Δ_F.

(ii) The structural group of the bundle is the group of isomorphisms of Δ_V and the foliation of N determined by a local product of the one leaf foliation of F and Δ_V is the restriction $\mathscr{D}_N$ of $\mathscr{D}$ to N.

We will show that if all singular leaf of $\mathscr{D}$ is tractable, then an (algebraic) holonomy groupoid $G(\mathscr{D})$ of D is defined and we have

THEOREM 2.2 *If each singular leaf of* $\mathscr{D}$ *is tractable, then* $G(\mathscr{D})$ *is a topological groupoid.*

The part of $G(\mathscr{D})$ outside singular leaves is a (non-Hausdorff) C^{∞}-manifold by the usual theory of regular foliations (see, e.g., [P], [Wi]), but $G(\mathscr{D})$ itself is not a manifold in general. We take examples of the foliations from those of symplectic leaves of Poisson structures and from some constructions of fibre bundles. In his construction of a singular foliation C^*-algebra, A. Sheu [Sh] uses the notion of holonomy of a locally simple foliation due to C. Ehresmann [E], but there are some foliations which are not locally simple. Our definition of holonomy can be applied to these.

In Section 1, we explain properties of a (generalized) foliation $\mathscr{D}$. In

Section 2, we construct the holonomy groupoid $G(\mathscr{D})$ of $\mathscr{D}$ and prove THEOREM 2.2. In Section 3, we show examples of holonomy groupoids from Poisson structures. The last section is devoted to constructing examples of holonomy groupoids for generalized foliations with nontrivial holonomy maps.

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1. Generalized foliations

Here we review basic facts about foliations with singularities from P. Dazord's work [D]. Let M be a paracompact Hausdorff C^∞-manifold and T_xM the tangent space of M at a point $x \in M$. Let $C^\infty(M)$ be the algebra of real valued C^∞-functions on M.

A *distribution* D of tangent subspaces of M is a collection of subspaces:

$$D=\{D_x \subset T_xM \mid x \in M\}.$$

Let $\rho(x)$ denote the dimension of D_x and $C^\infty(D)$ denote the $C^\infty(M)$-module of C^∞-vector fields on M, the value of which belongs to D_x for each $x \in M$.

D is called a *C^∞-distribution*, if for each $x \in M$ one can find a finite number of elements of $C^\infty(D)$,

$$X_1, \ldots, X_k$$

such that D_x is generated by $\{X_i(x) \mid 1 \leqq i \leqq k\}$ where $k=k(x)$ depends on x (equivalently, each element of $\mathscr{D}$ is a value of a local section). An *integral manifold* F of D is a connected immersed submanifold of M such that for each $x \in F$,

$$T_xF=D_x.$$

A *C^∞-foliation* of M (in a generalized sense) is a C^∞-distribution $\mathscr{D}$ on M with the condition that for each point $x \in M$, there exists an integral manifold through x.

If $f: M \to N$ is a diffeomorphism of C^∞-manifolds M and N, and X a C^∞-vector field on N, then a C^∞-vector field f_*X on M is defined by

$$(f_*X)(y)=f_*(X(f^{-1}(y))) \qquad y \in N.$$

The integrability conditions are stated as follows in the formulation by P. Dazord [D].

THEOREM 1.1 [*Sus*], [*D*]. *Let D be a C^∞-distrubution on a manifold M. The following properties are equivalent.*

(i) *For each point $x \in M$, there exists an integral manifold through x.*

(ii) *For each point $x \in M$, there exists a unique maximal integral manifold through x.*

(iii) *D is invariant by the flow of each vector field of $C^\infty(D)$.*

(iv) *There is a Lie subalgebra $\mathscr{H}$ contained in the Lie algebra of C^∞-vector fields of M such that*

a) *at each point x, D_x is equal to the values at x of the fields in $\mathscr{H}$.*

b) *for each $X \in \mathscr{H}$ with a flow germ φ_t and for each $Y \in \mathscr{H}$, $(\varphi_t)_* Y$ is a germ of vector field of $\mathscr{H}$.*

Let $\mathscr{D}$ be a C^∞-foliation of M. By THEOREM 1.1, (ii), for each point $x \in M$, there exists a unique maximal integral manifold through x, which we call a *leaf* of $\mathscr{D}$. Local structures of $\mathscr{D}$ are examined on the basis of THEOREM 1.1.

