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On serial quasi-hereditary rings

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Dedicated to Professor Tosiro Tsuzuku on his 60th birthday

In their Carleton Lecture Note [3], V. Dlab and C. M. Ringel studied the quasi-hereditary rings initiated by E. Cline, B. Parshall and L. Scott [1] and applied to the representation theory of algebras. The quasi-hereditary algebras generalize the hereditary algebras and have the finite global dimension. But not all algebras of finite global dimension are quasi-hereditary, though the algebras of global dimension 2 are quasi-hereditary [3]. In fact, they showed an example of a non-quasi-hereditary algebra of global dimension 4 and dominant dimension $\geqq 2$. Taking account of these facts, Dlab posed a question in [2] whether the algebras of global dimensaion 3 are quasi-hereditary. The aim of this note is to show that serial Artinian rings (= Nakayama rings) of global dimension 3 are quasi-hereditary, and to answer in the negative to his question by showing an example of an algebra, without any heredity ideals, whose global dimension and dominant dimension are three.*) In the first two sections, we shall give two remarks concerning the refinement of heredity chains and Morita invariance of the quasi-hereditarity of rings. In the final section, some examples will be given and some problem, which is naturally arised from those examples, will be discussed.

Throughout this note, all rings are semi-primary and, unless specified otherwise, all modules are right modules. Denoted by add M we understand the category of modules which are isomorphic to direct summands of direct sums of copies of M. For a given ring A, the Jacobson radical will be denoted by N.

1. Refinement of heredity chains

In this section we shall show that all heredity chains are refined to heredity chains with the same length as the number of simple modules.

We first recall from [3] the definition of heredity chains. Let A be a

*) Our example was announced in a lecture of Dlab at the Conference of Representation Theory of Algebras held at the Banach Center (Warsaw, April, 1988).

semi-primary ring and N the Jacobson radical. An ideal J of A is said to be a heredity ideal of A if $J^2=J$, $JNJ=0$ and J is projective as a left or right A-module. This implies that the ideal J is projective on both sides. A semi-primary ring A is said to be quasi-hereditary if there is a chain $0=J_0\subset J_1\subset\cdots\subset J_{t-1}\subset J_t\subset\cdots\subset J_m=A$ of ideals of A such that, for any $1\leq t\leq m$, J_t/J_{t-1} is a heredity ideal of A/J_{t-1}. Such a chain is called a heredity chain. An idempotent e is called a heredity idempotent when AeA is a heredity ideal. For two idempotents e and f, we say that f contains e if $e=ef=fe$. Two orthogonal idempotents e and f are said to be purely orthogonal if eA does not contain any summand isomorphic to a summand of fA. An idempotent e is said to be basic if it is a sum of orthogonal and nonisomorphic primitive idempotents or, equivalently, eA is basic.

Now we begin by stating a supplementary lemma to [3] Statement 6.

LEMMA 1.1. *Suppose that I and J are idempotent ideals of A such that $J\subset I$ and $J=AeA$ for a basic idempotent e. Then there is an idempotent f purely orthogonal to e such that $I=A(e+f)A$.*

PROOF. Let $I=Af'A$ for a basic idempotent f'. Since I_A is generated by $f'A$ and eA is a summand of I_A, eA is also generated by $f'A$. It follows that eA is isomorphic to a summand of $f'A$, because eA is basic. Hence, f' is a sum of orthogonal idempotents e' and f such that $e'\simeq e$, so that $I=A(e+f)A$.

LEMMA 1.2. *Suppose that I is a heredity ideal of A and $I=AeA$ with an idempotent e, and let e_1 be an idempotent contained in e such that e_1 and $e-e_1$ are purely orthogonal. Then $I=Ae_1A\oplus A(e-e_1)A$, and both Ae_1A and $A(e-e_1)A$ are heredity ideals.*

