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The Bochner type curvature tensor of contact Riemannian structure

Shukichi TANNO

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§ 0. Introduction.

Let (M, η) be a contact manifold with a contact form η. A contact manifold (M, η) with a Riemannian metric g associated with η is called a contact Riemannian manifold. The notion of contact Riemannian structure is wider than the notion of strongly pseudo-convex, integrable CR-structure, because the former satisfying the integrability condition $Q=0$ (cf. (2.1)) corresponds to the latter. One of the important problems in the study of contact manifolds (M, η) is to find differential geometric properties which are independent of the choice of contact forms $f\eta$, f being positive functions on M. So, we study gauge transformations of contact Riemannian structure.

The Bochner curvature tensor of a Kaehlerian manifold is related to the pseudo-conformal invariant of the 4-th order of nondegenerate, integrable CR-structure (Chern-Moser [2], Tanaka [5], Webster [7]). Sakamoto and Takemura [3] also discussed the curvature invariant by the method of almost contact structure tensors (ϕ, ξ, η). In the case of contact Riemannian manifolds we can also define the Bochner type curvature tensor (B^u_{zxy}) for the subspace P_x of the tangent space T_xM to M at each point x, where P_x is defined by $\eta=0$ (cf. (5.10)). The purpose of this paper is to prove the following.

THEOREM. *The Bochner type curvature tensor* (B^u_{zxy}) *of a contact Riemannian manifold* (M, η, g) *is invariant by gauge transformations* $(\eta\rightarrow\tilde{\eta}=\sigma\eta)$ *of contact Riemannian structure, if and only if the CR-structure corresponding to* (η, ϕ) *is integrable.*

After preliminaries in § 1~§ 4, in § 5 we define (B^u_{zxy}) for P so that its change by gauge transformations is natural and it has a generalized form of the Chern-Moser-Tanaka invariant. As a matter of fact the expression of (B^u_{zxy}) contains Q. In some steps we follow the paper [3] by Sakamoto and Takemura.

§ 1. Contact Riemannian manifolds.

An m-dimensional manifold M is a contact manifold if it admits a 1-form η' such that $\eta'\wedge(d\eta')^n\neq 0$ everywhere on M, where $m=2n+1$. By a function or tensor field on M we mean a smooth one. We fix a 1-form η among $\{f\eta' : f$ being positive functions on $M\}$, which is called a contact form associated with the contact structure. Then we have a unique vector field ξ such that

$$\eta(\xi)=1, \quad L_\xi\eta=0,$$

where L_ξ denotes the Lie derivation by ξ. It is well known that there is a Riemannian metric g and a (1, 1)-tensor field ϕ such that

$$g(\xi, X)=\eta(X), \quad 2g(X, \phi Y)=d\eta(X, Y),$$
$$\phi\phi X=-X+\eta(X)\xi,$$

where X and Y are vector fields on M. g is called a Riemannian metric associated with η. The next relations follow from the above:

$$\phi\xi=0, \quad \eta(\phi X)=0,$$
$$g(X, Y)=g(\phi X, \phi Y)+\eta(X)\eta(Y),$$
$$d\eta(X, \phi Y)=-d\eta(\phi X, Y).$$

Concerning the Riemannian connection ∇ with respect to g, the following hold:

$$\nabla_\xi\eta=0, \quad \nabla_\xi\xi=0, \; \nabla_\xi\phi=0,$$
$$\nabla_r\xi^r=0, \quad \nabla_r\phi^r_j=-2n\eta_j, \quad \xi^r\nabla_i\eta_r=0,$$
$$\nabla_r\eta_s\phi^r_i\phi^s_j=-\nabla_j\eta_i.$$

The last equality implies that $\nabla_r\eta_i\phi^r_j$ and $\nabla_i\eta_r\phi^r_j$ are symmetric with respect to i and j. Here the indices $i, j, k, \cdots, r, s$ run from 1 to $m=2n+1$.

We define $p=(p_{ij})$ by $2p_{ij}=\nabla_i\eta_j+\nabla_j\eta_i=L_\xi g_{ij}$. Then

$$\nabla_i\eta_j=p_{ij}+\phi_{ij}$$

holds. It is also verified that $\phi^i_r p^r_j=-p^i_r\phi^r_j$ holds.

§ 2. Contact Riemannian structure and CR-structure.

The notion of strongly pseudo-convex, pseudo-hermitian structure is equivalent to the notion of contact Riemannian structure (cf. for example, Tanaka [5], Tanno [6]). In [6] we defined a (1, 2)-tensor field Q on a contact Riemannian manifold by

(2.1) $Q^i_{jk}=\nabla_k\phi^i_j+\xi^i\phi^r_j\nabla_k\eta_r+\phi^i_r\nabla_k\xi^r\eta_j.$

Let (M, η, g) be a contact Riemannian manifold. We define J by $J=\phi|P$, where P denotes the subbundle of the tangent bundle TM defined by $\eta=0$. Then (M, η, J) is a strongly pseudo-convex, integrable CR manifold, if and only if $Q=0$.

