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Second order hyperbolic equations with time-dependent singularity or degeneracy

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Introduction

Let H be a Hilbert space with norm $\|\cdot\|$, and let Λ be a non-negative self-adjoint operator in H. Let S_1, S_2, t_0, α and ν be real numbers with $S_1\leqq 0\leqq S_2$, $S_1\leqq t_0\leqq S_2$, $\alpha'>-1$ and $-2\alpha-1<\nu<1$. We are concerned with the well-posedness of the following singular or degenerate hyperbolic equation in H :

$$\left.\begin{array}{ll}(0.1) & u''(t)+\phi^2(t)\Lambda u(t)+\psi(t)u'(t)+\Xi(t)u(t)=f(t) \\ & \qquad\qquad\qquad\qquad\qquad\qquad \text{on } (t_0, S_2), \\ (0.2) & u(t_0)=u_0,\ |t|^{\nu}u'(t)|_{t=t_0}=u_1,\end{array}\right\}\ (\mathrm{WE})$$

where u' is the t-derivative in the sense of vector-valued derivative, ϕ and ψ are functions on $[S_1, S_2]$ to $[0, +\infty]$ satisfying the following;

(0.3) $\phi(\cdot)\in W_{\mathrm{loc}}^{2,\infty}((S_1, S_2)\backslash\{0\})$,
(0.4) $C^{-1}|t|^{\alpha}\leqq\phi(t)\leqq C|t|^{\alpha}$,
(0.5) $|\phi'(t)|\leqq C|t|^{\alpha-1}$, $|\phi'(t)|\leqq C|t|^{\alpha-2}$,

for a.e. t on (S_1, S_2), with some positive constant C,

(0.6) $\psi(t)-\nu/t\in L^1(S_1, S_2)$,

We note that ϕ takes value 0 or ∞ at $t=0$. That is, the singularity or the degeneracy of (0.1) occurs at $t=0$, which may be initial time ($t_0=0$) or not ($t_0\neq 0$). Especially if $2\alpha>-1$, we can take $\nu=0$. In [15], we showed the well-posednes of (WE) in the space $H=L^2(\Omega)$, where Ω is a bounded domain in $\boldsymbol{R}^n$ with smooth boundary, $\Lambda=-\Delta$ with homogeneous Dirichlet boundary condition, $2\alpha>-1$, $\nu=0$, $\phi(t)=t^{\alpha}$, $\psi=f=0$, $\Xi=0$. The purpose of this paper is to generalize the above theorem. For this purpose, we first prove an abstract theorem on the well-posedness of non-homogeneous evolution equation, which generalizes the abstract theorem on that of homogeneous equation in [15] (see Theorem 2). Then we solve (WE) by applying this abstract theorem (see Theorem 1).

Equation (0.1) with $t_0=0$ is studied by various authors: see Carroll-

Showalter [2] and Lacomblez [7]; Bernardi [1], and Coppoletta [3] for $\alpha<0$; Imai [4], Ivrii [5], Kubo [6], Oleinik [8], Protter [9], Sakamoto [10], Segala [11] and Taniguchi [12] for concrete partial differential equations with $\alpha>0$ and Λ being dependent on t: and the references quoted there and in [13]. (Here we note that [1] and [3] obtained more irregular solutions to more irregular equation (0.1) than our setting.) But in their results, the regularity of a solution of (WE) is lost at t_0 $(=0)$. Hence, the initial data needs strong regularity. Hence they did not show the well-posedness in H itself.

The main difference between above results and this paper is that the sum of the space regularity of a solution u and that of $|t|^{\nu}u'$ is conserved for all t, whether the singularity (or the degeneracy) occurs at initial time or en route. In particular, the initial condition is weaker than that of the known results. More precisely, let D_β $(\beta\geqq 0)$ denote the domain of Λ^β with its graph norm and let D_β $(\beta<0)$ denote the dual space of $D_{-\beta}$. For an arbitrary real number κ, we define a product space:

$$\pi_t^\kappa=\begin{cases} D_{(1/2)+\kappa}\times D_\kappa & \text{for } t\neq 0,\\ D_{\gamma+\kappa}\times D_{\sigma+\kappa} & \text{for } t=0,\end{cases}$$

where γ and σ are real number with $\gamma+\sigma=1/2$ determined by α and ν (see (1.1) and (1.2)). Then we show that for every $(u_0, u_1)\in\pi_{t_0}^\kappa$, (WE) has a unique solution u with $(u(t), |t|^\nu u'(t))\in\pi_t^\kappa$ for every $t\geqq t_0$. Thus the sum of the space regularity of u and $|t|^\nu u'$ is conserved $1/2+2\kappa$ for every $t\geqq t_0$. In other words, the well-posedness of (WE) in $D_{(1/2)+\kappa}+D_\kappa$ holds in some sense.

We apply the result of this paper to quasilinear degenerate hyperbolic equations in [14].

§ 1. Notations and result

First we describe notations and definitions.

Let X and Y be Banach spaces. For an operator A from Y to X, the norm $\|A\|_{Y,X}$ is defined by $\|A\|_{Y,X}=\sup\{\|Ay\|_X\,;\, y\in Y,\ \|y\|_Y=1\}$, which may be ∞. The dual space of X is denoted by X^*. The duality map of X into X^* is denoted by J_X.