PROOF. The second assertion is an easy consequence of the first. Since $I=Ae_1A+A(e-e_1)A$, it then suffices to show that $Ae_1A\cap A(e-e_1)A=0$. Now, I_A is isomorphic to a summand of a direct sum of copies of eA_A, because I_A is projective and generated by eA. Hence, $I_A=P\oplus Q$, where P and Q are isomorphic to summands of direct sums of copies of e_1A and $(e-e_1)A$, respectively. For our aim it suffices to show that $Ae_1A=P$ and $A(e-e_1)A=Q$. Since $e-e_1$ and e_1 are purely orthogonal and $eNe=0$, we have that $(e-e_1)Ae_1=(e-e_1)Ne_1=0$, hence $Qe_1=0$. Therefore it follows that $Ie_1=Pe_1$ and so $Ae_1A\subset P$. Conversely, let $P=\oplus_i u_ie_iA$, where every e_i is a primitive idempotent contained in e_1 and $u_ie_iA\simeq e_iA$ canonically. Then, since $u_ie_i=u_ie_ie_i\in Pe_1A$, we have that $P\subset Pe_1A$, which implies that $P\subset Ie_1A\subset Ae_1A$. In consequence, we have that $P=Ae_1A$, as desired. Similarly we know that $Q=A(e-e_1)A$.

PROPOSITION 1.3. *Any heredity chain is refined to a maximal heredity chain of the same length as the number of simple modules.*

PROOF. Let e and f be basic and purely orthogonal idempotents, and $e=\sum_{i=1}^{s} e_i$ a sum of orthogonal primitive idempotents. Let $I=A(e+f)A$ and $J=AfA$, and assume that $\bar{I}:=I/J$ is a heredity ideal in $\bar{A}:=A/J$. Then, by Lemma 1.1, it suffices to show that the inclusion $J\subset I$ is refined to a chain $J=J_0\subset J_1\subset\cdots\subset J_s=I$ such that every J_j/J_{j-1} is a heredity ideal in A/J_{j-1}, where $J_j=A(f+\sum_{i=1}^{j} e_i)A$. For any one of e_i, say e_1, we know from Lemma 1.2 that $\bar{I}=\bar{A}\bar{e}_1\bar{A}\oplus\bar{A}(\bar{e}-\bar{e}_1)\bar{A}$, and both $\bar{A}\bar{e}_1\bar{A}$ and $\bar{A}(\bar{e}-\bar{e}_1)\bar{A}$ are heredity ideals of $\bar{A}$. This implies that, putting $J_1=A(e_1+f)A$, J_1/J is a heredity ideal in A/J. Moreover, since $I/J_1\simeq\bar{I}/\bar{J}_1\simeq\bar{A}(\bar{e}-\bar{e}_1)\bar{A}$ as $\bar{A}$-modules, I/J_1 is a projective $\bar{A}$-module. Hence I/J_1 is clearly a heredity ideal of A/J_1. Thus, by induction, we have a desired refinement.

LEMMA 1.4. *Let P and P' be projective A-modules, and assume that e is a heredity idempotent such that P belongs to* add(eA). *Then, for any morphism $f: P\to P'$ such that* $\ker f$ *is small in P, the morphism $P\to P'eA$ induced from f is a splittable monomorphism.*

PROOF. We may assume that e and $1-e$ are purely orthogonal, which implies that $eA(1-e)=eN(1-e)$, where $N=\operatorname{rad} A$. First observe that $P=PeA$. Then $f(P)\cap P'eN=[f(P)eA(1-e)+f(P)eAe]\cap P'eN=f(P)eN(1-e)+(f(P)eAe\cap P'eN)$. Moreover, $f(P)eAe\cap P'eN\subset P'eNe=0$. As a consequence, we have that $f(P)\cap P'eN=f(P)N$ and hence $f(P)/f(P)N\simeq(f(P)+P'eN)/P'eN$, which is a summand of $P'eA/P'eN$. Hence there is a splittable epimorphism $p: P'eA/P'eN\to f(P)/f(P)N$ so that the following diagram is commutative

$$\begin{array}{ccccc} P & \longrightarrow & f(P) & \longrightarrow & f(P)/f(P)N \\ & & \cap\downarrow & & \uparrow p \\ & & P'eA & \longrightarrow & P'eA/P'eN, \end{array}$$

where all morphisms except p are natural. Since the composite $P\to f(P)\to f(P)/f(P)N$ is a projective cover, we therefore know that the morphism $P\to P'eA$ induced from f is a splittable monomorphism, because AeA_A and so $P'eA_A$ are projective.

As an easy application, we can give another proof of the following proposition which characterizes the hereditary rings in terms of the refinement of chains of ideals [3].

