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On H-separable extensions of primitive rings II

Dedicated to Professor Kazuhiko Hirata on his 60th birthday

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Introduction. Throughout this paper every ring is assumed to have the identity, and all subrings of a ring will contain the identity of the ring, unless otherwise stated. Let B be a strongly primitive ring and A an H-separable extension of B, and suppose A is left B-finitely generated projective. In [13] it is shown that in this case A is also strongly primitive if and only if $A\mathfrak{z}A\cap B=\mathfrak{z}$, where $\mathfrak{z}$ is the socle of B. The aim of this paper is to detail the structure of A and B which satisfy the above condition. Let furthermore I and $\mathfrak{m}$ be faithful minimal left ideals of A and B, respectively, and denote the double centralizers of ${}_AI$, ${}_BI$ and ${}_B\mathfrak{m}$ by A^*, $\tilde{B}$ and B^*, respectively. Then there exists a ring isomorphism Φ of B^* to $\tilde{B}(\subseteq A^*)$ such that $\Phi(b)=b$ for each $b\in B$, and A^* is an H-separable extension of $\tilde{B}(\cong B^*)$ (Theorem 3.3), that is, the right full linear ring A^* is an inner Galois extension of the right full linear ring B^* (See Theorem 4 [11]). We will also treat the inner Galois theory of full linear rings in §4. Let A be a right full linear ring with its center C, D a simple C-subalgebra of A with $[D:C]<\infty$ and $B=V_A(D)$. Denote the class of right full linear subrings R of A such that R contains B and the simple left ideal of A is a finite direct sum of faithful simple left R-modules by $\mathscr{L}$, and the class of simple C-subalgebras of $V_A(B)$ by $\mathscr{D}$. We already know that there exists a duality between $\mathscr{L}$ and $\mathscr{D}$. We will show that a right full linear subring R of A containing B is in $\mathscr{L}$ if and only if A is left or right B-projective (Theorem 4.1). §1 is the preparation for §2, and in §2 we will introduce some fundamental properties of strongly primitive rings. Let R be a ring and M a flat left R-module, and denote the Gabriel topology of R consisting of right ideals $\mathfrak{a}$ of R such that $\mathfrak{a}M=M$ by $\mathfrak{F}$. As K. Morita showed in [5], there is a ring isomorphism θ of $R_{\mathfrak{F}}$, the ring of quotients of R with respect to $\mathfrak{F}$, to a subring of $R^*=\mathrm{Bic}({}_RM)$ In [3] the author gave a simpler proof of this theorem. Here we will determine $\mathrm{Im}\,\theta$ completely, and show that $\mathrm{Im}\,\theta$ consists of elements r^* of R^* such that $\mathfrak{a}r^*\subset\tilde{R}$ for some $\mathfrak{a}$ in $\mathfrak{F}$, where $\tilde{R}$ is the image of the canonical map of R to R^* (Theorem 1.1). By applying this theorem to the strongly primitive ring, we can obtain a generalization of the last part of Theorem 3 [1], that is, if R is a strongly primitive ring with its socle $\mathfrak{z}$ and a faithful minimal left ideal $\mathfrak{m}$, the above map θ induces the isomorphism of End$({}_{\mathfrak{z}}{}_R)$ to $R^*=$Bic$({}_R\mathfrak{m})$. This means that, regarding R as a subring of R^* by the canonical map, $\mathfrak{z}$ becomes a left ideal of R^*, and the map σ of R^* to End$(\mathfrak{z}_R)$ such that $\sigma(r^*)(a)=r^*a$, for each $r^*\in R^*$ and $a \in\mathfrak{z}$, is an isomorphism (Theorem 2.1).