THEOREM 1.2 [*D*]. *For every point m of M with a C^∞-foliation $\mathscr{D}$, there is an open neighborhood U of x and a local coordinate map ψ: $U \xrightarrow{\cong} W \times V$ as follows: W and V are neighborhoods of origins in $\mathbf{R}^p$ and $\mathbf{R}^q$ respectively for*

$$p = \rho(m), \qquad p + q = \dim M,$$

and ψ carries the foliation induced on U to the product foliation,

$$\mathbf{R}^p \times \Delta_V,$$

where Δ_V is a foliation on V with a point leaf at the origin of $\mathbf{R}^q$ and $\mathbf{R}^p$ is the one leaf foliation.

PROOF: From the condition of THEOREM 1.1 (iv), a) it follows that if $\rho(m) = p > 0$, there exist p C^∞-vector fields,

$$\{X_i | 1 \leq i \leq p\},$$

belonging to $\mathscr{H}$ such that $X_1(m), \ldots, X_p(m)$ form a base of D_m. Let V_0 be a submanifold of M through m such that

$$D_m \oplus T_m V_0 \cong T_m M.$$

Let W denote a neighborhood of the origin O in $\mathbf{R}^p$, V a neighborhood of m in V_0 and ψ a map,

$$\psi : W\times V\to M,$$

defined by

$$\psi(x, y)=\psi^{(p)}_{x_p}\circ\cdots\circ\psi^{(1)}_{x_1}(y),$$

where $\psi^{(i)}_{x_i}$ is the flow of X_i for the parameter x_i, $x=(x_1, \ldots, x_p)\in W$ and $y\in V$. Since the differential map of ψ at $(0, m)$ is an isomorphism, one can assume by restricting eventually W and V that ψ is a diffeomorphism,

$$\psi : W\times V\xrightarrow{\cong} U\subset M,$$

where U is an open set of M containing m.

Let $\mathscr{D}_U$ denote the restriction of $\mathscr{D}$ to U and $\tilde{\mathscr{D}}_U$ the foliation $\psi^*\mathscr{D}_U$, that is, the pullback of $\mathscr{D}_U$ by ψ. The vector fields $\partial/\partial x_i$, $1\leqq i\leqq p$ are C^∞-vector fields of $\tilde{\mathscr{D}}_U$ which are invariant by the translation parallel to W. Thus we have,

$$\tilde{\mathscr{D}}_U(x, y)=\mathbf{R}^p\times\Delta(y),$$

where Δ is a distribution on V which admits an integral manifold through any point of V. Obviously Δ is generated by $\pi_*X(0, y)$ where X are C^∞-vector fields of $\tilde{\mathscr{D}}_U$ and $\pi : W\times V\to V$ is the projection. By THEOREM 1.1, (i), Δ is C^∞-foliation on V. q. e. d.

We call U a *foliation coordinate neighborhood* and ψ a *local foliation coordinate system* (or *chart*). Also we call a leaf of $\mathscr{D}_U$ a *plaque*. Foliation coordinate neighborhoods of a point form a fundamental system of neighborhoods. $\mathscr{D}_U$ is the pullback of Δ_V by the submersion $\pi_U=\pi\cdot\psi : U\to V$.

In general, plaques other than the leaf of m are not (locally closed) submanifolds of M. This is a difference from the regular case. From THEOREM 1.2, it follows that each plaque of U is diffeomorphic to the product of the plaque of m and a leaf of Δ_V. In particular, $\rho=\dim\mathscr{D}$ is lower semi-continuous.

Let M' be an immersed submanifold of M transverse to the foliation $\mathscr{D}$ and U an open foliation coordinate neighborhood of $m\in M'$. The submersion $\pi_U : U\to V$ associated with a local foliation chart ψ induces a submersion π'_U of the neighborhood $M'_U=U\cap M'$ of m in V and the restriction $\mathscr{D}|M'_U$ is the pullback $\pi'^*_U\Delta_V$. This shows from the definition of foliation that the restriction $\mathscr{D}_{M'}$ is a foliation.

COROLLARY 1.3 [D]. *Let M_1 and M_2 be two submanifolds of M passing through m and being transverse to $\mathscr{D}$ at m. If we have*

$$\dim M_2 = \dim M_1 + r \qquad (r \geq 0),$$

then the foliation $\mathscr{D}_{M_2}$ of M_2 is locally isomorphic, on a neighborhood of m, to the product foliation $\mathbf{R}^r \times \mathscr{D}_{M_1}$ of $\mathbf{R}^r \times M_1$.