Let $m=0,1$. For a closed interval I in $\boldsymbol{R}$, $AC^m(I\,;\,X)$ denotes the set of functions in $C^m(I\,;\,X)$ all of whose derivatives of order $\leqq m$ are absolutely continuous on I (as an X-valued function). For a subset I of $\boldsymbol{R}$, $AC^m_{\mathrm{loc}}(I\,;\,X)$ denotes the set of functions belonging to $AC^m(I'\,;\,X)$ for all closed interval $I'\subset I$. $AC^0_{\mathrm{loc}}(I\,;\,X)$ is denoted by $AC_{\mathrm{loc}}(I\,;\,X)$

Let $\Lambda=\int_0^\infty \lambda\ dE_\lambda$ be the spectral decomposition of Λ.

For a nonnegative number κ, we define Hilbert space D_κ as $D_\kappa=D(\Lambda^\kappa)$, the domain of Λ^κ, with the graph norm $\|\cdot\|_\kappa$ of Λ^κ, where $\Lambda^\kappa=\int_0^\infty \lambda^\kappa dE_\lambda$. For a negative number κ, we define

$$D_\kappa=(D_{-\kappa})^*.$$

We put

(1.1) $\gamma=(\alpha+2-\nu)/\{4(\alpha+1)\}\ (>1/4),\ \gamma'=\min\ (\gamma, 1/2),$

(1.2) $\sigma=(\alpha+\nu)/\{4(\alpha+1)\}\ (>-1/4),\ \sigma'=\min\ (\sigma, 0).$

Here we note that $\gamma+\sigma=1/2$. For each real number κ, we define product spaces

$$\pi_t^\kappa=\begin{cases} D_{(1/2)+\kappa}\times D_\kappa & \text{for } t\neq 0,\\ D_{\gamma+\kappa}\times D_{\sigma+\kappa} & \text{for } t=0,\end{cases}$$

REMARK 1.1 The sum of the space regularity in the product space π_t^κ is constantly $1/2+2\kappa$ for every t.

We assume that Ξ and f satisfy the following.

(H1) For every $y\in D_{1+\eta}$, $\Xi(\cdot)y$ is a $D_{(1/2)+\eta}$-valued measurable function on (S_1, S_2) with

(1.3) $\|\Xi(t)\|_{D_{1+\eta}, D_{(1/2)+\eta}}\leqq b(t),$

(1.3)' $\|\Xi(t)\|_{D_{(1/2)+\gamma'+\eta}, D_{\gamma'+\eta}}\leqq b(t),$

for some non-negative function $b(\cdot)$ satisfying

(1.4) $|t|^{-\alpha}b(t)\in L^1(S_1, S_2)$ if $\alpha+\nu\geqq 0$.

(1.5) $|t|^{\nu}b(t)\in L^1(S_1, S_2)$ if $\alpha+\nu<0$.

(H2) f is a $D_{(1/2)+\eta}$-valued function on $[S_1, S_2]$ satisfying

(1.6) $|t|^{(-\alpha+\nu)/2}f(t)\in L^1(S_1, S_2;\ D_{(1/2)+\eta})$ if $\alpha+\nu\geqq 0$.

(1.7) $|t|^{\nu}f(t)\in L^1(S_1, S_2;\ D_{(1/2)+\eta})$ if $\alpha+\nu<0$.

Now we describe our main result:

THEOREM 1. *Let η be an arbitrary fixed real number and let γ'', and σ'' be arbitrary numbers with $\gamma''\leqq\gamma'+\eta$ and $\sigma''\leqq\sigma'+\eta$. Assume* (0.3)~(0.6), (H1) *and* (H2) *for η. Then for every $(u_0, u_1)\in\pi_{t_0}^{(1/2)+\eta}$, there exists a unique solution u of* (WE) *in the following sense;*

$$(u(t), |t|^\nu u'(t))\in\pi_t^\eta \textit{ for every } t\in[t_0, S_2].$$

$$u\in C([t_0, S_2]; D_{\gamma''})\cap L^\infty([t_0, S_2]; D_{\gamma'+(1/2)+\eta})$$
$$\cap AC_{loc}([t_0, S_2]\backslash\{0\}; D_{(1/2)+\eta})\cap AC^1_{loc}([t_0, S_2]\backslash\{0\}; D_\eta),$$
$$|t|^\nu u'(t)\in C([t_0, S_2]; D_{\sigma''})\cap L^\infty([t_0, S_2]; D_{\sigma'+(1/2)+\eta}),$$
(0.1) *holds in* D_η *a. e. on* (t_0, S_2),

$u(t_0)=u_0$, $|t|^\nu u'(t)|_{t=t_0}=u_1$ (*so that* $u'(0)=0$ *if* $t=0$ *and* $\nu<0$).

Furthermore, the following estimates hold:

$$\sup_{t_0\leqq t\leqq S_2}(\|u(t)\|_{\gamma'+(1/2)+\eta}+|t|^\nu\|u'(t)\|_{\sigma'+(1/2)+\eta}+|t|^{\alpha^+/2}\|u(t)\|_{1+\eta})<\infty, \tag{1.8}$$
$$\|u''(t)\|_\eta\leqq C_1(|t|^{2\alpha-\{(1/2)(\alpha+\nu)^+/(1-\nu)\}}+b(t)+|t|^{-\nu}|\psi(t)|+\|f(t)\|_\eta), \tag{1.9}$$

for some positive constant C_1. *Here we write* $\tau^+=\max\{\tau, 0\}$ *for real number* τ.