1. Let R be a ring and M a left R-module. Assume that M is R-flat, and let $\mathfrak{F}$ be the set of right ideals $\mathfrak{a}$ of R such that $\mathfrak{a}M=M$. Then $\mathfrak{F}$ is a Gabriel topology on R, and as is shown in [6] we can construct the rings $R_{(\mathfrak{F})}=\varinjlim_{\mathfrak{a}\in\mathfrak{F}}\mathrm{Hom}(\mathfrak{a}_R, R_R)$ and $R_{\mathfrak{F}}=\varinjlim_{\mathfrak{a}\in\mathfrak{F}}\mathrm{Hom}(\mathfrak{a}_R, R/t(R)_R)$, where $t(R)$ is the $\mathfrak{F}$-torsion submodule of R, namely, $t(R)=\{x\in R|x\mathfrak{a}=0$ for some $\mathfrak{a}\in\mathfrak{F}\}$. For any $m\in M$ and $x\in R_{\mathfrak{F}}$, if x is represented by $\xi:\mathfrak{a}_R\to R/t(R)_R$ with $\mathfrak{a}\in\mathfrak{F}$, then we have $m=\Sigma a_im_i$ with $a_i\in\mathfrak{a}$ and $m_i\in M$, since $m\in M=\mathfrak{a}M$. Then we can define $xm=\Sigma\xi(a_i)m_i$, and by this definition we can make M a left $R_{\mathfrak{F}}$-module such that $R_{\mathfrak{F}}\otimes_RM\cong M$, via $x\otimes m\to xm$, for $x\in R_{\mathfrak{F}}$ and $m\in M$, and Hom$({}_{R_{\mathfrak{F}}}M, {}_{R_{\mathfrak{F}}}N)=Hom({}_RM, {}_RN)$ for any $R_{\mathfrak{F}}$-module N. (See [11]). Let $S=$ Hom$({}_RM, {}_RM)$ and $R^*=$ Bic$({}_RM)=$ Hom(M_S, M_S). There exists a ring homomorphism θ of $R_{\mathfrak{F}}$ to R^* such that $\theta(x)(m)=xm$, for $x\in R_{\mathfrak{F}}$ and $m\in M$, since $S=$Hom$({}_{R_{\mathfrak{F}}}M, {}_{R_{\mathfrak{F}}}M)$. θ is an injection, since $t(R)=Ann({}_RM)$. Denote the canonical ring homomorphisms of R to R^* and R to $R_{\mathfrak{F}}$ by ι and ψ, respectively. Then $\iota=\theta\psi$. Now we have the completion of theorems 1.4 and 1.6 [5] as follows (See also Theorem 1 [11]).

THEOREM 1.1. *With the same notation as above, $R_{\mathfrak{F}}$ is isomorphic to the subring of R^* consisting of all elements r^* of* R* *such that $r^*\mathfrak{a}\subset Im\ \iota$ for some $\mathfrak{a}\in\mathfrak{F}$, namely, $Im\theta=\pi^{-1}(t(R^*/Im\ \iota))$, where π is the canonical map of R^* to $R^*/Im\ \iota$.*

PROOF. Since $Cok\psi$ is $\mathfrak{F}$-torsion and $\theta\psi=\iota$, $Im\theta/Im\iota$ is also $\mathfrak{F}$-torsion. Thus $Im\theta\subset\pi^{-1}(t(Cok\iota))$. Let $r^*\in\pi^{-1}(t(Cok\iota))$. This means that there exists $\mathfrak{a}\in\mathfrak{F}$ such that $r^*\mathfrak{a}\subset Im\iota$. But we have $Im\iota=R/t(R)$, since $Ker\iota=Ann({}_RM)=t(R)$. Therefore, for each $a\in\mathfrak{a}$ there exists an $\bar{r}\in R/t(R)$ such that $\bar{r}m=(r^*a)(m)=r^*(am)$ for each $m\in M$, that is, $r^*a=\bar{r}\in R/t(R)$. Thus we have an R-homomorphism ξ of $\mathfrak{a}$ to $R/t(R)$ such that $\xi(a)=r^*a\in R/t(R)$. Let x be the element of $R_{\mathfrak{F}}$ represented by ξ, and let $m=\Sigma a_im_i$ with $a_i\in\mathfrak{a}$ and $m_i\in M$. Then $xm=\Sigma\xi(a_i)m_i=\Sigma(r^*a_i)m_i=r^*(\Sigma a_im_i)=r^*(m)$, for each $m\in M$. This means $r^*=x\in Im\theta$. Thus we have $\pi^{-1}(t(Cok\iota))\subset Im\theta$, and consequently, $Im\theta=$

$\pi^{-1}(t(Cok\iota))$.

COROLLARY 1.1. (Proposition 8.5 XI [6]). *If M is R-finitely generated projective, then θ is an isomorphism, i. e., $R_{\mathfrak{F}} \cong Bic({}_RM)$.*

PROOF. Since M is R-finitely generated projective, we have $R^* \otimes_R M \cong M$, via $r^* \otimes m \to r^*(m)$, for any $r^* \in R^*$ and $m \in M$. Thus we have $R^*/Im\iota \otimes_R M = 0$, which means that $R^*/Im\iota$ is $\mathfrak{F}$-torsion. Then we have that $Im\theta = R^*$ by Theorem 1.1.