PROPOSITION 1.5 (Dlab-Ringel). *A semi-primary ring is hereditary if and only if every chain of idempotent ideals can be refined to be a heredity chain.*

PROOF. First, assume that a semi-primary ring A is hereditary and let $J\subset I$ be a chain of idempotent ideals. It follows from Lemma 1.1 that there are primitive idempotents e_i, $1\leq i\leq s$, and an idempotent e such that the set $\{e, e_i|1\leq i\leq s\}$ is of orthogonal idempotents and $J=J_0\subset\cdots\subset J_{s-1}\subset J_s=I$, where $J_0=AeA$ and $J_i=A(e+e_1+\cdots+e_i)A$ for $i\geq 1$. Take any i and let $\bar{A}=A/J_{i-1}$. Since $\bar{A}$ is hereditary and $\bar{e}_i\bar{N}\bar{e}_i=0$, it is then obvious that $J_i/J_{i-1}(=\bar{A}\bar{e}_i\bar{A})$ is a heredity ideal in $\bar{A}$.

To show the converse, we shall show that N_A is projective. Let $\{e_i|0\leq i\leq n\}$ be a complete set of orthogonal primitive idempotents and $p:\oplus_{i=0}^{n}P_i\longrightarrow N_A$ a projective cover such that $P_i\in\mathrm{add}(e_iA)$. Since from our assumption every e_i is a heredity idempotent, we can assume that $j\leq i$ if $e_jAe_i\neq 0$. Now put $P_s'=P_0\oplus\cdots\oplus P_{s-1}$ and $e_s'=e_0+\cdots+e_{s-1}$ $(1\leq s\leq n+1)$. Then, $p(P'_{s+1})=p(P_s')+p(P_s)$ and $N=p(P_{n+1}')$, where by Lemma 1.4 every $p(P_s)$ is projective. To show the projectivity of N_A, we shall show by induction on s that every $p(P_s')$ is projective. Now let $\bar{A}=A/Ae_s'A$ and $\bar{N}=(N+Ae_s'A)/Ae_s'A$. Since $e_jAe_i=0$ for $i<j$, all e_iA $(i\geq s)$ are canonically considered as $\bar{A}$-modules, so that $P_i\in\mathrm{add}(\bar{e}_i\bar{A})$ $(i\geq s)$. On the other hand, it is easily seen that the morphism $\bar{p}:\oplus_{i\geq s}P_i\rightarrow\bar{N}$, which is naturally induced from p, is a projective cover. The composite $f:P_s\rightarrow\oplus_{i\geq s}P_i\xrightarrow{\bar{p}}\bar{N}\rightarrow\bar{A}_{\bar{A}}$ has the small kernel, and $\bar{A}\bar{e}_s\bar{A}$ is a heredity ideal of $\bar{A}$ because by the assumption on the refinement the chain $Ae_s'A\subset Ae_{s+1}'A$ is a consecutive part of a heredity chain. It therefore follows from Lemma 1.4 that f is a monomorphism, which implies that $p(P_s)\cap Ae_s'A=0$. Thus we have that $p(P_{s+1}')=p(P_s)+p(P_s')$ is a direct sum since $p(P_s')\subset Ae_s'A$, which implies that $p(P_{s+1}')$ is a projective A-module.

2. Morita invariance

In this section, we shall show that the quasi-hereditarity of semi-primary rings is Morita invariant.

PROPOSITION 2.1. *Suppose that e is an idempotent of A such that $A=AeA$. Then, an ideal I of A is a heredity ideal if and only if eIe is a heredity ideal of eAe.*

PROOF. We denote by B the ring eAe, and choose an idempotent f such that $f\leq e$ and $I=AfA$. Then, $J:=eIe$ is an idempotent ideal BfB of B. Since the rings A and B are Morita equivalent, we have that Ie is a projective B-module, because I_A is projective and $Ie \simeq I \otimes_A Ae$ as B-modules. This clearly implies that J is a heredity ideal of B, because J is a summand of Ie, and $f(\mathrm{rad}\, B)f \subset e(\mathrm{rad}\, A)e = 0$.

Conversely, assume that $J(:=eIe)$ is a heredity ideal of B. Let f be an idempotent contained in e such that $J=BfB$, and let $I=AfA$. Since eA_A is a generator, A_A is isomorphic to a summand of a direct sum, say $\oplus eA$, of copies of eA. This implies that I_A is isomorphic to a summand of $\oplus JA_A$. On the other hand, JA is a projective A-module, since $JA = BfA \simeq BfB \otimes_B eA$ as A-modules and A is Morita equivalent to B. Thus we know that I is a projective A-module.