Assume moreover that ψ, Ξ *and* f *satisfy the following*;

$\psi\in C^1([t_0, S_2]\backslash\{0\}; [0,\infty])$, $f\in C([t_0, S_2]\backslash\{0\}; D_{(1/2)+\eta})$,

$\Xi(\cdot)$ *is strongly continuous on* $[t_0, S_2]\backslash\{0\}$, *as an operator from* $D_{1+\eta}$ *to* D_η. *Then*

$$u\in\bigcap_{i=0}^{2} C^i([t_0, S_2]\backslash\{0\}; D_{\{(2-i)/2\}+\eta}). \tag{1.10}$$

REMARK 1.2. If $\alpha+\nu\geqq 0$ and

$$\operatorname*{ess.\,sup}_{t_0<t<S_2}(|t|^{-(\alpha+\nu)/(1-\nu)}+b(t)+|\psi(t)|+\|f(t)\|)<\infty,$$

then

$$u\in W^{1,\infty}((t_0, S_2); D_{(1/2)+\eta})\cap W^{2,\infty}((t_0, S_2); D_\eta).$$

In fact, that $\alpha+\nu\geqq 0$ means $\sigma'\geqq 0$. Thus, (1.8) and (1.9) imply the assertion.

We reduce this theorem to the case that $\alpha>-1/2$ and $\nu=0$. For the sake of this we change the variables as follows:

$$t(s)=|s|^{\beta-1}s \quad (\beta=1/(1-\nu)\ (>0)),\qquad v(s)=u(t(s)), \tag{1.11}$$

for $S_1\leqq s\leqq S_2$. Then it is easy to see that (WE) is transformed into the following equation for $v(s)$:

$$\left.\begin{array}{r} v''(s)+(-t''(s)/t'(s)+t'(s)\psi(t(s)))v'(s) \\ +t'^2(s)\phi^2(t(s))\Lambda v(s)+t'^2(s)\Xi(t(s))v(s)=t'^2(s)f(t(s)) \\ \textit{for } s_0<s<S_2', \\ v(s_0)=u_0,\ v'(s_0)=\beta u_1, \end{array}\right\}\ \text{(WE)'}$$

where $s_0=|t_0|^{-\nu}t_0$ and $S_2'=S_2^{-\nu+1}$. We show that the equation (WE)' (s-invariant) satisfies the assumption of Theorem 1 with α replaced by

(1.12) $\alpha'=\alpha\beta+\beta-1=(\alpha+\nu)/(1-\nu)\ (>-1/2)$

and $\nu=0$. From (0.3)~(0.5), (1.11) and (1.12), it follows that the function $\tilde{\phi}: s\to|t'(s)|\phi(t(s))$ satisfies the assumption (0.3)~(0.5) with α replaced by α'. Using the relations: $1-\beta=-\beta\nu$, we have

$$\begin{aligned}&(t''(s)/t'(s)+t'(s)\psi(t(s)))ds\\&=t'(s)(-\nu/t(s)+\psi(t(s)))ds=(-\nu/t+\psi(t))dt.\end{aligned}$$

Thus by (0.6), the function $\tilde{\psi}: s\to -t''(s)/t'(s)+t'(s)\psi(t(s))$ belongs to $L^1(-1,1)$. That is, $\tilde{\psi}$ satisfies (0.6) with $\nu=0$. Inequality (1.3) means

$$\|t'^2(s)\Xi(t(s))\|_{D_{1+\eta,(1/2)+\eta}}\leqq t'^2(s)b(t(s)),$$

So it remains only to prove that functions

$$\tilde{b}: s\to t'^2(s)b(t(s))\ (\in[0,\infty])$$

and

$$\tilde{f}: s\to t'^2(s)f(t(s))\ (\in H)$$

satisfy (1.4)~(1.7) with α and ν replaced by α' and 0, respectively. For the sake of this, we have only to note the following relations which follow from (1.11) and (1.12);

$$\begin{aligned}&|s|^{-\alpha'}t'^2(s)b(t(s))ds=\beta|t|^{-\alpha}b(t)dt,\\&t'^2(s)b(t(s))ds=\beta|t|^{\nu}b(t)dt,\\&|s|^{-(\alpha'/2)}t'^2(s)f(t(s))ds=\beta|t|^{(-\alpha+\nu)/2}f(t)dt,\\&\alpha'>0\quad\text{if and only if}\quad\alpha+\nu>0.\end{aligned}$$

We have proved that (WE)' satisfies the assumption of theorem 1 with α and ν replaced by α' and 0 respectively.

Here we note that the value of γ in (1.1) (resp. σ in (1.2)) with substituted α' for α and 0 for ν equals original γ in (1.1) (resp. σ in (1.2)). We also note that $v'(s)=\beta|t|^{\nu}u'(t)$. Thus it is easy to see that (WE)' has a unique solution v (s-invariant) if and only if (WE) has a unique solution u (t-invariant) in the sense of Theorem 1. We also see that the additional condition and estimates except (1.9) in Theorem 1 are satisfied by original one if and only if they are satisfied by transformed one. The estimate (1.9) immediately follows from (0.1), (0.4), (1.3)' and (1.8), by noting that $\sigma'+1/2\geqq0$ and $\gamma'\geqq0$. Therefore, it suffices to show Theorem 1 except (1.9) in the case that $\alpha>-1/2$ and $\nu=0$. We shall prove this by using an abstract theorem for generating an evolution operator, which we describe in the next section.

§ 2. Abstract linear evolution equations

In this section, we study a linear evolution equation in a Banach space Z with norm $\|\cdot\|_Z$;

$$(\mathrm{CP}\,;\,F)_s \quad du(t)/dt+A(t)u(t)=F(t) \text{ for } s\leqq t\leqq T, \quad u(s)=y,$$

where $0\leqq s<T$, $\{A(t)\}_{t\in[0,T]}$ is a family of linear operators in Z and $F(t)$ is a Z-valued function on $[0, T]$. In [13] we obtained unique solutions to $(\mathrm{CP}\,;\,0)_s$. Using this theorem, we shall show the well-posedness of $(\mathrm{CP}\,;\,F)_s$ for non-zero function F.