COROLLARY 1.2. *Let M be a faithful finitely generated projective R-module, and $\mathfrak{a}$ the trace ideal of M in R. Then we have an isomorphism ρ of $Hom(\mathfrak{a}_R, \mathfrak{a}_R)$ to R^* such that $\rho(\xi)(m) = \Sigma\xi(a_i)m_i$ for each $\xi \in Hom(\mathfrak{a}_R, \mathfrak{a}_R)$ and $m \in M$, where $m = \Sigma a_i m_i$ with $a_i \in \mathfrak{a}$ and $m_i \in M$. Moreover, $\mathfrak{a}$ is a left ideal of R^*, regarding R as a subring of R^* by the usual way, and the inverse map σ of ρ is given by $\sigma(r^*)(a) = r^*a$, for each $r^* \in R^*$ and $a \in \mathfrak{a}$.*

PROOF. Since M is R-projective, we have $\mathfrak{a}^2 = \mathfrak{a}$ and $\mathfrak{a}M = M$. $\mathfrak{a}$ is contained in every right ideal belonging to $\mathfrak{F}$. Hence we have $R_{(\mathfrak{F})} = Hom(\mathfrak{a}_R, \mathfrak{a}_R)$. But $t(R) = Ann({}_RM) = 0$, since M is R-faithful. Therefore we have $R^* \cong R_{\mathfrak{F}} = R_{(\mathfrak{F})} = Hom(\mathfrak{a}_R, \mathfrak{a}_R)$. Next, since $R^*/R(=R^*/Im\iota)$ is $\mathfrak{F}$-torsion, we have $r^*\mathfrak{a} \subset R$ for each $r^* \in R^*$. But $\mathfrak{a} = \mathfrak{a}^2$. Hence $r^*\mathfrak{a} = (r^*\mathfrak{a})\mathfrak{a} \subset R\mathfrak{a} = \mathfrak{a}$. Thus $\mathfrak{a}$ is a left ideal of R^*. Note that $r^*a = b \in \mathfrak{a}$, for $a \in \mathfrak{a}$, means that $r^*(am) = bm$ for each $m \in M$. Therefore if we define $\sigma(r^*)(a) = r^*a$ for $r^* \in R^*$ and $a \in \mathfrak{a}$, we have $(\rho\sigma(r^*))(m) = \Sigma\sigma(r^*)(a_i)m_i = \Sigma(r^*a_i)(m_i) = r^*(\Sigma a_i m_i) = r^*(m)$ for each $r^* \in R^*$ and $m \in M$, where $m = \Sigma a_i m_i$ with $a_i \in \mathfrak{a}$ and $m_i \in M$. Thus we have $\rho\sigma = 1_{R^*}$ and $\sigma = \rho^{-1}$.

2. Now we will apply the results of § 1 to the theory on strongly primitive rings. For a few moments we do not assume that all rings have the identities. A ring R is said to be strongly primitive if R has a faithful minimal left ideal. In this case R has also a faithful minimal right ideal, and the left socle of R coincides with the right socle and is the smallest non zero ideal of R. It is shown in Lemma 2 [1] that the typical examples of strongly primitive rings are subrings of a left (or right) full linear ring which contain the socle of it. Here we will give a generalization of it with a simpler proof.

PROPOSITION 2.1. *Let R be a strongly primitive ring with the socle $\mathfrak{z}$. Then every subring of R which contains $\mathfrak{z}$ is also a strongly primitive ring.*

PROOF. Let $\mathfrak{l}$ be a faithful minimal left ideal of R. $\mathfrak{l}$ is a left ideal of $\mathfrak{z}$. Let $\mathfrak{n}$ be a non zero left ideal of $\mathfrak{z}$ contained in $\mathfrak{l}$. $\mathfrak{z}$ is faithful as right R-module. Hence $\mathfrak{z}\mathfrak{n}$ is a non zero left ideal of R with $\mathfrak{z}\mathfrak{n}\subset\mathfrak{n}\subset\mathfrak{l}$. Then we have $\mathfrak{z}\mathfrak{n}=\mathfrak{n}=\mathfrak{l}$. Thus $\mathfrak{l}$ is a minimal left ideal of $\mathfrak{z}$. Then $\mathfrak{l}$ is a faithful minimal left ideal of every subring of R containing $\mathfrak{z}$. (See §2. 4 [4]).