Generalizing the canonical affine connection (due to Tanaka [5]) on a nondegenerate integrable CR manifold, we defined in [6] the generalized Tanaka connection $^*\nabla$ on a contact Riemannian manifold (M, η, g) by

(2.2) $^*\Gamma^i_{jk}=\Gamma^i_{jk}+\eta_j\phi^i_k-\nabla_j\xi^i\eta_k+\xi^i\nabla_j\eta_k,$

where Γ^i_{jk} denote the coefficients of the Riemannian connection ∇. The connection $^*\nabla$ on a contact Riemannian manifold (M, η, g) is a unique linear connection satisfying the following:

(i) $^*\nabla\eta=0,\quad {}^*\nabla\xi=0,\quad {}^*\nabla g=0,$
(ii) $^*T(X, Y)=d\eta(X, Y)\xi \quad X, Y\in\Gamma(P),$
(iii) $^*T(\xi, \phi Y)=-\phi{}^*T(\xi, Y) \quad Y\in\Gamma(P)$ or $Y\in\Gamma(TM),$
(iv) $^*\nabla_X\phi\cdot Y=Q(Y, X) \quad X, Y\in\Gamma(TM),$

where $\Gamma(P)$ denotes the space of all sections of the bundle P.

LEMMA 2.1. *The following holds:*

(2.3) $^*\nabla_i\phi_{jk}+{}^*\nabla_j\phi_{ki}+{}^*\nabla_k\phi_{ij}=0.$

PROOF. By (2.2) we get

$$^*\nabla_i\phi_{jk}=\nabla_i\phi_{jk}+\eta_j\phi^r_k\nabla_i\eta_r+\phi_{jr}\nabla_i\xi^r\eta_k.$$

Since $\nabla_i\phi_{jk}+\nabla_j\phi_{ki}+\nabla_k\phi_{ij}=0$ and $\phi^r_k\nabla_i\eta_r$ is symmetric with respect to k and i, we obtain (2.3).

LEMMA 2.2. *The following holds:*

(2.4) $\phi^r_j\phi^s_k{}^*\nabla_r\phi^i_s=-{}^*\nabla_j\phi^i_k.$

PROOF. Using (2.3), we calculate the following:

$$\begin{aligned}
\phi^r_j\phi^s_k({}^*\nabla_r\phi_{sl}+{}^*\nabla_s\phi_{rl})&=-\phi^r_j{}^*\nabla_r\phi^s_k\cdot\phi_{sl}-{}^*\nabla_s\phi^r_j\cdot\phi^s_k\phi_{rl}\\
&=\phi^r_j\phi^s_l({}^*\nabla_s\phi_{kr}+{}^*\nabla_k\phi_{rs})+\phi^s_k\phi^r_l({}^*\nabla_r\phi_{js}+{}^*\nabla_j\phi_{sr})\\
&=\phi^r_j\phi^s_l{}^*\nabla_s\phi_{kr}-\phi^r_j{}^*\nabla_k\phi^s_l\cdot\phi_{rs}-{}^*\nabla_r\phi^s_k\cdot\phi^r_l\phi_{js}-\phi^s_k{}^*\nabla_j\phi^r_l\cdot\phi_{sr}\\
&=-{}^*\nabla_k\phi_{jl}-{}^*\nabla_j\phi_{kl},\\
\phi^r_j\phi^s_k({}^*\nabla_r\phi_{sl}-{}^*\nabla_s\phi_{rl})&=-\phi^r_j\phi^s_k{}^*\nabla_l\phi_{rs}\\
&=\phi^r_j{}^*\nabla_l\phi^s_k\cdot\phi_{rs}\\
&={}^*\nabla_l\phi_{jk}\\
&={}^*\nabla_k\phi_{jl}-{}^*\nabla_j\phi_{kl}.
\end{aligned}$$

Then adding the two results, we get (2.4).

LEMMA 2.3. *Q satisfies the following:*

$$\eta_r Q^r_{jk}=Q^i_{jr}\xi_r=Q^i_{rk}\xi^r=0,$$
$$Q^r_{rk}=Q^r_{jr}=Q^i_{rs}g^{rs}=0,$$
$$Q^r_{js}\phi^s_r=Q^r_{sk}\phi^s_r=Q^i_{rs}\phi^{rs}=0,$$
$$Q^i_{jk}=-g^{ir}g_{js}Q^s_{rk},$$
$$\phi^i_r Q^r_{jk}=-\phi^r_j Q^i_{rk},$$
$$\phi^r_j Q^i_{rk}=\phi^r_k Q^i_{jr}.$$

PROOF. is easy. The last identity follows from (2.4).

§ 3. Gauge transformations of contact Riemannian structure.

Let (M, η, g) be a contact Riemannian manifold and let $\sigma=\exp(2\alpha)$ be a positive function on M. Corresponding to $\tilde{\eta}=\sigma\eta$ we define a gauge transformation of contact Riemannian structure by

$$\tilde{\xi}^i=\frac{1}{\sigma}(\xi^i+\zeta^i),\quad \zeta^i=\frac{1}{2\sigma}\phi^i_j\sigma^j=\phi^i_j\alpha^j,$$
$$\tilde{\phi}^i_j=\phi^i_j+\frac{1}{2\sigma}(\sigma^i-\xi\sigma\cdot\xi^i)\eta_j=\phi^i_j+\hat{\alpha}^i\eta_j,$$
$$\tilde{g}_{ij}=\sigma(g_{ij}-\eta_i\zeta_j-\eta_j\zeta_i)+\sigma(\sigma-1+\|\zeta\|^2)\eta_i\eta_j,$$

where $\sigma_i=\nabla_i\sigma$, $\alpha_i=\nabla_i\alpha$ and $\hat{\alpha}^i=\alpha^i-\xi\alpha\cdot\xi^i$. The geometric object with respect to $\tilde{\eta}$ corresponding to the geometric object K with respect to η is denoted by $\tilde{K}$. Then the inverse matrix $(\tilde{g}^{jk})$ and $\tilde{\phi}_{ij}$ are given by

$$\sigma(\tilde{g}^{jk}-\tilde{\xi}^j\tilde{\xi}^k)=g^{jk}-\xi^j\xi^k, \tag{3.1}$$
$$2\tilde{\phi}_{ij}=\sigma_i\eta_j-\sigma_j\eta_i+2\sigma\phi_{ij}. \tag{3.2}$$