PROOF: Let $\pi_U^{(i)}: U \cap M_i \to V$ denote the submersion induced by π_U: $U \to V$. Then we have obviously

$$(\pi_U^{(2)})^{-1} V \cong \mathbf{R}^r \times (\pi_U^{(1)})^{-1} V,$$

and as in the above argument, $\mathscr{D}_{M_i}$ is locally a pullback $\pi_U^{(i)*}\Delta_V$. q. e. d.

In particular, if $r=0$, then $\mathscr{D}_{M_1}$ and $\mathscr{D}_{M_2}$ are locally isomorphic. If M' is the image of V in the local foliation chart ψ, then $\mathscr{D}_{M'}$ is isomorphic to Δ_V. This yields that the germ at the origin O in $\mathbf{R}^q$, defined by Δ_V depends, up to a diffeomorphism, on the point m and it is called the *germ of transverse structure of $\mathscr{D}$ at m.* If $\rho(m)=0$, then Δ_V defines just the germ of $\mathscr{D}$ at m.

2. Holonomy groupoids

Let M be a manifold with a C^∞-foliation $\mathscr{D}$. Since every point of the leaf F of $m \in M$ is attained starting from m by a product of flows tangent to $\mathscr{D}$, germs of transverse structures at all points of F are isomorphic, and therefore there is a unique germ Δ_F of transverse structure. It is called the *germ of transverse structure of the leaf F.*

A leaf F is called *regular* if its germ of transverse structure is trivial, and it is called *singular* otherwise. A leaf is regular if and only if it has a neighborhood on which $\mathscr{D}$ induces a regular foliation, or equivalently, ρ is constant. A point is called *regular* if it belongs to a regular leaf. We say that a singular leaf F is *tractable* if F has a saturated neighborhood N in M with following properties:

(i) N is isomorphic to a fibre bundle over F, $\pi_F: N \to F$ having a fibre V with a foliation Δ_V which is a representative of Δ_F.

(ii) The structural group of the bundle is the group of isomorphisms of Δ_V and the foliation of N determined by a local product of one leaf foliation of F and Δ_V is the restriction $\mathscr{D}_N$ of $\mathscr{D}$ to N.

In the following, we consider a foliation $\mathscr{D}$ whose singular leaves are all tractable.

EXAMPLE 2.1: We mention some simple examples of foliations with tractable singular leaves.

(1) Let $\mathbf{R}^p$ denote a p-dimensional coordinate space. The set of all concentric spheres S_r^{p-1} with radii r around the origin O makes a foliation. Its only singular leaf is the origin, which is obviously tractable. We denote this foliation by $\mathscr{S}$.

(2) We denote the one leaf foliation of $\mathbf{R}^q$ by the same symbol and the point foliation of $\mathbf{R}^k$ by $\{\mathbf{R}^k\}$. The product foliation $\mathscr{S}\times\mathbf{R}^q\times\{\mathbf{R}^k\}$ has the set of singular leaves $\{O\}\times\mathbf{R}^q\times\{\mathbf{R}^k\}$.

A tractable singular leaf F of a foliation $\mathscr{D}$ has a saturated neighborhood which is isomorphic to a bundle with a fibre which is a foliated manifold V. We take the associated V/Δ_V-bundle. Then any continuous curve of F obviously determines an isomorphism from the leaf space of the source point to that of the target point. Hence one can define the *holonomy map* with respect to the germ of V/Δ_V, associated with a curve on F by that isomorphism. By the elementary theory of fibre bundles this map is determined up to homotopy of the curves fixing end points. Under the assumption that each singular leaf of C^∞-foliation $\mathscr{D}$ is tractable, one can construct a holonomy groupoid $G(\mathscr{D})$ by quite a similar way to the regular case.

For a leaf L of $\mathscr{D}$, let

$$\lambda : [0,1]\to L$$

be a continuous curve with end points $\lambda(0)=x$ and $\lambda(1)=y$. Let V_m denote a sufficiently small manifold of dimension dim M—dim L transverse to L at $m\in L$ and $H_{x,y}^{\lambda}: V_x/\Delta_{V_x}\to V_y/\Delta_{V_y}$ the holonomy map germ associated with λ, which depends only on the homotopy class $\bar{\lambda}$ relative to $\{0,1\}$.