LEMMA 2.2. *Suppose that e is an idempotent of A such that $A = AeA$. Then a chain of ideals of A, $0=I_0 \subset I_1 \subset \cdots \subset I_m = A$, is heredity if and only if $0 = eI_0e \subset eI_1e \subset \cdots \subset eI_me = eAe$ is a heredity chain of eAe.*

PROOF. Let I be an ideal and $\bar{A} = A/I$. Then by induction on m, the lemma follows from Proposition 2.1, taking account of the fact that $\bar{A} = \bar{A}\bar{e}\bar{A}$ and $\bar{e}\bar{A}\bar{e}$ is canonically isomorphic to eAe/eIe.

Assume now that A is Morita equivalent to a ring B. Then, as well known, their basic rings are isomorphic and generated by idempotents. Hence, the following statement is an immediate consequence of the above lemma.

PROPOSITION 2.3. *Suppose that two semi-primary rings A and B are Morita equivalent. Then, A is quasi-hereditary if and only if so is B.*

3. Serial quasi-hereditary rings

Throughout this section, all rings will be serial Artinian rings.

PROPOSITION 3.1. *Let A be a connected serial Artinian ring and $N = \mathrm{rad}\, A$. Then the following statements are equivalent.*

(1) *A is quasi-hereditary.*

(2) *Either there is a simple projective module or there is a morphism $f: e_1A \to e_2A$ with primitive idempotents e_1, e_2 such that $\mathrm{Im}\, f = e_2N$ and $\mathrm{Ker}\, f$ is projective, that is, there is a simple module with projective dimension 2.*

(3) *There is an indecomposable projective module, say eA, such that every non-zero morphism form eA to a projective module is a monomorphism.*

Moreover, in the statement (2), let eA be a module with an idempotent e which is isomorphic to $\mathrm{Ker}\, f$. Then, AeA in (2) as well as in (3) is a heredity ideal of A.

PROOF. We assume that A contains no simple projective modules because, otherwise, all assertions are obviously valid.

(1) $\Longrightarrow$ (3) By Proposition 1.3 there is a heredity and primitive idempotent e. The assertion (3) for the module eA is then clear from Lemma 1.4.

(3) $\Longrightarrow$ (2) Assume that eA is a module given in (3). Let e_1A be an indecomposable projective module of maximal length which contains a submodule, say P, isomorphic to eA. Since A contains no simple projectives, there is an indecomposable projective module e_2A and a non-zero morphism $f: e_1A \to e_2A$ such that $f(e_1A) = e_2N$. From the maximalilty of e_1A, f is not a monomorphism and so the restriction of f to P is not monomorphic. Hence the composite $eA \simeq P \xrightarrow{f} e_2A$ is not a monomorphism. It therefore follows from assumption that $P \subset \operatorname{Ker} f$, which implies that $\operatorname{Ker} f$ is projective because any module containing a projective submodule must be projective.

(2) $\Longrightarrow$ (3) Let e_1, e_2 and f be the idempotents and the morphism given in the statement (2). Let now e be an idempotent such that eA is isomorphic to $\operatorname{Ker} f$, g a non-monomorphism from eA to a projective module P, and $i: P \to I$ an injective hull of P. We claim that g is a zero map. The composite $ig: eA \to I$ is extended to e_1A, say $g': e_1A \to I$. Since g' is not an isomorphism, it is easily seen that there is a morphism $h: e_2A \to I$ such that $g' = hf$ because I is projective and f is a source map (= a minimal left almost split map) in the category of finitely generated projective modules (see the proof of [6] Theorem 1). Hence $\operatorname{Ker} f \subset \operatorname{Ker} g'$ so that $g=0$, as desired.

(3) $\Longrightarrow$ (1) From the assumption, the endomorphism ring of eA is a division ring and hence $eNe=0$. Next, to show that AeA_A is projective, take an idempotent e_1 such that $e_1AeA \neq 0$. Since e_1AeA is a homomorphic image of eA, there is a morphism from eA to e_1A, $eA \to e_1AeA \hookrightarrow e_1A$, which is a monomorphism by assumption. We therefore have an isomorphism $eA \simeq e_1AeA$, which implies that e_1AeA is projective. Thus we conclude that AeA is a heredity ideal and A is quasi-hereditary, because the quiver of A/AeA is a tree so that A/AeA is obviously quasi-hereditary.