LEMMA 3.1. *The difference* $(\tilde{W}^i_{jk})$ *of* $(\tilde{\Gamma}^i_{jk})$ *and* (Γ^i_{jk}) *is given by*

$$\tilde{W}^i_{jk}=\delta^i_j\alpha_k+\delta^i_k\alpha_j+\tilde{\xi}^i B_{jk}+C^i_{jk}, \tag{3.3}$$

where we have put

$$\begin{aligned}2B_{jk}=&-2p_{jk}-(\nabla_j\zeta_k+\nabla_k\zeta_j)+2(\alpha_j\zeta_k+\alpha_k\zeta_j)-2\xi\alpha\cdot g_{jk}\\&+(1/2)(\nabla_j\|\zeta\|^2\eta_k+\nabla_k\|\zeta\|^2\eta_j)-2\|\zeta\|^2(\alpha_j\eta_k+\alpha_k\eta_j)\\&+\eta_j(\nabla_\xi\zeta_k+\nabla_\zeta\eta_k+\nabla_\zeta\zeta_k)+\eta_k(\nabla_\xi\zeta_j+\nabla_\zeta\eta_j+\nabla_\zeta\zeta_j)\\&+[2(1+\|\zeta\|^2)\xi\alpha-\nabla_\xi\|\zeta\|^2-\nabla_\zeta\|\zeta\|^2]\eta_j\eta_k,\\2C^i_{jk}=&2\xi^i p_{jk}+2(\phi^i_j\zeta_k+\phi^i_k\zeta_j)-2\hat{\alpha}^i g_{jk}\\&-2(\sigma-1+\|\zeta\|^2)(\phi^i_j\eta_k+\phi^i_k\eta_j)+2\hat{\alpha}^i(\eta_j\zeta_k+\eta_k\zeta_j)\\&+\eta_j(\nabla^r\zeta_k-\nabla_k\zeta^r)(\delta^i_r-\xi^i\eta_r)+\eta_k(\nabla^r\zeta_j-\nabla_j\zeta^r)(\delta^i_r-\xi^i\eta_r)\\&+[2(1-2\sigma-\|\zeta\|^2)\hat{\alpha}^i-\nabla^i\|\zeta\|^2+\nabla_\xi\|\zeta\|^2\xi^i]\eta_j\eta_k.\end{aligned}$$

PROOF. We calculate the following:

$$2\widetilde{W}^i_{jk}=\tilde{g}^{ia}(\nabla_j\tilde{g}_{ak}+\nabla_k\tilde{g}_{aj}-\nabla_a\tilde{g}_{jk}).$$

For example we obtain

$$\begin{aligned}\tilde{g}^{ia}\nabla_j\tilde{g}_{ak}=&2\alpha_j\delta^i_k+\xi^i\nabla_j\eta_k+(\sigma-1+\|\zeta\|^2)\nabla_j\xi^i\eta_k\\&-\nabla_j\xi^i\zeta_k-\nabla_j\zeta^r(\delta^i_r-\xi^i\eta_r)\eta_k\\&+\tilde{\xi}^i[-\nabla_j\eta_k-\nabla_j\zeta_k-\nabla_j\eta_a\zeta^a\zeta_k+(\sigma_j+(1/2)\nabla_j\|\zeta\|^2)\eta_k\\&+(\sigma+\|\zeta\|^2)\zeta^a\nabla_j\eta_a\eta_k].\end{aligned}$$

Then (3.3) is verified.

LEMMA 3.2. *B_{jk} and C^i_{jk} are symmetric with respect to j, k and satisfy the following:*

$$B_{jr}\tilde{\xi}^r=0,\quad \eta_rC^r_{jk}=p_{jk},$$
$$C^r_{jr}=0\ (\textit{and hence, } \widetilde{W}^r_{jr}=(m+1)\alpha_j).$$

LEMMA 3.3. *The defference $({}^*\widetilde{W}^i_{jk})$ of $({}^*\widetilde{\Gamma}^i_{jk})$ and $({}^*\Gamma^i_{jk})$ is given by*

$$\begin{aligned}{}^*\widetilde{W}^i_{jk}=&\delta^i_j\alpha_k+\delta^i_k\alpha_j+\zeta^i\phi_{jk}+\phi^i_j\zeta_k+\phi^i_k\zeta_j-\hat{\alpha}^ig_{jk}\\&-\|\zeta\|^2(\phi^i_j\eta_k+\phi^i_k\eta_j)-\xi\alpha\cdot\delta^i_j\eta_k+(\xi^i+\zeta^i)(2\alpha_j\eta_k-\alpha_k\eta_j)\\&-\nabla_j\zeta^i\eta_k-\xi^i\eta_k\zeta^rp_{jr}+(1/2)(\nabla^r\zeta_k-\nabla_k\zeta^r)(\delta^i_r-\xi^i\eta_r)\eta_j\\&+\hat{\alpha}^i(\eta_j\zeta_k+2\eta_k\zeta_j)+(1/4)[\nabla_\xi\|\zeta\|^2\xi^i-\nabla^i\|\zeta\|^2\\&+4(1-2\|\zeta\|^2)\hat{\alpha}^i+2\nabla_\zeta\eta_r\zeta^r\xi^i+2(\nabla_\zeta\xi^i+\nabla_\xi\zeta^i+\nabla_\zeta\zeta^i)]\eta_j\eta_k.\end{aligned}\tag{3.4}$$