First, we describe some definitions described in [13].

Let $\{W_t\}_{t\in[0,T]}$ be a family of Banach spaces in a Banach space Z with norms $\{\|\cdot\|_{W_t}\}$.

DEFINITION 1. We say that $\|\cdot\|_{W_t}$ is *differentiable* at t if the following holds; W_{t+h} equals W_t as a linear space for sufficiently small $|h|$ with $t+h\in[0, T]$ and $(\|x\|_{W_{t+h}}-\|x\|_{W_t})/h$ is convergent as h tends to 0, uniformly with respect to x in each bounded subset of W_t. The limit of the above is denoted by $\frac{d}{dt}\|x\|_{W_t}$.

DEFINITION 2. A two-parameter family $\{U(t, s)\,;\,0\leqq s\leqq t\leqq T\}$ of operators in Z is said to be an *evolution operator* on $\{W_t\}$ if it satisfies the following: for $0\leqq s\leqq r\leqq t\leqq T$,

(i) $U(t, s)$ is a bounded linear operator on W_s into W_t,

(ii) $U(t, t)=I$ on W_t and $U(t, r)U(r, s)=U(t, s)$ on W_s.

Now, we describe the assumtions in this section.

Let Γ be a closed subset of $[0, T]$ which has at most countable numbers. Let $\{X_t\}_{t\in[0,T]}$ and $\{Y_t\}_{t\in[0,T]}$ be families of Banach spaces in Z with norms $\{\|\cdot\|_{X_t}\}$ and $\{\|\cdot\|_{Y_t}\}$ respectively such that Y_t is continuously and densely imbedded in X_t for each t. Here we note that X_t (resp. Y_t) is not necessarily equivalent to X_s (resp. Y_s) if $s\neq t$.

(S. 1) There are constants C_i, $i=1, 2, 3$, and $\theta\in(0, 1]$ such that $\|\cdot\|_Z\leqq C_1\|\cdot\|_{X_t}\leqq C_2\|\cdot\|_{Y_t}$, $\|\cdot\|_{X_t}\leqq C_3\|\cdot\|_{Y_t}^{1-\theta}\|\cdot\|_Z^{\theta}$, for $0\leqq t\leqq T$.

(S. 2) If t_n tends to $t\in[0, T]$ from the left and $\{y_n\in Y_{t_n}\}$ is a sequence such that $\sup_n\|y_n\|_{Y_{t_n}}<\infty$ and y_n converges to y in Z, then y belongs to Y_t with

$$\|y\|_{X_t}\leqq\limsup_{n\to\infty}\|y_n\|_{X_{t_n}}, \quad \|y\|_{Y_t}\leqq\limsup_{n\to\infty}\|y_n\|_{Y_{t_n}}.$$

(S. 3) For each $t\in(0, T)\backslash\Gamma$, $\|x\|_{X_s}$ (resp. $\|x\|_{Y_s}$) is differentiable with derivative bounded uniformly with respect to s near t and x in every bounded set in X_t (resp. Y_t).

(S. 4) For every $t\in\Gamma$ and $\varepsilon>0$, if $h>0$ is sufficiently small, then there exists a linear operator P on Y_t into Y_{t+h} such that

$$\|P\|_{X_t, X_{t+h}} \text{ and } \|P\|_{Y_t, Y_{t+h}}<1+\varepsilon, \quad \|(I-P)\|_{Y_t, Z}<\varepsilon.$$

Let $\{A(t)\}_{t\in[0,T]}$ be a family of linear operators in Z which satisfies the following conditions;

(A. 1) For each $t\in[0, T]\backslash\Gamma$, $A(t)$ is a closed operator in X_t with $Y_t\subset D(A(t))$ $(\subset X_t)$, and if λ is sufficiently large, λ belongs to the resolvent set of $A(t)$ and $(A(t)+\lambda I)^{-1}Y_t$ is densely included in Y_t.

(A. 2) (*Weak stability condition*) There are integrable functions ω_X and ω_Y which are continuous at every point of $[0, T]\backslash\Gamma$ and satisfy the following. If $t\in[0, T]\backslash\Gamma$, then for every $x\in Y_t$ and $y\in D(A(t)|_{Y_t})=\{y\in Y_t\,;\, A(t)y\in Y_t\}$, there are $x^*\in J_{X_t}(x)$ and $y^*\in J_{Y_t}(y)$ such that

$$\frac{d}{dt}\|x\|_{X_t}^2\leqq 2\mathrm{Re}(A(t)x, x^*)+\omega_X(t)\|x\|_{X_t}^2, \tag{2.1}$$

$$\frac{d}{dt}\|y\|_{Y_t}^2\leqq 2\mathrm{Re}(A(t)y, y^*)+\omega_Y(t)\|y\|_{Y_t}^2, \tag{2.2}$$

(A. 3) For each $t\in[0, T]\backslash\Gamma$ and each $y\in Y_t$, $A(s)y$ is right continuous at t in X_t.

(A. 4) $\|A(t)\|_{Y_t, X_t}$ is dominated by an integrable function $\xi(t)$ which is continuous at every point of $[0, T]\backslash\Gamma$.

Let $F(\cdot)$ be a Z-valued function with $F(t)\in X_t$ a. e. t on $(0, T)$.

DEFINITION 3. In the above situation, we say that $u(\cdot)\in C([s, T]\,; Z)$ is a *solution* of $(\mathrm{CP}\,; F)_s$ with $y\in Y_s$, if

(i) $u(t)\in Y_t$ for every $t\in[s, T]$ and $u(s)=y$.