Let $\mu : [0,1]\to L$ be another curve with $\mu(0)=x$ and $\mu(1)=y$, and μ^{-1} its inverse curve. We define a relation $\lambda\sim\mu$ by the equation,

$$H_{x,x}^{\lambda\cdot\mu^{-1}}=id.$$

This is an equivalence relation; the equivalence class of λ is denoted by $[\lambda]$.

Let $G(\mathscr{D})$ be the set of triples,

$$g=(x, y, [\lambda]),$$

where x, $y\in L$, L is a leaf of $\mathscr{D}$, and $\lambda : [0,1]\to L$ is a continuous curve with $\lambda(0)=x$ and $\lambda(1)=y$. The maps s, $r : G(\mathscr{D})\to M$ are defined by

$$s((x, y, [\lambda]))=x, \qquad r((x, y, [\lambda]))=y$$

and are called *source* and *target* map respectively. $G(\mathscr{D})$ is a groupoid over M by the usual composition and inverse operations obtained from those of curves:

$$(y, z, [\lambda_1])\cdot(x, y, [\lambda_2])=(x, z, [\lambda_1\cdot\lambda_2])),$$
$$(x, y, [\lambda])^{-1}=(y, x, [\lambda^{-1}])),$$

where λ^{-1} is the inverse curve of λ.

We will introduce a topology in $G(\mathscr{D})$ by defining fundamental systems of neighborhoods of points. For a point belonging to a regular leaf, its neighborhoods are the same as in the case of a regular foliation: For $g=(x, y, [\lambda])$, there exists a sequence of foliation coordinate neighborhoods $\{U_i|0\leq i\leq k\}$ and a partition of $[0,1]$, $0=t_0<\cdots<t_{k+1}=1$ such that if $U_i\cap U_j\neq\phi$ then $U_i\cup U_j$ is contained in a foliation coordinate neighborhood and $\lambda([t_i, t_{i+1}])\subset U_i$ for all $0\leq i<k+1$. We call the sequence $\{U_i|0\leqq i\leq k\}$ a *chain subordinated* to λ. A neighborhood of g is the set $U_{g,\lambda}$ of $(x', y', [\nu])$ such that $x'\in U_0$, $y'\in U_k$ and $\{U_i|0\leq i\leq k\}$ is a chain subordinated to ν.

Let $a=(u, v, [\lambda])$ be a point of $G(\mathscr{D})$ such that $\lambda([0,1])$ is contained in a singular leaf F, which is, of course, tractable by our assumption. Let $\{U'_i|0\leq i\leq k\}$ be a sequence of open sets in F, which is a chain subordinated to λ for one leaf foliation. Let Δ_V be a representative of Δ_F in a transverse submanifold V of dimension, dim $M-$dim F in M. We note that $\pi_F^{-1}(U'_i)\cong U'_i\times V$ and the sequence $\{\pi_F^{-1}(U'_i)|0\leq i\leq k\}$ is *a chain subordinated to* λ for $\mathscr{D}$ in a generalized sense. We define a neighborhood of a in $G(\mathscr{D})$ by the set $U_{a,\lambda}$ of $(u', v', [\nu])$ such that $u'\in\pi_F^{-1}(U'_0)$, $v'\in\pi_F^{-1}(U'_k)$ and $\{\pi_F^{-1}(U'_i)|0\leq i\leq k\}$ is a chain subordinated to ν.

THEOREM 2.2 *If $\mathscr{D}$ is a C^∞-foliation of M and each singular leaf of $\mathscr{D}$ is tractable, then $G(\mathscr{D})$ is a topological groupoid.*

PROOF: For a regular leaf, holonomy maps are defined without any restriction to them, but for a singular leaf, holonomy maps are not defined in general. In fact, we can not define a notion of germs of leaf spaces of transverse structures in a usual way for points of a singular leaf of a foliation which is not locally simple (see EXAMPLE 4.1). For a tractable singular leaf F, there is a neighborhood N of F, isomorphic to a bundle with a fibre V which has a saturated foliation Δ_V. The germ of a leaf space of the transverse structure Δ_F is defined to be the germ of V/Δ_V and holonomy maps for F are defined by the flat structure of the associated V/Δ_V-bundle.