THEOREM 3.2. *Assume that A is a serial Artinian ring whose global dimension is three. Let $P_2 \to P_1 \to P_0 \to M \to 0$ be a minimal projective resolution of an indecomposable module M such that P_2 is not zero, and let e be an idempotent such that eA is isomorphic to the smallest projective sub-*

module of P_2. Then AeA is a heredity ideal of A, and A is quasi-hereditary.

PROOF. From results in Section 2 we can assume that A is basic and connected. Let $u: P_2 \to P_1$ be the given morphism and $i: eA \to P_2$ an embedding. Since injective hulls of projective modules are projective, there exists an idempotent e_1 such that e_1A is an injective hull of eA, and let $k: eA \to e_1A$ be the embedding which is extended to a monomorphism $j: P_2 \to e_1A$.

(a) We first assume that top $e_1A(= e_1A/e_1N)$ is injective. To show that AeA is a heredity ideal, by Proposition 3.1 it suffices to show that every non-zero morphism $g: eA \to Q$ is monomorpic for any indecomposable projective module Q. But we can obviously assume that Q is projective and injective. Then, clearly there exists a morphism $h: e_1A \to Q$ such that $g=hk$. Since top e_1A is injective, h should be an epimorphism and so an isomorphism. Thus we know that g is a monomorphism.

(b) Next we assume that top e_1A is not injective. There is then a morphism $f: e_1A \to e_2A$ such that $f(e_1A)=e_2N$. By Proposition 3.1 again, it suffices to show that Ker f is a projective submodule and we consider the case where $j(P_2)$ is not contained in Ker f. Thus we have that Ker $f \subsetneqq j(P_2)$ and, since e_1A is injective, Ker f is not zero. It follows that there are canonical isomorphisms $\mathrm{soc}(e_2A/fj(P_2)) \simeq \mathrm{soc}\,(\mathrm{Cok}\,u) \simeq \mathrm{soc}\,P_0$. Since $fj(P_2) \neq 0$ and P_0 is projective, it is easily seen that the isomorphism $\mathrm{soc}(e_2A/fj(P_2)) \simeq \mathrm{soc}\,P_0$ is extended to a monomorphism $v': e_2A/fj(P_2) \subsetneqq P_0$. In consequence, denoting by v the composite of morphisms $e_2A \to e_2A/fj(P_2) \xrightarrow{v'} P_0$, there exists the following exact sequence

$$0 \longrightarrow \mathrm{Ker}\,f \longrightarrow P_2 \xrightarrow{fj} e_2A \xrightarrow{v} P_0 \longrightarrow \mathrm{Cok}\,v \longrightarrow 0,$$

because Ker $f \subsetneqq j(P_2)$ and so $j^{-1}(\mathrm{Ker}\,f) \simeq \mathrm{Ker}\,f$. Since the projective dimension of Cok v is less than 4, it therefore follows that Ker f is projective. This completes the proof of the theorem.

4. Examples

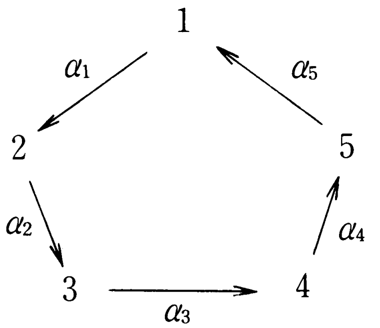
All algebras given in the examples in this section will be defined by quivers with relations over a field. By dom. dim A we understand the dominant dimension of a ring A.

(1) First we shall note that, in Theorem 3.2, Ker u does not necessarily generate a heredity ideal, though Ker u is projective. This is seen, for instance, by the algebra A_1 given in Figure 1. This algebra has a

unique heredity idempotent e_5, and by taking top e_5A_1 as M in Theorem 3.2, we have that Ker $u \simeq e_4A_1$, where the idempotents e_i correspond to the vertices i.

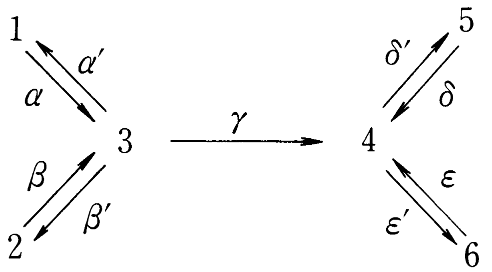
(2) We next consider the algebra A_2 defined by the quiver with relations in Figure 2. Then it is easily seen that dom. dim A_2=gl. dim $A_2=3$. But, A_2 has no heredity primitive idempotents so that A_2 has no heredity ideals (Lemma 1.2).