(ii) For all t except at most countably many points of (s, T), there is $\delta_t>0$ such that u belongs to $AC([t-\delta_t, t+\delta_t]\,; X_t)$ with

$$du(r)/dr+A(r)u(r)=F(r) \text{ in } X_t \quad \text{a. e. on } (t-\delta_t, t+\delta_t).$$

Now we state a theorem in [13].

THEOREM A (Theorem 2.1 in [13]). *Assume the conditions* (S. 1) $\sim$(S. 4), (A. 1)$\sim$(A. 4). *Then there exists an evolution operator* $\{U(t, s)\,; 0\leqq s\leqq t\leqq T\}$ *on* $\{X_t\}$ *and on* $\{Y_t\}$ *with the following three properties.*

(i) $\|U(t, s)\|_{X_s, X_t}\leqq\exp\int_s^t\omega_X(r)\,dr$, $\|U(t, s)\|_{Y_s, Y_t}\leqq\exp\int_s^t\omega_Y(r)\,dr$, *for* $0\leqq s\leqq t\leqq T$.

(ii) *If* Y_t *is a separable Banach space for every* $t\in[0, T]\backslash\Gamma$, *then for each* $s\in[0, T]$ *and* $y\in Y_s$, $u(\cdot)=U(\cdot, s)y$ *is a unique solution of* $(\mathrm{CP}; 0)_s$ *with* $\sup_{s\leqq t\leqq T}\|u(t)\|_{Y_t}<\infty$. *Furthermore*, $u(\cdot)$ *is in* $AC([s, T]; Z)$ *with*

$$u(t)-u(s)+\int_s^t A(r)u(r)dr=0 \text{ in } Z \text{ for } s\leqq t\leqq T.$$

Using Theorem A, we have the next theorem.

THEOREM 2. *Assume the same situation as in Theorem* A (ii) *and assume moreover that* $D(A(t))$ (*the domain of* $A(t)$ *as an operator in* X_t) $=Y_t$ *for all* $t\in[0, T]$. *Let* $U(t, s)$ *be the evolution operator given by Theorem* A. *Let* F *be a* Z*-valued function on* $[0, T]$ *with* $F(t)\in Y_t$ *a. e. on* $(0, T)$, *and with the following properties.*

(i) *There exists a sequence of* Z*-valued step functions* $\{F_m\}$ *such that*

$$F_m(t)\rightarrow F(t) \text{ in } X_t \text{ as } m\rightarrow\infty \text{ for a. e. } t \text{ on } (0, T),$$

(ii) $\|F(t)\|_{Y_t}\leqq\varsigma(t)$ *on* $[0, T)$, *for some* $\varsigma\in L^1(0, T)$.

Then for every $y\in Y_s$ $(s\in[0, T])$,

$$u(t)=U(t, s)y+\int_s^t U(t, r)F(r)dr$$

is a unique solution of $(\mathrm{CP}; F)_s$ *with* $\sup_{s\leqq t\leqq T}\|u(t)\|_{Y_t}<\infty$.

Furthermore $u(\cdot)$ *is an absolutely continuous* Z*-valued function on* $[s, T]$.

REMARK 2.1 Assume that for interval $[\tau_1, \tau_2]\subset[0, T]$, there exists a positive constant d such that

$$d^{-1}\|\cdot\|_{\tau_1}\leqq\|\cdot\|_{X_t}\leqq d\|\cdot\|_{\tau_1} \text{ for } \tau_1\leqq t\leqq\tau_2.$$

Then, $u(\cdot)$ is an X_{τ_1}-valued absolutely continuous function on $[\tau_1, \tau_2]$.

This immediately follows from the second inequality of (S. 1) and the absolute continuity of $U(\cdot, s)y$ in Z.

REMARK 2.2 Assume that there are Banach spaces $\tilde{X}_i$, $\tilde{Y}_i (i=1, \ldots, n)$ and that $[0, T]$ is divided into finite intervals $\{I_i\}_{i=1,\cdots,n}$ with the following properties; for each i, $X_t\sim\tilde{X}_i$ and $Y_t\sim\tilde{Y}_i$ as Banach spaces a. e. t on I_i, and $F(\cdot)$ is $\tilde{X}_i$-measurable on I_i. Then (i) is satisfied.

In fact, $\tilde{X}_i$-measurability on I_i means the existence of step functions $\{F_{i,m}\}$ such that

$$F_{i,m}(t)\to F(t) \text{ in } \tilde{X}_i \text{ as } m\to\infty \text{ for a. e. } t \text{ on } I_i.$$

By the denseness of Y_t in X_t, we can assume that $F_{i,m}(t)\in\tilde{Y}_i$ a. e. on I_i. If we put $F_m(t)=F_{i,m}(t)$ for $t\in I_i$ $(i=1,\dots,n)$, then $\{F_m\}$ satisfies (i).