Let U_{a_1,λ_1} and U_{a_2,λ_2} be neighborhoods of $a_1=(u_1, v_1, [\lambda_1])$ and $a_2=(u_2,$

$v_2, [\lambda_2])$ in $G(\mathscr{D})$. Let $\{U_{1,i}|0\leq i\leq k_1\}$ and $\{U_{2,i}|0\leq i\leq k_2\}$ be chains subordinated to λ_1 and λ_2 in the definitions of U_{a_1,λ_1} and U_{a_2,λ_2} respectively. If $b=(x, y, [\mu])$ is a point of $U_{a_1,\lambda_1}\cap U_{a_2,\lambda_2}$, one can define a chain $\{W_i|0\leq i\leq k\}$ subordinated to μ such that W_i is an open set of M contained in some U_{1,j_1} and in some U_{2,j_2}.

The neighborhood $U_{b,\lambda}$ of b obtained from the chain $\{W_i|0\leq i\leq k_1\}$ is contained in U_{a_1,λ_1} and U_{a_2,λ_2}. Thus the family $\mathscr{D}=\{U_{a,\lambda}\}$ satisfies conditions of a fundamental system of neighborhoods and defines a topology of $G(\mathscr{D})$.

It is obvious that the source and target maps $r, s: G(\mathscr{D})\to M=G^0(\mathscr{D})$ are continuous with respect to this topology of $G(\mathscr{D})$. The continuity of the groupoid multiplication map,

$$G^{(2)}(\mathscr{D})=\{(a_1, a_2)\in G(\mathscr{D})\times G(\mathscr{D})|r(a_1)=s(a_2)\}\to G(\mathscr{D})$$

defined by $(a_1, a_2)\mapsto a_1\cdot a_2$ follows from the construction of a chain of coordinate neighborhoods subordinated to a composite curves. The continuity of the map to the inverse element,

$$G(\mathscr{D})\to G(\mathscr{D}), \qquad a\mapsto a^{-1}$$

is checked more easily. q. e. d.

3. Examples from Poisson structures

Important examples of generalized foliations are those of symplectic leaves in Poisson manifolds. Let M be a C^∞-manifold and $C^\infty(M)$ an algebra of real valued C^∞-function on M. A Poisson structure on M is defined as a Lie algebra structure $\{,\}$ on $C^\infty(M)$ satisfying the Leibnitz identity,

$$\{fg, h\}=f\{g, h\}+\{f, h\}g.$$

The manifold M equipped with such a structure is called a *Poisson manifold.*

For $f, g\in C^\infty(M, \mathbf{R})$, we have

$$\{f, g\}=\Lambda(df, dg),$$

where Λ is the Poisson tensor field. Let T^*M denote the cotangent bundle of M. For $x\in M$ and $\alpha\in T^*_xM$, we define $\#\alpha\in T_xM$ by the relation,

$$<\beta, \#\alpha>=\Lambda_x(\alpha, \beta)$$

for all $\beta\in T^*_xM$. This correspondence determines a vector bundle morphism,

$$\#: T^*M \to TM.$$

The subset $D=\#(T^*M)$ is a C^∞-distribution and according to A. Kirillov [K1], D defines a C^∞-foliation $\mathscr{D}$ and each leaf of $\mathscr{D}$ with its Poisson structure is a symplectic immersed submanifold. (See, e. g., C. M. Marle [M].)

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $\mathfrak{g}^*$ the dual of $\mathfrak{g}$. For $f, g \in C^\infty(\mathfrak{g}^*, \mathbf{R})$, we set

$$\{f, g\}(x) = <x, [df(x), dg(x)]>.$$

This gives a Poisson structure on $\mathfrak{g}^*$, which was defined by S. Lie and F. A. Berezin. Each leaf of the C^∞-foliation of $\mathfrak{g}^*$ associated with this Poisson structure is a coadjoint orbit of G. (See, e. g., [K2], [Ko], [M] and [So].)

We will examine our holonomy groupoids of foliations of $\mathfrak{g}^*$ by coadjoint orbits of G for some Lie groups mentioned in A. Weinstein [We]. It is noted that the holonomy groupoid of the foliation outside singular leaves is a (non-Hausdorff) manifold.