PROOF. First we notice the following:

$$\begin{aligned}2\sigma C^i_{jr}\tilde{\xi}^r=&2(1-\sigma)\phi^i_j-\nabla_\zeta\eta_r\zeta^r\xi^i\eta_j+\xi^i(\nabla_\zeta\eta_j-\nabla_\xi\zeta_j)\\&+\nabla^i\zeta_j-\nabla_j\zeta^i-(1/2)(\nabla^i\|\zeta\|^2-\nabla_\xi\|\zeta\|^2\xi^i)\eta_j-2\hat{\alpha}^i\zeta_j\\&-[2(\sigma-\|\zeta\|^2)\hat{\alpha}^i+\nabla_\zeta\xi^i+\nabla_\xi\zeta^i+\nabla_\zeta\zeta^i]\eta_j,\end{aligned}\tag{3.5}$$

where we have used the following:

$$\begin{aligned}\eta_j\nabla^i\zeta_r\xi^r&=-\eta_j\zeta^r\nabla^i\eta_r\\&=-\eta_j\zeta^r(\nabla_r\xi^i+2\phi^i_r)\\&=-\eta_j\nabla_\zeta\xi^i+2\hat{\alpha}^i\eta_j.\end{aligned}$$

Now we calculate ${}^*\widetilde{W}^i_{jk}={}^*\widetilde{\Gamma}^i_{jk}-{}^*\Gamma^i_{jk}$ by

$${}^*\widetilde{\Gamma}^i_{jk}=\widetilde{\Gamma}^i_{jk}+\tilde{\eta}_j\tilde{\phi}^i_k-\widetilde{\nabla}_j\tilde{\xi}^i\tilde{\eta}_k+\tilde{\xi}^i\widetilde{\nabla}_j\tilde{\eta}_k$$

(${}^*\Gamma^i_{jk}$, resp.) using Lemma 3.2, (3.5) and

$$\widetilde{\nabla}_j\tilde{\xi}^i=\nabla_j\tilde{\xi}^i+\widetilde{W}^i_{js}\tilde{\xi}^s,$$

etc. and obtain (3.4).

A (local) frame $\{e_i\}=\{e_0=\xi, e_u\}$ is called P-related, if $e_u\in P$, $1\leqq u\leqq 2n$. From now on, the indices u, v, w, x, y, and z run from 1 to $2n$, and the components of tensors are ones with respect to a P-related frame.

COROLLARY 3.4. $*\tilde{W}^0_{xy}=*\tilde{W}^0_{0y}=0$ *and*

$$(3.6)\qquad *\tilde{W}^u_{xy}=\delta^u_x\alpha_y+\delta^u_y\alpha_x+\zeta^u\phi_{xy}+\phi^u_x\zeta_y+\phi^u_y\zeta_x-\alpha^u g_{xy},$$

$$(3.7)\qquad *\tilde{W}^u_{0y}=\xi\alpha\cdot\delta^u_y+\zeta_y\alpha^u-\|\zeta\|^2\phi^u_y-\alpha_y\zeta^u-(1/2)(*\nabla_y\zeta^u-*\nabla^u\zeta_y).$$

PROOF. (3.6) follows from (3.4). To prove (3.7) it suffices to notice that $*\nabla_x\zeta_y=\nabla_x\zeta_y$ holds.

The next relation is necessary in the following:

$$(3.8)\qquad *\nabla_x\alpha_y-*\nabla_y\alpha_x=-2\xi\alpha\cdot\phi_{xy}.$$

COROLLARY 3.5. $\tilde{Q}^u_{xy}=Q^u_{xy}$ *holds.*

PROOF. By (3.6) we obtain

$$\begin{aligned}\tilde{Q}^u_{xy}-Q^u_{xy}&=*\tilde{\nabla}_y\tilde{\phi}^u_x-*\nabla_y\phi^u_x\\&=*\tilde{\nabla}_y\phi^u_x-*\nabla_y\phi^u_x\\&=*\tilde{W}^u_{yv}\phi^v_x-*\tilde{W}^v_{yx}\phi^u_v=0.\end{aligned}$$

§ 4. The curvature tensors of the generalized Tanaka connection.

Since $*\nabla\xi=*\nabla\eta=0$, with respect to a P-related frame we have

$$(4.1)\qquad *R^i_{0kl}=*R^0_{jkl}=0.$$

If we write $*R_{ijkl}=g_{ir}*R^r_{jkl}$, then $*\nabla g=0$ implies

$$*R_{ijkl}+*R_{jikl}=0.$$

By (2.2) the torsion $*T$ of $*\nabla$ satisfies the following:

$$(4.2)\qquad *T^0_{xy}=2\phi_{xy},\quad *T^u_{0y}=p^u_y,\quad *T^u_{xy}=*T^0_{0y}=0.$$