PROOF. We assume that $\omega_X\equiv\omega_Y\equiv 0$ without losing generality. Let t^* be an arbitrary element of $[0, T]\backslash\Gamma$. Then by (S. 3) and the closedness of Γ, there is an interval $[t_1, t_2]$ with the following two properties;

$$[t_1, t_2]\ni t^*,\ [t_1, t_2]\cap\Gamma=\emptyset,$$

$$(2.3)\quad \begin{cases} X_t\sim X_{t^*} \text{ with } d^{-1}\|\cdot\|_{X_{t^*}}\leqq\|\cdot\|_{X_t}\leqq d\|\cdot\|_{X_{t^*}}, \\ \qquad\qquad\left|\dfrac{d}{dt}\|\cdot\|_{X_t}\right|\leqq d\|\cdot\|_{X_t}, \\ Y_t\sim Y_{t^*} \text{ with } d^{-1}\|\cdot\|_{Y_{t^*}}\leqq\|\cdot\|_{Y_t}\leqq d\|\cdot\|_{Y_{t^*}}, \end{cases}$$

for $t_1\leqq t\leqq t_2$, with some positive constant d. By the same reason as Remark 2. 11,

$$(2.4)\quad U(\cdot, s)y\in AC([t_1, t_2];\ X_{t^*})$$

for every fixed $s\in[0, T]$ and $y\in Y_s$. By the assumption, we can take a subset Θ of $[0, T]$ satisfying;

$$[0, T]\backslash\Theta \text{ has measure } 0,\ \Theta\cap\Gamma=\emptyset,$$

$$(2.5)\quad F(s)\in Y_s \text{ and } F_m(s)\to F(s) \text{ in } X_s \text{ for } s\in\Theta.$$

We put

$$\Upsilon=\{(t, s)\in[t_1, t_2]\times[0, T];\ s\leqq t\},$$
$$\Upsilon_\Theta=\{(t, s)\in[t_1, t_2]\times\Theta;\ s\leqq t\}.$$

(1) First we prove that $U(t, s)F(s)$ is an X_{t^*}-valued integrable function with respect to (t, s) on Υ. It can be written as

$$F_m(s)=F_m(s_{m,j}) \text{ for } s\in[s_{m,j-1}, s_{m,j}),$$

where $s_{m,0}=0<\cdots<s_{m,j}<s_{m,j+1}<\cdots<s_{m,N_m}=t$. We define a function $G_{m,t}$ by

$$G_m(t, s)=U(t, s_{m,j})F_m(s_{m,j}) \text{ for } s\in[s_{m,j-1}, s_{m,j}),$$

Then by (2.4), G_m becomes an X_{t^*}-valued measurable function with respect to (t, s). Thus for the measurability of G, it suffices to show that

$$(2.6)\quad G_m(t, s)\to U(t, s)F(s) \text{ in } X_{t^*} \text{ for every } (t, s)\in\Upsilon_\Theta,$$

i. e., a. e. on Υ. For every $(t, s)\in\Upsilon_\theta$, it is written as

$$G_m(t, s)-U(t, s)F(s)=U(t, s_{m,j})F(s_{m,j})-U(t, s)F(s)$$

$$=U(t, s_{m,j})F_m(s_{m,j})-F(s))+U(t, s_{m_j})(I-U(s_{m,j}, s))F(s)$$

with some partition point $s_{m,j}$. Thus by (i) of theorem A, we have

$$\begin{aligned}(2.7)\qquad &\|G_m(t, s)-U(t, s)F(s)\|_{X_t}\\ &\leqq M\{\|F(s_{m,j})-F(s)\|_{X_{s_{m,j}}}+\|(I-U(s_{m,j}, s))F(s)\|_{X_s}\},\end{aligned}$$

for some positive constant M. The right-hand side of (2.7) tends to 0 as $m\to\infty$, by (2.4), (2.5) and the continuity of norm $\|\cdot\|_{X_r}$ at $r=s$ ($\notin\Gamma$). Thus (2.6) holds, and therefore the integrability of $U(t, s)F(s)$ immediately follows from (i) of Theorem A and assumption (ii).

(2) We prove that $A(t)U(t, s)F(s)$ is integrable with respect to (t, s) on Υ. By assumptions (2.3) and (A. 2), if $\omega>0$ is large enough, then

$$(2.8)\qquad 0\leqq \mathrm{Re}(A(t)x, x^*)+\omega\|x\|_{X_t}^2 \text{ for some } x^*\in J_{X_t}(x),$$

for every $x\in Y_t$, $t\in[t_1, t_2]$. From (2.8) and the assumption (A. 1), we easily see that $A(t)+\omega I$ is m-accretive in X_t for every $t\in[t_1, t_2]$. Thus, for every $t\in[t_1, t_2]$,

$$J_\varepsilon(t)=\{I+\varepsilon(A(t)+\omega I)\}^{-1}$$

exists and satisfies the following;

$$(2.9)\qquad J_\varepsilon(t)\to I \text{ as } \varepsilon\to 0+$$

in the strong topology of bounded operators in X_t ($\sim X_{t^*}$),

$$(2.10)\qquad \|J_\varepsilon(t)\|_{X_t, X_t}\leqq 1.$$

We put

$$A_\varepsilon(t)=A(t)J_\varepsilon(t)\subset J_\varepsilon(t)A(t).$$

Then it follows from (2.9) that

$$A_\varepsilon(t)x\to A(t)x \text{ in } X_t \text{ as } \varepsilon\to 0+,$$

for every $x\in D(A(t))$, $t\in[t_1, t_2]$. Hence

$$(2.11)\qquad A_\varepsilon(t)U(t, s)F(s)\to A(t)U(t, s)F(s) \text{ in } X_{t^*} \text{ as } \varepsilon\to 0+,$$

for every $(t, s)\in\Upsilon_\Theta$, and thus for a. e. (t, s) on Υ. We show that $A_\varepsilon(t)U(t, s)F(s)$ is measurable with respect to (t, s) on Υ. By (A. 3), (2.3) and (2.10), $A_\varepsilon(t)$ is strongly right-continuous with respect to t as a bounded operator in X_{t^*}. Hence in the same way as in (2.7), $A_\varepsilon(t)G_m(t, s)$ converges to $A_\varepsilon(t)G(t, s)$ as $m\to\infty$. Thus by the same reason as the proof of (1), $A_\varepsilon(t)U(t, s)F(s)$ is X_{t^*}-measurable with respect to (t, s), and so is $A_\varepsilon(t)U(t, s)F(s)$ by (2.11). By (A. 4) and the assumption (ii), the integrability follows.