The next theorem is a generalization of the last part of Theorem 3 [1].

THEOREM 2. 1. *Let R be a strongly primitive ring with the socle $\mathfrak{z}$ and $\mathfrak{l}$ a faithful minimal left ideal of R. Denote the double centralizer of ${}_R\mathfrak{l}$ by R^*. Then $\mathfrak{z}$ is a left ideal of R^*, and the map σ of R^* to Hom$(\mathfrak{z}_R, \mathfrak{z}_R)$ defined by $\sigma(r^*)(x)=r^*x$, for $r^*\in R^*$ and $x\in\mathfrak{z}$, is an isomorphism.*

PROOF. By Theorem 1 [1] we have $\mathfrak{l}=Re$ for some primitive idempotent e of R. Hom$({}_RRe, {}_RRe)=eRe$ and $R\subset R^*=$Hom(Re_{eRe}, Re_{eRe}). Of course, Re is R^*-faithful. Let R' be the subring of R^* generated by R and the identity of R^*. Then we have $R'R=RR'=R$, and consequently, $R'e=Re$, and see that Re is faithful minimal left ideal of R'. Thus R' is also strongly primitive. Next, let $R'f$ be any minimal left ideal of R' with $f^2=f\in R'$. Since $R'e\cong R'f$, there exist $x, y\in R'$ such that $f=fyeexf$ and $e=exffye$. Then $f\in R'RR'=R$, and $R'f=Rf\subset\mathfrak{z}$. This means that the socle of R' coincides with $\mathfrak{z}$. Moreover since $Re=R'e$, we have $eRe=eR'e$, and see that the double contralizer of ${}_{R'}R'e$ coincides with R^*, while Hom$(\mathfrak{z}_R, \mathfrak{z}_R)=Hom(\mathfrak{z}_{R'}, \mathfrak{z}_{R'})$. Therefore we can assume that R has the identity. Then $\mathfrak{l}=Re$ is R-faithful finitely generated projective, and $\mathfrak{z}$ coincides with the trace ideal of ${}_R\mathfrak{l}$ in R, since every two minimal left ideals are isomorphic. Now we can apply Corollary 1. 2.

COROLLARY 2. 1. *With the same notation as Theorem 2. 1, we have that $\mathfrak{z}R^*$ coincides with the socle of R^*.*

PROOF. Let $\mathfrak{z}^*$ be the socle of R^*. Since $\mathfrak{z}R^*$ is an ideal of R^* by Theorem 2. 1, we have $\mathfrak{z}R^*\supset\mathfrak{z}^*$. Let f be any primitive idempotent of R. Then $Re\cong Rf$ and $R^*f\cong R^*\otimes_RRf\cong R^*\otimes_RRe\cong Re$ as R^*-module. Thus R^*f is a minimal left ideal of R^*, and we have $f\in\mathfrak{z}^*$. This means that $\mathfrak{z}\subset\mathfrak{z}^*$ and $\mathfrak{z}R^*\subset\mathfrak{z}^*$. Now we have $\mathfrak{z}R^*=\mathfrak{z}^*$.

LEMMA 2. 1. *Let R be a left primitive ring and M a faithful simple left R-module. Then, for each non zero idempotent e of R, eM is a faithful simple left eRe-module. Thus eRe is also left primitive.*

PROOF. It is obvious that eM is eRe-faithful, since M is R-faithful. Let N be a non zero submodule of ${}_{eRe}eM$. Then $0\neq ReN\subset M$, and we have $M=ReN$, since M is R-simple. Then $eM=eReN=N$, which means

$\mathfrak{m}$ which contains no non zero ideal. Let $\mathfrak{a}=Tr(A_B)$, the trace ideal of A_B. Under our hypotheses we have $\mathfrak{a}\neq 0$. If $A\mathfrak{m}=A$, we have $f(A)=f(A\mathfrak{m})=f(A)\mathfrak{m}\subset\mathfrak{m}$ for any f in $\mathrm{Hom}(A_B, B_B)$. This means that $0\neq\mathfrak{a}\subset\mathfrak{m}$, a contradiction. Thus we have $A\mathfrak{m}\neq A$, and there exists a maximal left ideal L of A such that $A\mathfrak{m}\subset L$ and $L\cap B=\mathfrak{m}$. Suppose that L contains a non zero ideal I of A. Then we have $I=A(I\cap B)$ or $I=(I\cap B)A$ by Theorems 3.1 and 4.1 [8]. Hence we have $0\neq I\cap B\subset\mathfrak{m}$, a contradiction. Thus A has a maximal left ideal which contains no proper ideal.