The first Bianchi identity is

$$\mathfrak{S}[*R(X, Y)Z]=\mathfrak{S}[*T(*T(X, Y), Z)+(*\nabla_X *T)(Y, Z)]$$

where $\mathfrak{S}$ denotes the cyclic sum with respect to X, Y and Z. So we obtain the following:

$$(4.3)\qquad *R^u_{zxy}+*R^u_{xyz}+*R^u_{yzx}=2(\phi_{xy}p^u_z+\phi_{yz}p^u_x+\phi_{zx}p^u_y).$$

$$(4.4)\qquad *R^u_{z0y}+*R^u_{yz0}=-*\nabla_y p^u_z+*\nabla_z p^u_y.$$

By the Ricci identity:

$$*\nabla_k*\nabla_l\phi^i_j-*\nabla_l*\nabla_k\phi^i_j=*R^i_{rkl}\phi^r_j-*R^r_{jkl}\phi^i_r-*T^r_{kl}*\nabla_r\phi^i_j,$$

we obtain

(4.5) $\quad {}^*R^u_{vxy}\phi^v_z - \phi^u_v {}^*R^v_{zxy} = {}^*\nabla_x Q^u_{zy} - {}^*\nabla_y Q^u_{zx}.$

The second Bianchi identity:

$$\mathfrak{S}[({}^*\nabla_X {}^*R)(Y, Z)] = \mathfrak{S}[{}^*R(Z, {}^*T(X, Y))]$$

and (4.2) give

(4.6) $\quad {}^*\nabla_x {}^*R^u_{wyz} + {}^*\nabla_y {}^*R^u_{wzx} + {}^*\nabla_z {}^*R^u_{wxy} = -2\phi_{xy} {}^*R^u_{w0z} - 2\phi_{yz} {}^*R^u_{w0x} - 2\phi_{zx} {}^*R^u_{w0y},$

(4.7) $\quad {}^*\nabla_0 {}^*R^u_{wyz} + {}^*\nabla_y {}^*R^u_{wz0} + {}^*\nabla_z {}^*R^u_{w0y} = -{}^*R^u_{wvz} p^v_y + {}^*R^u_{wvy} p^v_z.$

We denote by $(4.3)_{wzxy}$ the identity obtained from (4.3) by lowering the index u by g_{uw} and calculate $(4.3)_{wyzx} - (4.3)_{zwxy} - (4.3)_{xzyw} + (4.3)_{yxwz}$ to get

(4.8) $\quad {}^*R_{wzxy} - {}^*R_{xywz} = -2\phi_{xz} p_{yw} + 2\phi_{xw} p_{yz} + 2\phi_{yz} p_{xw} - 2\phi_{yw} p_{xz}.$

Similarly, we denote by $(4.4)_{wzy}$ the identity obtained from (4.4) by lowering the index u by g_{uw}. Then calculating $(4.4)_{wzy} + (4.4)_{ywz} - (4.4)_{zyw}$. we obtain

(4.9) $\quad {}^*R_{wy0z} = {}^*\nabla_y p_{wz} - {}^*\nabla_w p_{yz}.$

We apply (4.8) to ${}^*R_{wzuv}\phi^u_x\phi^v_y$, apply (4.5) to the result, and again apply (4.8) to the result. Then we obtain

(4.10) $\quad {}^*R_{wzuv}\phi^u_x\phi^v_y = {}^*R_{wzxy} - 2g_{xz} p_{vw}\phi^v_y + 2g_{xw} p_{vz}\phi^v_y + 2g_{yz} p_{uw}\phi^u_x - 2g_{yw} p_{uz}\phi^u_x + 2\phi_{xz} p_{yw} - 2\phi_{xw} p_{yz} - 2\phi_{yz} p_{xw} + 2\phi_{yw} p_{xz} + {}^*\nabla_z Q^v_{xw}\phi_{vy} - {}^*\nabla_w Q^v_{xz}\phi_{vy}.$

The Ricci tensor $({}^*R_{jl})$ is defined by ${}^*R_{jl} = {}^*R^r_{jrl}$. By (4.1) and (4.9) we obtain ${}^*R_{0y} = 0$ and ${}^*R_{x0} = {}^*\nabla_u p^u_x$. By (4.8) and $\phi^v_x p_{vy} = \phi^v_y p_{vx}$, we can verify that

(4.11) $\quad {}^*R_{xy} = {}^*R_{yx}$

holds. As in the Kaehlerian geometry we define $({}^*k_{xy})$ by

(4.12) $\quad {}^*k_{xy} = \frac{1}{2}\phi^u_v {}^*R^v_{uxw}\phi^w_y.$

LEMMA 4.1. *The relation between* ${}^*k_{xy}$ *and* ${}^*R_{xy}$ *is given by*

(4.13) $\quad {}^*k_{xy} = {}^*R_{xy} + (m-3) p_{xu}\phi^u_y - \phi^u_v {}^*\nabla_u Q^v_{yx} + \phi^u_v {}^*\nabla_x Q^v_{yu}.$

PROOF. By (4.3), (4.5) and (4.11) we obtain

$$\begin{aligned}{}^*R_{xy} &= {}^*R^v_{yvx}\\ &= -\phi^v_u{}^*R^u_{wvx}\phi^w_y + \phi^v_u{}^*\nabla_v Q^u_{yx} - \phi^v_u{}^*\nabla_x Q^u_{yv}\\ &= \phi^v_u({}^*R^u_{vxw} + {}^*R^u_{xwv})\phi^w_y - 2\phi^v_u(\phi_{wv}p^u_x + \phi_{vx}p^u_w)\phi^w_y\\ &\quad + \phi^v_u{}^*\nabla_v Q^u_{yx} - \phi^v_u{}^*\nabla_x Q^u_{yv}\\ &= 2{}^*k_{xy} + {}^*R^u_{xwv}\phi^w_y\phi^v_u + \phi^v_u{}^*\nabla_v Q^u_{yx} - \phi^v_u{}^*\nabla_x Q^u_{yv}.\end{aligned}$$