(3) We show that

$$\frac{d}{dt}\int_0^t U(t, s)F(s)ds + A(t)\int_0^t U(t, s)F(s) = F(t) \tag{2.12}$$

for a.e. t on (t_1, t_2). First we show that

$$\begin{aligned} \frac{1}{h}\{\int_0^{t+h} U(t+h, s)F(s)ds - \int_0^t U(t, s)F(s)ds\} \\ \to -\int_0^t A(t)U(t, s)F(s)ds + F(t) \end{aligned} \tag{2.13}$$

as $h \to 0+$, for a.e. t on (t_1, t_2). Let $h > 0$. We have

$$\begin{aligned} &\frac{1}{h}\{\int_0^{t+h} U(t+h, s)F(s)ds - \int_0^t U(t, s)F(s)ds\} \\ &= \frac{1}{h}\int_t^{t+h}(U(t+h, s)F(s) - F(s))ds + \frac{1}{h}\int_t^{t+h} F(s)ds \\ &\quad + \frac{1}{h}\int_0^t (U(t+h, s) - U(t, s))F(s)ds \\ &= -\frac{1}{h}\int_t^{t+h}\int_s^{t+h} A(r)U(t, s)F(s)drds + \frac{1}{h}\int_t^{t+h} F(s)ds \\ &\quad - \frac{1}{h}\int_0^t\int_t^{t+h} A(r)U(t, s)F(s)drds, \end{aligned} \tag{2.14}$$

since $U(t, s)F(s)$ is a solution of $(\mathrm{CP}; 0)_s$ with $y = F(s)$ (see (ii) of Definition 3). We estimate the right-hand side of (2. 14). By assumption (ii) of Theorem, (A. 4) and result (i) of Theorem A,

$$\|\frac{1}{h}\int_t^{t+h}\int_s^{t+h} A(r)U(t, s)F(s)drds\| \leqq \frac{1}{h}\int_t^{t+h} \xi(s)ds\int_t^{t+h} \varsigma(s)ds, \tag{2.15}$$

which tends to 0 as $h \to 0$. By the assumption, we easily see that $F(t)$ is integrable with respect to t on (t_1, t_2). Hence

$$\frac{1}{h}\int_t^{t+h} F(s)ds \to F(t) \text{ as } h \to 0 \text{ for a.e. } t \text{ on } (t_1, t_2). \tag{2.16}$$

Fubini's theorem implies

$$\begin{aligned} \frac{1}{h}\int_0^t\int_t^{t+h} A(r)U(r, s)F(s)drds &= \frac{1}{h}\int_t^{t+h}\int_0^t A(r)U(r, s)F(s)dsdr \\ &= \frac{1}{h}\int_t^{t+h}\int_0^r A(r)U(r, s)F(s)dsdr \end{aligned} \tag{2.17}$$

$$+\frac{1}{h}\int_t^{t+h}\int_r^t A(r)U(r,s)F(s)\,dsdr.$$
$$\to\int_0^t A(t)U(t,s)F(s)ds \text{ as } h\to 0+,$$

Here we used that $\int_0^r A(r)U(r,s)F(s)ds$ is integrable with respect to r, and the estimate similar to (2.15). Equality (2.14) combined with (2.15)$\sim$(2.17) yields (2.13). Convergence (2.13) with " $h\to 0+$ " replaced by " $h\to 0-$ " holds similarly. Therefore, using that $A(t)$ is closed in X_t for every $t\in[t_1, t_2]$, we obtain (2.12).

The above and the definition of $u(t)$ imply that $u(t)$ is a solution of (CP; $F)_s$.

The uniqueness holds by Theorem A, and the rest is easily seen.

§ 3. The existence of an evolution operator for (WE)

In this section, we consider (WE) with $\psi=0$, $\Xi=0$, $f=0$ and $\nu=0$. Then by putting $v(t)=u'(t)$, (WE) in D_η is transformed into the following;

$$\left.\begin{aligned} &dU(t)/dt+A(t)u(t)=0 \qquad \text{for } t_0<t<S_2,\\ &U(t_0)=\begin{pmatrix}u_0\\u_1\end{pmatrix}\ (\in\pi_{t_0}^{(1/2)+\eta}),\end{aligned}\right\}\ \text{(E)}$$

where

$$U(t)=\begin{pmatrix}u(t)\\v(t)\end{pmatrix},\ A(t)=\begin{pmatrix}0 & -I\\ \phi^2(t)\Lambda & 0\end{pmatrix}.$$

For each real number κ, we shall define Hilbert spaces $\{X_t^\kappa\}$ with $X_t^\kappa\sim\pi_t^\kappa$ for $-1\leqq t\leqq 1$ and Z^κ so as to apply Theorem A to (E). For $\lambda>1$, we define t_λ by

(3.1) $\qquad 8C^3t_\lambda^{-\alpha-1}=\lambda^{1/2}$.