Let $\{X_i | 1 \leq i \leq n\}$ be a basis of $\mathfrak{g}$ with $n = \dim \mathfrak{g}$ and $x_1, \ldots, x_n$ the linear functions corresponding to these basis elements.

EXAMPLE 3.1: $G=SO(3)$. One can take a basis $\{X_1, X_2, X_3\}$ of the Lie algebra $\mathfrak{g}=\mathfrak{so}(3)$ such that Poisson brackets of x_i are given by

$$\{x_1, x_2\}=x_3, \quad \{x_2, x_3\}=x_1, \quad \{x_3, x_1\}=x_2.$$

The manifold M is $\mathfrak{so}(3)^* \cong \mathbf{R}^3$. Leaves of $\mathscr{D}$ are coadjoint orbits of $SO(3)$ in $\mathfrak{so}(3)^*$ which are concentric spheres:

$$x_1^2+x_2^2+x_3^2=c>0, \quad c \in \mathbf{R}.$$

The origin $O=(0,0,0) \in \mathbf{R}^3$ is the only singular leaf which is obviously tractable and all holonomy maps are trivial.

The holonomy groupoid $G(\mathscr{D})$ of the foliation $\mathscr{D}$ is described as follows: The holonomy groupoid of the regular part of $\mathscr{D}$ is

$$S^2 \times S^2 \times (\mathbf{R}_+ \backslash \{0\})$$

and hence $G(\mathscr{D})$ is the cone over $S^2 \times S^2$,

$$C(S^2 \times S^2) \cong (S^2 \times S^2 \times \mathbf{R}_+)/(S^2 \times S^2 \times \{0\}).$$

EXAMPLE 3.2: $G=SL(2, \mathbf{R})$. One can take a basis $\{X_1, X_2, X_3\}$ of the Lie algebra $\mathfrak{g}=\mathfrak{sl}(2, \mathbf{R})$ such that Poisson brackets of x_i are given by

$$\{x_1, x_2\} = -x_3, \quad \{x_2, x_3\} = x_1, \quad \{x_3, x_1\} = x_2.$$

The manifold M is $\mathfrak{sl}(2, \mathbf{R})^* \cong \mathbf{R}^3$. Leaves of $\mathscr{D}$ are coadjoint orbits of $SL(2, \mathbf{R})$ in $\mathfrak{sl}(2, \mathbf{R})^*$ which are the origin, one sheet hyperboloids, two sheet hyperboloids and circular cones:

$$\{(0, 0, 0)\},$$
$$x_1^2 + x_2^2 - x_3^2 = c \neq 0, \ c \in \mathbf{R},$$
$$x_1^2 + x_2^2 - x_3^2 = 0, \ x_3 \neq 0.$$

The origin $O = (0, 0, 0)$ is the only singular leaf, which is obviously tractable, and all holonomy maps are trivial.

The holonomy groupoid $G(\mathscr{D})$ of the foliation $\mathscr{D}$ is described as follows: The holonomy groupoid of the regular part of $\mathscr{D}$ is the disjoint union with an appropriate topology,

$$(S^1 \times \mathbf{R} \times S^1 \times \mathbf{R} \times \mathbf{R}) \cup (B^2 \times B^2 \times \mathbf{R}^+) \cup (B^2 \times B^2 \times \mathbf{R}^-)$$
$$\cup (S^1 \times \mathbf{R}^+ \times S^1 \times \mathbf{R}^+) \cup (S^1 \times \mathbf{R}^- \times S^1 \times \mathbf{R}^-),$$

where B^2 is the open 2-disk and $\mathbf{R}^\pm = \mathbf{R}_\pm \backslash \{0\}$. The holonomy groupoid $G(\mathscr{D})$ is the disjoint union with an appropriate topology,

$$(S^1 \times \mathbf{R} \times S^1 \times \mathbf{R} \times \mathbf{R}) \cup (B^2 \times B^2 \times \mathbf{R}^+) \cup (B^2 \times B^2 \times \mathbf{R}^-)$$
$$\cup (S^1 \times \mathbf{R}^+ \times S^1 \times \mathbf{R}^+) \cup (S^1 \times \mathbf{R}^- \times S^1 \times \mathbf{R}^-) \cup *,$$

where $*$ is the only element of $G(\mathscr{D})$ obtained from the leaf O.

4. Various examples

First of all, we mention one more example of a generalized foliation with trivial holonomy maps, which is not locally simple.