(3) For the algebra A_3 given in Figure 3 ([3] Part 1, Example) it holds that dom. dim A_3=gl. dim $A_3=4$. This shows that the global dimension 3 in Theorem 3.2 is the best possible dimension for serial Artinian rings. (See Lemma 4.2 below.)



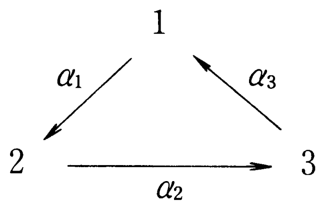
$$0=\alpha_3\alpha_2\alpha_1 \\ =\alpha_4\alpha_3\alpha_2 \\ =\alpha_2\alpha_1\alpha_5$$

Figure 1



$$0=\beta'\beta=\gamma\alpha=\alpha'\beta \\ =\varepsilon'\gamma=\delta'\delta=\delta'\varepsilon \\ \alpha\alpha'=\beta\beta',\ \delta\delta'=\varepsilon\varepsilon'$$

Figure 2



$$0=\alpha_2\alpha_1\alpha_3=\alpha_1\alpha_3\alpha_2\alpha_1$$

Figure 3

(4) Considering the algebras A_2, A_3 and the other examples (e.g. [3]), it is very natural to ask when the algebras of dominant dimension ≥ 2 are quasi-hereditary. In the following we answer this question for serial artinian rings.

LEMMA 4.1. *Let A be a semi-primary ring, G a generator-cogenerator, and R the endomorphism ring of G. Then, an indecomposable summand X of G is simple if and only if every non-zero morphism from e_XR to an indecomposable projective module e_YR is monomorphic. Here, e_Z denotes the idempotent of R corresponding to a summand Z of G.*

PROOF. We denote by **Mod** A the category of all right A-modules. Since the functor $\mathrm{Hom}_A(G, -): \mathbf{Mod}\, A \to \mathbf{Mod}\, R$ is fully faithful, for any non-zero morphism $f: e_XR \to e_YR$ there is a non-zero morphism $f': X \to Y$ such that $f = \mathrm{Hom}(G, f')$, where X and Y are indecomposable summands of G. Hence, in case X is simple, f is clearly monomorphic because so is f'. Conversely, suppose that an indecomposable summand X of G contains a simple submodule S with $S \subsetneqq X$. Take a projective cover $p: P \to S$ and an indecomposable summand I of an injective hull of X/S, and let $u: P \to X$, $v: X \to I$ be the canonical morphisms factoring through S and X/S, respectively. Since G is a generator-cogenerator, we can assume that P and I are summands of G and so u and v are non-zero elements of e_XRe_P and e_IRe_X, respectively. Let now $f: e_XR \to e_IR$ be a morphism defined by a multiplication of v. It then follows from assumption that v is a monomorphism and hence u is zero, because $f(u) = vu = 0$, a contradiction.

LEMMA 4.2. *Let R be the endomorphism ring of a generator-cogenerator A-module G, and assume that R is connected and serial Artinian. Then, R is quasi-hereditary if and only if G has a simple summand.*

PROOF. Since the heredity chains are determined by primitive idempotents (Proposition 1.3), this immediately follows from Proposition 3.1 and Lemma 4.1.

PROPOSITION 4.3. *Let R be a serial Artinian ring of dominant dimension greater than 1, and e an idempotent such that eR is minimal faithful. Then the following assertions hold.*

(i) *R is quasi-hereditary if and only if the left eRe-module eR has a simple summand.*

(ii) *If the Loewy length of every indecomposable summand of eRe_{eRe} is greater than 2, then R has no heredity ideals.*

PROOF. Let A be the ring eRe and $G = eR$. Then G is a generator-cogenerator as an A-module and $R = \mathrm{End}_A(G)^{op}$ ([5] and [4] §4.3, Proposition 1). The assertion (i) is therefore clear from Lemma 4.2. For (ii), assume that $\ell(eRe)\geq 3$, where $\ell(X)$ denotes the minimal Loewy length of indecomposable summands of a module X. By [6], we know that G belongs to the category add($A\oplus I\oplus I/\mathrm{soc}\, I$), where I is the injective hull of the top of ${}_AA$. It therefore follows from the assumption that the minimal Loewy length of indecomposable objects in add($A\oplus I\oplus I/\mathrm{soc}\, I$) is greater than one, so that G has no simple summands. In consequence, our assertion follows from (i).

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