PROPOSITION 3.1. *If R is a left (or right) primitive ring, then for any finitely generated projective left R-module M, $End({}_RM)$ is also a left (resp. right) primitive ring.*

PROOF. This is clear by Lemma 2.1 and Theorem 3.1, since $M_n(R)$ is an H-separable extension of R.

PROPOSITION 3.2. *Let B be a left (or right) primitive ring and A an H-separable extension of B. Assume that A is left B-finitely generated projective. Then $D(=V_A(B))$ is a semiprime ring without proper central idempotent. In particular if C is a field, D is a simple artinian ring.*

PROOF. By assumption $\mathrm{End}({}_BA)$ is a left (resp. right) primitive ring. Therefore it has neither non zero nilpotent ideal nor proper central idempotent. But there exists a ring isomorphism η of $D\otimes_C A^\circ$ to $\mathrm{End}({}_BA)$ such that $\eta(d\otimes a^\circ)(x)=dxa$ for any $a, x\in A$ and $d\in D$, since A is H-separable over B. Then if $\mathfrak{a}$ is a nilpotent ideal of D, $\mathfrak{a}\otimes A^\circ$ must be zero in $D\otimes_C A^\circ$. Therefore, for each $a\in\mathfrak{a}$, $\eta(a\otimes 1^\circ)(A)=aA=0$. This implies $\mathfrak{a}=0$. For the same reason we have that, if e is a central idempotent of D, $e=\eta(e\otimes 1^\circ)(1)=0$. The rest of the proof is obvious, since D is finitely generated as C-module.

The next lemma is a paraphrase of Proposition 4 [13].

LEMMA 3.1. *Let A and B be strongly primitive rings with their socles S and $\mathfrak{z}$, respectively. Suppose that A is left (or right) B-projective. Then we have either $B\cap S=0$ or $B\cap S=\mathfrak{z}$ and $S=A\mathfrak{z}A$.*

PROOF. Suppose that $B\cap S\neq 0$. Since S and $\mathfrak{z}$ are the smallest non zero ideal of A and B, respectively, we have $S\subset A\mathfrak{z}A$ and $\mathfrak{z}\subset B\cap S$. On the other hand we have $B\cap S\subset\mathfrak{z}$ by Proposition 4 [13]. Hence we have $\mathfrak{z}=B\cap S\subset S$, and $A\mathfrak{z}A\subset S$. Then we have $S=A\mathfrak{z}A$.

THEOREM 3.2. *Let A, B, S and $\mathfrak{z}$ be as in Lemma 3.1. Assume furthermore that A is an H-separable extension of B. Then we have $\mathfrak{z}=$*

that eM is eRe-simple. (See Proposition 3. 7. 1 [4]).

PROPOSITION 2. 2. *Let R be a strongly primitive ring with the socle $\mathfrak{z}$ and e a non zero idempotent of R. Then eRe is also a strongly primitive ring with the socle $e\mathfrak{z}e$.*

PROOF. By Theorem 1 [1], Re contains a faithful minimal left ideal $\mathfrak{l}$ of R. Then by the above lemma $e\mathfrak{l}=e\mathfrak{l}e$ is a minimal faithful left ideal of eRe. Thus eRe is strongly primitive. Let $\mathfrak{a}\,(=e\mathfrak{a}e)$ be any non zero ideal of eRe. Then $Re\mathfrak{a}eR$ contains $\mathfrak{z}$. Hence we have $e\mathfrak{z}e\subset eRe\mathfrak{a}eRe=\mathfrak{a}$. Thus $e\mathfrak{z}e$ is the smallest non zero ideal of eRe. Then $e\mathfrak{z}e$ coincides with the socle of eRe.

Hereafter we assume again that all rings have the identities.