EXAMPLE 4.1: Let $S_{\lambda,\mu}$ be a circle in $\mathbf{R}^2$:

$$\lambda(x_1^2 + x_2^2 - 2x_2) + \mu x_2 = 0$$

The set $\{S_{\lambda,\mu} | \lambda, \mu \in \mathbf{R}\}$ defines a generalized foliation $\mathscr{D}$ of $\mathbf{R}^2$ with $D_0 = \{0\}$. $\mathscr{D}$ has the only singular leaf $\{O\}$.

A foliation of a manifold M is *locally simple* by definition (see [E]), if each point x of M has an open neighborhood V such that for the fundamental system of open neighborhoods U of x in V, the maps of the leaf spaces $\tilde{U}$ of U to the leaf space $\tilde{V}$ of V, induced by inclusion maps, are homeomorphisms onto open sets of $\tilde{V}$. In our foliation $\mathscr{D}$, it is obvious that the origin $O \in \mathbf{R}^2 = M$ does not satisfy the condition of local simplicity. In fact, any funndamental system of neighborhoods of O contains open sets U, V such that $U \subsetneq V$ and the map $\tilde{U} \to \tilde{V}$ is not injective.

The holonomy groupoid $G(\mathscr{D}|_{\mathbf{R}^2\setminus\{O\}})$ of the regular part of $\mathscr{D}$ is diffeomorphic to the manifold

$$((S^1\setminus\{q\})\times(S^1\setminus\{q\})\times(S^1\setminus\{q,-q\}))\cup((S^1\setminus\{q,-q\})\\ \times(S^1\setminus\{q,-q\})\times\{-q\})\subset S^1\times S^1\times S^1,$$

where S^1 is the unit circle and $q=(0,1)\in S^1$. In the quotient space $Q=(S^1\times S^1\times S^1)/((S^1\times S^1\times\{q\})\cup(\{q\}\times\{q\}\times S^1))$, we denote the point $[\{q\}\times\{q\}\times\{q\}]$ by $*$. Then $G(\mathscr{D})$ is the disjoint union

$$G(\mathscr{D}|_{\mathbf{R}^2\setminus\{O\}})\cup *$$

and $*$ is the element represented by the point leaf $\{O\}$.

Examples in Section 3 obtained from foliations of coadjoint orbits of Lie groups in the dual of its Lie algebra, have all trivial holonomy maps. However, some generalized foliations with nontrivial holonomy maps are constructed as follows:

EXAMPLE 4.2: Let $\mathscr{D}$ be the foliation of EXAMPLE 3.2 and E an $\mathbf{R}^3$-bundle associated with the nontrivial $\mathbf{Z}_2$-bundle over S^1, where the nontrivial element of $\mathbf{Z}_2$ acts on $\mathbf{R}^3$ as a symmetry with respect to x_1x_2-plane. Let $\mathscr{D}_E$ denote the foliation of E obtained from $\mathscr{D}$ by taking a local product with $\mathbf{R}$. The zero-section $F\cong S^1$ of E is a singular leaf. The associated $\mathbf{R}^3/\mathscr{D}$-bundle over F is flat, that is, it has the discrete structural group $\mathbf{Z}_2$ and hence the holonomy group of F is $\mathbf{Z}_2$ from the definition of holonomy maps in Section 2. Thus F has a nontrivial holonomy map.

Let α, β be generators of the first and the second factor of $\mathbf{Z}^2\cong\pi_1(S^1\times S^1)$. We define a homomorphism h: $\pi_1(S^1\times S^1)\longrightarrow\mathbf{Z}_2$ by

$$h(\alpha)=h(\beta)=1\in\mathbf{Z}_2$$

Let K be the set of continuous curves γ: $[0,1]\to S^1$ and $\bar{K}$ the space of equivalence classes in K by the holonomy of curves, fixing end points. $\bar{K}$ is regarded as the quotient space of $\mathbf{R}^2$ by the $h^{-1}(0)$-action, which is diffeomorphic to a torus. Since $\mathbf{Z}_2$ is a group of diffeomorphisms of $\mathbf{R}^3$ preserving the foliation $\mathscr{D}$, it acts on the manifold $G(\mathscr{D}|_{\mathbf{R}^3\setminus\{O\}})$ and hence $\pi_1(S^1\times S^1)\cong\mathbf{Z}^2$ acts on $G(\mathscr{D}|_{\mathbf{R}^3\setminus\{O\}})$ through the homomorphism h. The holonomy groupoid of the regular part $\mathscr{D}|_{E\setminus F}$ is the bundle associated with the covering map $\mathbf{R}^2\to\mathbf{R}^2/h^{-1}(0)\cong\bar{K}$:

$$G(\mathscr{D}|_{E\setminus F})=G(\mathscr{D}|_{\mathbf{R}^3\setminus\{O\}})\times_{h^{-1}(0)}\mathbf{R}^2\\ \cong G(\mathscr{D}|_{\mathbf{R}^3\setminus\{O\}})\times\bar{K}.$$

The holonomy groupoid $G(\mathscr{D}_E)$ is of the form of disjoint union

$$G(\mathscr{D}|_{E\backslash F})\cup\bar{K}$$

which is homeomorphic to $G(\mathscr{D})\times\bar{K}$.

A similar construction is made with the foliation $\mathscr{D}$ of EXAMPLE 3.1 and a holonomy groupoid of a locally nonsimple foliation is obtained.

EXAMPLE 4.3: Let $\mathscr{D}$ be the same foliations of $\mathbf{R}^3$ as in EXAMPLE 3.1 and 3.2. We define a C^∞-diffeomorphism $f:\ \mathbf{R}^3\xrightarrow{\cong}\mathbf{R}^3$ by

$$f(x_1, x_2, x_3)=e(x_1, x_2, x_3).$$

where $e>1$. We identify points of $\mathbf{R}^3\times\{0\}$ in $\mathbf{R}^3\times[0,1]$ to points of $\mathbf{R}^3\times\{1\}$ by the diffeomorphism

$$(x_1, x_2, x_3, 0)\mapsto(f(x_1, x_2, x_3), 1).$$

Since f preserves the foliation $\mathscr{D}$, one obtains a foliation $\mathscr{D}_S$ on $S=\mathbf{R}^3\times S^1$ from $\mathscr{D}$ by taking a local product with the one leaf foliation of $\mathbf{R}$. The identification image F_S of $\{O\}\times[0,1]$ is the only singular leaf of $\mathscr{D}_S$. The associated $\mathbf{R}^3/\mathscr{D}$-bundle over F_S is flat and its holonomy group is $\mathbf{Z}$ which is again nontrivial.

We define a homomorphism $h_S:\ \pi_1(S^1\times S^1)\rightarrow\mathbf{Z}$ by

$$h_S(\alpha)=-h_S(\beta)=1\in\mathbf{Z}$$

and an equivalence relation in K by making use of h_S in a similar way to that of EXAMPLE 4.2. We denote the resulting quotient space by $\bar{K}_S$ which is diffeomorphic to an open cylinder.

Since $\mathbf{Z}$ is a group of diffeomorphisms of $\mathbf{R}^3$ preserving the foliation $\mathscr{D}$, it acts on the manifold $G(\mathscr{D}|_{\mathbf{R}^3\backslash\{O\}})$ and hence $\pi_1(S_1\times S_1)\cong\mathbf{Z}^2$ acts on $G(\mathscr{D}|_{\mathbf{R}^3\backslash\{O\}})$ through the homomorphism h_S, the holonomy groupoid of the regular part $\mathscr{D}|_{S\backslash F_S}$ is the bundle associated with the covering map $\mathbf{R}^2\rightarrow\mathbf{R}^2/h_S^{-1}(0)\cong\bar{K}_S$:

$$\begin{aligned}G(\mathscr{D}_S|_{S\backslash F_S})&=G(\mathscr{D}|_{\mathbf{R}^3\backslash\{O\}})\times_{h_S^{-1}(0)}\mathbf{R}^2\\&\cong G(\mathscr{D}|_{\mathbf{R}^3\backslash\{O\}})\times\bar{K}_S.\end{aligned}$$

The holonomy groupoid $G(\mathscr{D}_S)$ is of the form of disjoint union

$$G(\mathscr{D}_S|_{S\backslash F_S})\cup\bar{K}_S$$

which is homeomorphic to $G(\mathscr{D})\times\bar{K}_S$.

A similar construction is made with the foliation $\mathscr{D}$ of EXAMPLE 4.1

and a holonomy groupoid of locally non-simple foliation is obtained.

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