Applying (4.10) to the second term of the last line we get (4.13). It follows from (4.12) that

$$(4.14)\qquad {}^*k_{uv}\phi^u_x\phi^v_y = {}^*k_{yx}.$$

By (4.13) and (4.14) we get

$$(4.15)\qquad \begin{aligned}{}^*R_{uv}\phi^u_x\phi^v_y &= {}^*R_{xy} + 2(m-3)p_{xu}\phi^u_y - \phi^u_v{}^*\nabla_u Q^v_{xy} + \phi^u_v{}^*\nabla_y Q^v_{xu}\\ &\quad + \phi^u_v{}^*\nabla_u Q^v_{zw}\phi^w_x\phi^z_y - \phi^u_v{}^*\nabla_w Q^v_{zu}\phi^w_x\phi^z_y.\end{aligned}$$

§ 5. Curvature tensor changes under gauge transformations.

By one of classical formulas, the difference of the curvature tensors ${}^*\tilde{R}$ and ${}^*R$ is given by

$${}^*\tilde{R}^u_{zxy} = {}^*R^u_{zxy} + {}^*\nabla_x{}^*\tilde{W}^u_{yz} - {}^*\nabla_y{}^*\tilde{W}^u_{xz} + {}^*\tilde{W}^k_{yz}{}^*\tilde{W}^u_{xk} - {}^*\tilde{W}^k_{xz}{}^*\tilde{W}^u_{yk} + {}^*T^k_{xy}{}^*W^u_{kz}.$$

Since ${}^*\nabla$ acts naturally on P-related frames, Corollamy 3.4 is enough to give the following:

$$(5.1)\qquad \begin{aligned}{}^*\tilde{R}^u_{zxy} - {}^*R^u_{zxy} &= {}^*\nabla_x a_z\delta^u_y - {}^*\nabla_y a_z\delta^u_x\\ &\quad + {}^*\nabla_x\zeta^u\phi_{yz} - {}^*\nabla_y\zeta^u\phi_{xz} + {}^*\nabla_x\zeta_z\phi^u_y - {}^*\nabla_y\zeta_z\phi^u_x\\ &\quad + {}^*\nabla_x\zeta_y\phi^u_z - {}^*\nabla_y\zeta_x\phi^u_z - g_{yz}{}^*\nabla_x a^u + g_{xz}{}^*\nabla_y a^u\\ &\quad + \delta^u_x(a_y a_z - \zeta_y\zeta_z - \|\zeta\|^2 g_{yz}) - \delta^u_y(a_x a_z - \zeta_x\zeta_z - \|\zeta\|^2 g_{xz})\\ &\quad + \zeta^u(-a_x\phi_{yz} + a_y\phi_{xz} - \zeta_x g_{yz} + \zeta_y g_{xz})\\ &\quad + \phi^u_x(a_y\zeta_z + \|\zeta\|^2\phi_{yz} + a_z\zeta_y) - \phi^u_y(a_x\zeta_z + |\zeta|^2\phi_{xz} + a_z\zeta_x)\\ &\quad - a^u(\zeta_x\phi_{yz} - \zeta_y\phi_{xz} - a_x g_{yz} + a_y g_{xz})\\ &\quad - \phi_{xy}(2\|\zeta\|^2\phi^u_z + {}^*\nabla_z\zeta^u - {}^*\nabla^u\zeta_z)\\ &\quad + \zeta_z Q^u_{yx} - \zeta_z Q^u_{xy} + \zeta_y Q^u_{zx} - \zeta_x Q^u_{zy} - \zeta^u{}^*\nabla_z\phi_{xy}.\end{aligned}$$

We define (A_{yz}) modifying one defined by Sakamoto-Takemura [3]:

$$(5.2)\qquad A_{yz} = {}^*\nabla_y a_z - a_y a_z + \zeta_y\zeta_z + (1/2)\|\zeta\|^2 g_{yz} + \xi a\cdot\phi_{yz}.$$

By (3.8) we see that (A_{yz}) is symmetric. Next we difine (G_{yz}) by $G_{yz} = A_{yv}\phi^v_z$. Then we obtain

$$G_{yz} = -{}^*\nabla_y\zeta_z + a_y\zeta_z + a_z\zeta_y + (1/2)\|\zeta\|^2\phi_{yz} - \xi a\cdot g_{yz} - a_v Q^v_{zy}.$$

Then (5.1) is rewritten as

$$(5.3)\quad \begin{aligned}{}^*\tilde{R}_{zxy}^{u}-{}^*R_{zxy}^{u}=&-A_{yz}\delta_x^u+A_{xz}\delta_y^u+G_{yz}\phi_x^u-G_{xz}\phi_y^u\\&-g_{yz}A_x^u+g_{xz}A_y^u-\phi_{yz}G_x{}^u+\phi_{xz}G_y{}^u\\&-(G_{xy}-G_{yx})\phi_z^u+\phi_{xy}(G_z{}^u-G^u{}_z)\\&+\alpha_vQ_{zy}^v\phi_x^u-\alpha_vQ_{zx}^v\phi_y^u-\phi_{yz}\alpha_vQ_{wx}^vg^{uw}+\phi_{xz}\alpha_vQ_{wy}^vg^{uw}\\&-\alpha_v(Q_{yx}^v-Q_{xy}^v)\phi_z^u+\alpha_v(Q_{wz}^v-Q_{zw}^v)g^{wu}\phi_{xy}\\&+\zeta_zQ_{yx}^u-\zeta_zQ_{xy}^u+\zeta_yQ_{zx}^u-\zeta_xQ_{zy}^u-\zeta^u{}^*\nabla_z\phi_{xy}.\end{aligned}$$