PROPOSITION 2. 3. *Let R be a strongly primitive ring and M a finitely generated projective left R-module. Then $End({}_RM)$ is also a strongly primitive ring.*

PROOF. $M_n(R)$, the $n\times n$-full matrix ring over R, is an H-separable extension of R and R-free of rank n^2. Moreover, $M_n(\mathfrak{z})$ is the smallest ideal of $M_n(R)$ with $M_n(\mathfrak{z})\cap R=\mathfrak{z}$, where $\mathfrak{z}$ is the socle of R. Therefore $M_n(R)$ is a strongly primitive ring by Theorem 1 [13]. By assumptiom M is a direct summand of a free R-module of rank n for some n, and there exists an idempotent e of $M_n(R)$ such that $\mathrm{End}({}_RM)=eM_n(R)e$. Then $\mathrm{End}({}_RM)$ is also a strongly primitive ring by Proposition 2. 2.

3. In this section we will deal with H-separable extensions of strongly primitive rings. We will use the same notation as the author's previous papers. In particular for an R-R-module M we denote $M^R=\{m\in M\,|\,rm=mr$ for any $r\in R\}$, and for any subring S of R $V_R(S)=R^S$, regarding R as an S-S-module. Throughout this section A will be a ring with the center C, B a subring of A and $D=V_A(B)$, the centralizer of B in A. A is an H-separable extension of B if and only if D is C-finitely generated projective and the map η of $A\otimes_B A$ to $\mathrm{Hom}({}_CD, {}_CA)$ defined by $\eta(a\otimes b)(d)=adb$, for $a, b\in A$ and $d\in D$, is an isomorphism.

THEOREM 3. 1. *Let B be a left primitive ring and A an H-separable extension of B. If furthermore A is right B-finitely generated projective, or B is a right B-direct summand of A, then A is also a left primitive ring.*

PROOF. A ring is left primitive if and only if it has a maximal left ideal which contains no non zero ideal. Thus B has a maximal left ideal

$S\cap B$ and $S=A\mathfrak{z}A=\mathfrak{z}A=Soc(_BA)$.

PROOF. Since A is H-separable over B and left B-finitely generated projective, we have $S=(S\cap B)A$ by Theorem 3.1 [8]. Hence $S\cap B\neq 0$, and we have $\mathfrak{z}=S\cap B$, $S=\mathfrak{z}A=A\mathfrak{z}A$ by Lemma 3.1. That $\mathfrak{z}A=Soc(_B(A)$ follows from the next lemma.

LEMMA 3.2. *Let R be a strongly primitive ring with the socle $\mathfrak{z}$ and M a projective left R-module. Then we have $Soc(_RM)=\mathfrak{z}M$. Every R-submodule of M is faithful.*

PROOF. By assumption there exist $f_i\in \mathrm{Hom}(_RM, {}_RR)$ and $m_i\in M$, for some index set $i\in\Lambda$, such that for each $m\in M$ $f_i(m)=0$ for almost all $i\in\Lambda$ and $m=\Sigma f_i(m)m_i$. Let N be any non zero R-submodule of M, and suppose $Ann(_RN)\neq 0$. Then $\mathfrak{z}\subset Ann(_RN)$ and $\mathfrak{z}N=0$. There exists at least one i such that $f_i(N)\neq 0$. Then $f_i(N)$ is a faithful left ideal of R. But we have $\mathfrak{z}f_i(N)=f_i(\mathfrak{z}N)=f_i(0)=0$, a contradiction. Thus every non zero R-submodule of M is faithful. Then if N is a simple R-submodule of M. we have $0\neq\mathfrak{z}N=N$. Hence $N\subset\mathfrak{z}M$, and $Soc(_RM)\subset\mathfrak{z}M\subset Soc(_RM)$.

In [13] it is shown that, in the case where A is an H-separable extension of a strongly primitive ring B and is left B-finitely generated projective, A is also strongly primitive if and only if $B\cap A\mathfrak{z}A=\mathfrak{z}$ holds (Theorem 1 [13]). In this situation we will detail the structure of A and B.

THEOREM 3.3. *Let B be a strongly primitive ring and A an H-separable extension of A. Assume that A is also strongly primitive and left B-finitely generated projective. Let I and $\mathfrak{m}$ be faithful minimal left ideals of A and B, respectively, and denote the double centralizers of $_AI$ and $_B\mathfrak{m}$ by A^* and B^*, respectively. Still more let $\tilde{B}$ be the double centralizer of $_BI$. Then we have*

(1) *$I\cong\overset{r}{\oplus}\mathfrak{m}$ for some positive integer r, and $End(_BI)$ is a simple artinian ring.*

(2) *There exists a ring isomorphism Φ of B^* to $\tilde{B}$ such that $\Phi(b)=b$ for any $b\in B$.*

(3) *$D\otimes_C C^*$ is a simple artinian ring and isomorphic to $V_{A^*}(\tilde{B})$, where C^* is the center of A^*.*

(4) *A^* is an H-separable extension of $\tilde{B}(\cong B^*)$.*

PROOF. (1). I is B-finitely generated, since A is left B-finitely generated, while we have $I\subset\mathfrak{z}A$ by Theorem 3.2, where $\mathfrak{z}$ is the socle of B. Hence we have (1). (2). This is immediate from (1), since there exists a canonical ring isomorphism of $\mathrm{Bic}(_B\mathfrak{m})$ to $\mathrm{Bic}(_B\oplus\mathfrak{m})$. (3). Put Δ

$=\mathrm{End}({}_AI)$, $\Gamma=\mathrm{End}({}_BI)$ and $\Lambda=\mathrm{End}(I)$. A and B are subrings of Λ, and we have $\Lambda^A=V_\Lambda(A)=\Delta$ and $\Lambda^B=V_\Lambda(B)=\Gamma$. It is obvious that the center of Δ coincides with C^*, the center of $\mathrm{End}(I_\Delta)(=A^*)$. Since A is H-separable over B, we have a ring isomorphism g of $D\otimes_C\Lambda^A$ to Λ^B such that $g(d\otimes\lambda)=d\lambda$ for each $d\in D$ and $\lambda\in\Lambda^A$. This means that $\Gamma=D\otimes_C\Delta$ $=(D\otimes_C C^*)\otimes_{C^*}\Delta$. Then since Γ is simple artinian and Δ is a division ring with its center C^*, we have that $D\otimes_C C^*$ is simple artinian by well known Noether-Krosch Theorem. Next, since $\tilde{B}=V_\Lambda(V_\Lambda(B))$, we have $V_\Lambda(\tilde{B})=$ $V_\Lambda(V_\Lambda(V_\Lambda(B)))=V_\Lambda(B)=\Gamma$. Then, $V_{A^*}(B)=\mathrm{Hom}({}_BI_\Delta, {}_BI_\Delta)=\mathrm{End}(I_\Delta)\cap\mathrm{End}$ $({}_BI)=A^*\cap\Gamma=A^*\cap V_\Lambda(\tilde{B})=V_{A^*}(\tilde{B})$, while $C^*=V_\Delta(\Delta)=\mathrm{End}({}_AI_\Delta)=V_{A^*}(A)$. On the other hand since A is an H-separable extension of B, we have a ring isomorphism $D\otimes_C V_{A^*}(A)\cong V_{A^*}(B)$ defined by the same way as the above map g. Then we have $D\otimes_C C^*\cong V_{A^*}(\tilde{B})$. (4). Since $\tilde{B}=V_\Lambda(\Gamma)=$ $V_\Lambda(D\Delta)=V_\Lambda(D)\cap V_\Lambda(\Delta)=V_\Lambda(D)\cap A^*=V_{A^*}(D)$, we have $V_{A^*}({}_{A^*}(\tilde{B}))=\tilde{B}$. Furthermore, $V_{A^*}(\tilde{B})$ is a simple C^*-algebra with $[V_{A^*}(\tilde{B}):C^*]=$ $[D\otimes_C C^*:C^*]<\infty$ by (3). Of course A^* and $\tilde{B}(\cong B^*)$ are right full linear rings. Then by Theorem 4 [11], A^* is an H-separable extension of $\tilde{B}$.