By a simple calculation we get

$$\begin{aligned}\phi_x^{v*}\nabla_v\alpha_y-{}^*\nabla_x\zeta_y&=\phi_x^v({}^*\nabla_y\alpha_v-2\xi\alpha\cdot\phi_{vy})-{}^*\nabla_x\zeta_y\\&=-{}^*\nabla_y\zeta_x-{}^*\nabla_x\zeta_y-2\xi\alpha\cdot g_{xy}-\alpha_vQ_{xy}^v\\&=-(\nabla_x\zeta_y+\nabla_y\zeta_x)-2\xi\alpha\cdot g_{xy}-\alpha_vQ_{xy}^v.\end{aligned}$$

Consequently, Lemmas 3.1 and 3.2 imply

$$(5.4)\quad \begin{aligned}2(\tilde{p}_{xy}-p_{xy})&=\tilde{\nabla}_x\tilde{\eta}_y+\tilde{\nabla}_y\tilde{\eta}_x-(\nabla_x\eta_y+\nabla_y\eta_x)\\&=\sigma[(\nabla_x\eta_y+\nabla_y\eta_x)-2(\tilde{\xi}^0B_{xy}+C_{xy}^0)]-(\nabla_x\eta_y+\nabla_y\eta_x)\\&=\nabla_x\zeta_y+\nabla_y\zeta_x+2\xi\alpha\cdot g_{xy}-2(\alpha_x\zeta_y+\alpha_y\zeta_x)\\&=-\phi_x^{v*}\nabla_v\alpha_y+{}^*\nabla_x\zeta_y-2(\alpha_x\zeta_y+\alpha_y\zeta_x)-\alpha_vQ_{xy}^v.\end{aligned}$$

Furthermore we get

$$\begin{aligned}G_{yz}+G_{zy}&=A_{yv}\phi_z^v+A_{vz}\phi_y^v\\&=\phi_y^{v*}\nabla_v\alpha_z-{}^*\nabla_y\zeta_z+2(\alpha_y\zeta_z+\alpha_z\zeta_y)-\alpha_vQ_{zy}^v\\&=-2(\tilde{p}_{yz}-p_{yz})-\alpha_v(Q_{yz}^v+Q_{zy}^v),\\A_{uv}\phi_y^u\phi_z^v-A_{yz}&=-2(\tilde{p}_{yv}-p_{yv})\phi_z^v-\alpha_v(Q_{yw}^v+Q_{wy}^v)\phi_z^w.\end{aligned}$$

In particular, we see that $Tr(G)=g^{uv}G_{uv}=0$ holds. By Lemma 2.3, (5.3) and the above relations, we obtain

$$(5.5)\quad \begin{aligned}{}^*\tilde{R}_{yz}-{}^*R_{yz}=&-(m+3)A_{yz}-Tr(A)g_{yz}+6(\tilde{p}_{yv}-p_{yv})\phi_z^v\\&+2\alpha_v(Q_{yw}^v+Q_{wy}^v)\phi_z^w,\end{aligned}$$

$$(5.6)\quad \sigma^*\tilde{S}-{}^*S=-4(n+1)Tr(A),$$

where ${}^*S=g^{xy*}R_{xy}$ is the (generalized) Tanaka-Webster scalar curvature. By (4.13) and (5.5) we obtain

$$\begin{aligned}{}^*\tilde{k}_{xy}&={}^*\tilde{R}_{xy}+(m-3)\tilde{p}_{xv}\phi_y^v-\phi_v^{u*}\tilde{\nabla}_uQ_{yx}^v+\phi_v^{u*}\tilde{\nabla}_xQ_{yu}^v\\&={}^*k_{xy}-(m+3)A_{xy}-Tr(A)g_{xy}+(m+3)(\tilde{p}_{xw}-p_{xw})\phi_y^w\\&\quad-\phi_v^u({}^*\tilde{\nabla}_uQ_{yx}^v-{}^*\nabla_uQ_{yx}^v-{}^*\tilde{\nabla}_xQ_{yu}^v+{}^*\nabla_xQ_{yu}^v)\\&\quad+2\alpha_v(Q_{xw}^v+Q_{wx}^v)\phi_y^w.\end{aligned}$$

By (2.3) and (3.6) we get

$$-\phi_v^u({}^*\tilde{\nabla}_u Q_{yx}^v - {}^*\nabla_u Q_{yx}^v - {}^*\tilde{\nabla}_x Q_{yu}^v + {}^*\nabla_x Q_{yu}^v)$$
$$= -\phi_v^u({}^*\tilde{W}_{uw}^v Q_{yx}^w - {}^*\tilde{W}_{uy}^w Q_{wx}^v - {}^*\tilde{W}_{ux}^w Q_{yw}^v - {}^*\tilde{W}_{xw}^v Q_{yu}^w + {}^*\tilde{W}_{xu}^w Q_{yw}^v)$$
$$= 2n\zeta_v Q_{yx}^v - 2a^v \phi_x^w {}^*\nabla_v \phi_{yw}$$
$$= 2n\zeta_v Q_{yx}^v + 2a_v(Q_{yw}^v - Q_{wy}^v)\phi_x^w.$$