REMARK. With the same notation as Theorems 3.2 and 3.3, let $I=$ $\oplus_{i=1}^r\mathfrak{m}_i$ with $\mathfrak{m}_i\cong\mathfrak{m}$ as left B-module and f_i the B-isomorphism of $\mathfrak{m}_i$ to $\mathfrak{m}$ for each i. The isomorphism Φ of B^* to $\tilde{B}$ in Theorem 3.3 (2) is given by $\Phi(b^*)(\Sigma m_i)=\Sigma(b^*(m_if_i))f_i^{-1}$, for each $b^*\in B^*$ and $m_i\in\mathfrak{m}_i$. On the other hand there is a ring isomorphism $\bar{\Psi}$ of $\mathrm{End}(\mathfrak{z}_B)$ to a subring of End $(\mathfrak{z}\otimes_BA_A)$ such that $\bar{\Psi}(f)(a\otimes x)=f(a)\otimes x$ for $f\in\mathrm{End}(\mathfrak{z}_B)$, $a\in\mathfrak{z}$ and $x\in A$. But we have $\mathfrak{z}\otimes_BA\cong\mathfrak{z}A=S$, since A is right B-flat. Then we obtain by $\bar{\Psi}$ a ring isomorphism Ψ of $\mathrm{End}(\mathfrak{z}_B)$ to a subring of $\mathrm{End}(S_A)$ such that $\Psi(f)(\Sigma a_ix_i)=\Sigma f(a_i)x_i$ for each $f\in\mathrm{End}(\mathfrak{z}_B)$, $a_i\in\mathfrak{z}$ and $x_i\in A$. Moreover, by Theorem 2.1 there exist ring isomorphisms σ and σ' of A^* to $\mathrm{End}(S_A)$ and B^* to $\mathrm{End}(\mathfrak{z}_B)$, respectively. For each $x\in I$ let $x=\Sigma m_i$ with $m_i\in$ m_i, and $m_i=\Sigma a_{ij}m_{ij}$ with $a_{ij}\in\mathfrak{z}$ and $m_{ij}\in\mathfrak{m}_i$ $(=\mathfrak{z}\mathfrak{m}_i)$. Then by direct computations we have $\Phi(\sigma'^{-1}(\xi))(x)=\Sigma_{i,j}\xi(a_{ij})m_{ij}=(\sigma^{-1}\Psi(\xi))(x)$ for each ξ $\in\mathrm{End}(z_B)$. Thus we have the following commutative diagram

$$\begin{array}{ccc} B^* & \xrightarrow{\quad\Phi\quad} & A^* \\ {\scriptstyle\sigma'}\downarrow & & \downarrow{\scriptstyle\sigma} \\ \mathrm{End}(z_B) & \xrightarrow[\quad\Psi\quad]{} & \mathrm{End}(S_A) \end{array}$$

4. In this short section we will deal with H-separable extensions of right full linear rings, which have closed relations with inner galois theory of full linear rings (See [1]).

Let B be a right full linear ring and A an H-separable extension of B. Then, A is also a right full linear ring, D is a simple C-algebra with $[D: C]<\infty$ and $B=V_A(D)$ (See Theorem 4 [13]). Let I be a faithful simple left ideal of A. Denote the class of right full linear subrings R of A such that R contains B and I is a finite direct sum of faithful simple left R-modules by $\mathscr{L}$, and the class of simple C-subalgebras of D by $\mathscr{D}$. Then by Theorems 36.2 and 36.4 [2], we obtain mutually inverse 1-1-correspondences between $\mathscr{L}$ and $\mathscr{D}$, namely, if $R\in\mathscr{L}$, then $V_A(R)\in\mathscr{D}$ and $R=V_A(V_A(R))$, and conversely if $E\in\mathscr{L}$, then $V_A(E)\in\mathscr{D}$ and $E=V_A(V_A(E))$. Concerning with this inner Galois theory we have.

THEOREM 4.1. *Let A, B, $\mathscr{L}$ and $\mathscr{D}$ be as above. Then for any right full linear subring R of A which contains B, the following three conditions are equivalent ;*

(a) *A is left R-finitely generated projective.*

(b) *A is right R-finitely generated projective.*

(c) $R\in\mathscr{L}$

PROOF. Firstly note that A is both left and right B-finitely generated free (See Theorem 4 [11]). Let S, $\mathfrak{z}$ and $\mathfrak{z}'$ be the socles of A, B and R, respectively. By Theorem 2 [13] we have $S=\mathfrak{z}A$ and $\mathfrak{z}=S\cap B\subset S\cap R\neq 0$. Now suppose (a) or (b). Then in either case $S\cap R=\mathfrak{z}'$ by Lemma 3.1. Then, $\mathfrak{z}\subset\mathfrak{z}'\subset S$, and $S=\mathfrak{z}A\subset\mathfrak{z}'A\subset S$. Thus we have $S=\mathfrak{z}'A$, which implies $R\in\mathscr{L}$. That (c) implies (a) is due to Theorem 36.2 [2], while that (c) implies (b) is shown in Theorem 4 [11]. Now we have proved the theorem.

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