Therefore we obtain

$$(5.7)\qquad {}^*\tilde{k}_{xy} = {}^*k_{xy} - (m+3)A_{xy} - Tr(A)g_{xy} + (m+3)(\tilde{p}_{xw} - p_{xw})\phi_y^w + (m+3)a_v\phi_x^u Q_{yu}^v.$$

Now we define (L_{xy}) and (N_{xy}) by

$$(5.8)\qquad L_{xy} = \frac{-1}{2(n+2)}{}^*k_{xy} + \frac{1}{8(n+1)(n+2)}{}^*Sg_{xy} + p_{xv}\phi_y^v,$$

and $N_{xy} = L_{xv}\phi_y^v$. Then we obtain

$$A_{xy} = \tilde{L}_{xy} - L_{xy} + a_v\phi_x^w Q_{yw}^v,$$
$$G_{xy} = \tilde{N}_{xy} - N_{xy} - a_v Q_{yx}^v.$$

Substituting the above into (5.3) we get

$$\begin{aligned}(5.9)\qquad {}^*\tilde{R}_{zxy}^u - {}^*R_{zxy}^u = &-(\tilde{L}_{yz} - L_{yz})\delta_x^u + (\tilde{L}_{xz} - L_{xz})\delta_y^u \\ &+ (\tilde{N}_{yz} - N_{yz})\phi_x^u - (\tilde{N}_{xz} - N_{xz})\phi_y^u \\ &- \tilde{g}_{yz}\tilde{L}_x{}^u + g_{yz}L_x{}^u + \tilde{g}_{xz}\tilde{L}_y{}^u - g_{xz}L_y{}^u - \tilde{\phi}_{yz}\tilde{N}_x{}^u + \phi_{yz}N_x{}^u \\ &+ \tilde{\phi}_{xz}\tilde{N}_y{}^u - \phi_{xz}N_y{}^u - (\tilde{N}_{xy} - \tilde{N}_{yx})\phi_z^u + (N_{xy} - N_{yx})\phi_z^u \\ &+ \tilde{\phi}_{xy}(\tilde{N}_z{}^u - \tilde{N}^u{}_z) - \phi_{xy}(N_z{}^u - N^u{}_z) \\ &+ a_v(-\phi_y^w Q_{zw}^v \delta_x^u + \phi_x^w Q_{zw}^v \delta_y^u - g_{yz}\phi_x^w Q_{tw}^v g^{tu} + g_{xz}\phi_y^w Q_{tw}^v g^{tu} \\ &- \phi_z^v Q_{yx}^u + \phi_z^v Q_{xy}^u - \phi_y^v Q_{zx}^u + \phi_x^v Q_{zy}^u - \phi^{uv} Q_{yz}^w g_{xw}).\end{aligned}$$

We define (B_{zxy}^u) by

$$\begin{aligned}(5.10)\qquad B_{zxy}^u = &{}^*R_{zxy}^u + L_{yz}\delta_x^u - L_{xz}\delta_y^u - N_{yz}\phi_x^u + N_{xz}\phi_y^u \\ &+ g_{yz}L_x{}^u - g_{xz}L_y{}^u + \phi_{yz}N_x{}^u - \phi_{xz}N_y{}^u \\ &+ (N_{xy} - N_{yx})\phi_z^u - \phi_{xy}(N_z{}^u - N^u{}_z).\end{aligned}$$

(B_{zxy}^u) is not a tensor on M, but it has meaning only for P.

THEOREM. *Let (M, η, g) be a contact Riemannian manifold. Then, for any gauge transformation $(\eta \to \tilde{\eta} = \sigma\eta)$ of contact Riemannian structure,*

$$\tilde{B}_{zxy}^u = B_{zxy}^u$$

holds with respect to a P-related frame, if and only if the CR-structure corresponding to (η, ϕ) is integrable.

PROOF. Assume that $\tilde{B}_{zxy}^u = B_{zxy}^u$ holds. By (5.9) we obtain

$$-\phi_y^w Q_{zw}^v \delta_x^u + \phi_x^w Q_{zw}^v \delta_y^u - g_{yz}\phi_x^w Q_{tw}^v g^{tu} + g_{xz}\phi_y^w Q_{tw}^v g^{tu}$$
$$-\phi_z^v Q_{yx}^u + \phi_z^v Q_{xy}^u - \phi_y^v Q_{zx}^u + \phi_x^v Q_{zy}^u - \phi^{uv} Q_{yz}^w g_{xw} = 0.$$

Contracting the last equality with respect to u and x, and operating ϕ_u^y to the result, we get

$$(2n-1)Q_{zu}^v + Q_{uz}^v = 0.$$

If $n=1$, $Q=0$ is a trivial consequence. So, we can assume that n is greater than 1. Then the above implies $Q=0$, and the CR-structure corresponding to (η, ϕ) is integrable. The converse is clear.

REMARK. By (7.1), (8.1) and (8.2) of [6] we obtain

$$*R_{zxy}^u = R_{zxy}^u + 2\phi_z^u \phi_{xy} + \nabla_x \xi^u \nabla_y \eta_z - \nabla_y \xi^u \nabla_x \eta_z,$$
$$*R_{xy} = R_{xy} + 2g_{xy} + \nabla_\xi p_{xy},$$
$$*S = S - R_{rs}\xi^r \xi^s + 4n.$$

Therefore, noticing $*\nabla_u Q_{xy}^v = \nabla_u Q_{xy}^v$, etc., we can rewrite (5.10) as the expression with respect to the Riemannian connection.

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Department of Mathematics
Tokyo Institute of Technology
Tokyo, Japan