we define the functions p°, q° and r° on $[S_1, S_2]\times[0,\infty)$ as follows:

$$p^\circ(t,\lambda)=\begin{cases}1 & \text{for } 0\leqq\lambda\leqq 1,\ S_1\leqq t\leqq S_2,\\ \lambda\{\phi(t_\lambda)(t+t_\lambda)+\phi(-t_\lambda)(t_\lambda-t)\}/(2t_\lambda) & \\ & \text{for } \lambda>1,\ |t|\leqq t_\lambda,\\ \lambda\phi(t) & \text{for } \lambda>1,\ |t|>t_\lambda.\end{cases}$$

$$q^\circ(t,\lambda)=\begin{cases}1 & \text{for } 0\leqq\lambda\leqq 1,\ S_1\leqq t\leqq S_2,\\ (2t_\lambda)/\{\phi(t_\lambda)(t+t_\lambda)+\phi(-t_\lambda)(t_\lambda-t)\} & \\ & \text{for } \lambda>1,\ |t|\leqq t_\lambda,\\ 1/\phi(t) & \text{for } \lambda>1,\ |t|>t_\lambda.\end{cases}$$

$$r^\circ(t,\lambda)=\begin{cases}0 & \text{for } 0\leqq\lambda\leqq 1,\ S_1\leqq t\leqq S_2,\\ 0 & \text{for } \lambda>1,\ |t|\leqq t_\lambda,\\ \frac{1}{2}\phi'(t)/\phi^2(t) & \text{for } \lambda>1,\ |t|>t_\lambda.\end{cases}$$

For each function $\nu^\circ=p^\circ,\ q^\circ,\ r^\circ$, we put

$$\tilde\nu(t,\lambda)=(\nu^\circ*\rho_{\varepsilon_\lambda})(t)=\int_{-1}^{1}\nu(s,\lambda)\rho_{\varepsilon_\lambda}(t-s)ds,$$

where ρ_ε is a Friedrichs mollifier and ε_λ is a positive number depending on λ and determined later in Propositions 3. 1 and 3. 2. We define

(3.2) $\quad g_1(t,\lambda)=2\{\tilde p'(t,\lambda)-2\phi^2(t)\lambda\tilde r(t,\lambda)\}/\tilde p(t,\lambda),$

(3.3) $\quad g_2(t,\lambda)=2\{\tilde q'(t,\lambda)+2\tilde r(t,\lambda)\}/\tilde q(t,\lambda),$

(3.4) $\quad g_3(t,\lambda)=4|\tilde r'(t,\lambda)+\tilde p(t,\lambda)-\phi^2(t)\lambda\tilde q(t,\lambda)|/(\tilde p\tilde q)^{1/2}(t,\lambda),$

$$g(t,\lambda)=\max\{g_1(t,\lambda),\ g_2(t,\lambda),\ g_3(t,\lambda)\},$$

$$G(t,\lambda)=\int_{-1}^{t}g(s,\lambda)ds,$$

and we put

$$\nu(t,\lambda)=e^{-G(t,\lambda)}\tilde\nu(t,\lambda)\qquad\text{for } \nu=p,\ q,\ r,$$

Using the above functions $p,\ q$ and r, we define Hilbert spaces X_t^κ and Z^κ for each real number κ and $-1\leqq t\leqq 1$.

$$X_t^\kappa=\{U=\binom{u}{v};\ \|U\|_{X_t^\kappa}^2=\int_0^\infty(\lambda+1)^{2\kappa}[p(t,\lambda)d(E_\lambda u,u)+q(t,\lambda)d(E_\lambda v,v)+2r(t,\lambda)d(E_\lambda u,v)]$$
$$(=\int_0^\infty(\lambda+1)^{2\kappa}\mu_{t,\lambda}(U))<\infty\},\ \text{with norm } \|\cdot\|_{X_t^\kappa}.$$

$$Z^\kappa=\{U=\binom{u}{v};\ \|U\|_{Z^\kappa}^2=\int_0^\infty(\lambda+1)^{2\kappa}[\lambda^{2\gamma'}d(E_\lambda u,u)+\lambda^{2\sigma'}d(E_\lambda v,v)]$$
$$(=\int_0^\infty(\lambda+1)^{\kappa}\underline{\mu}_\lambda(U))<\infty\},\ \text{with norm } \|\cdot\|_{Z^\kappa}.$$

Here we note that

$$\|U\|_{Z^\kappa}\leqq\|U\|_{X_t^\kappa}\leqq\|U\|_{X_t^{\kappa'}}\qquad\text{for } U=\binom{u}{v}\quad \kappa\leqq\kappa'.$$

PROPOSITION 3. 1. *If ε is sufficiently small, then there exists a positive constant a_1 for which the following holds :*

$$a_1^{-1}\|U\|_{X_t^\kappa}^{\circ 2}\leqq\|U\|_{X_t^\kappa}^2\leqq a_1\|U\|_{X_t^\kappa}^{\circ 2} \qquad \textit{for every } U=\begin{pmatrix}u\\ v\end{pmatrix},$$

where $\|U\|_{X_t^\kappa}^{\circ 2}=\int_0^\infty(\lambda+1)^{2\kappa}\{p^\circ(t, \lambda)d(E_\lambda u, u)+q^\circ(t, \lambda)d(E_\lambda v, v)\}$. *Thus* $\|\cdot\|_{X_t^\kappa}$ *actually defines the norm which is equivalent to* $\|\cdot\|_{X_t^\kappa}^\circ$.

REMARK 3. 1. The constant a_1 depends only on the constant C in (0.4) and (0.5), and not depend on ϕ itself.

PROOF . Using (0.4), (0.5) and (3.1), we have

$$|r^\circ(t, \lambda)|\leqq\frac{1}{8}\lambda^{1/2}=\frac{1}{8}(p^\circ q^\circ)^{1/2}(t, \lambda) \qquad \text{for every } t, \lambda.$$

Thus

$$(3.5) \qquad |\tilde{r}(t, \lambda)|\leqq\frac{1}{4}(\tilde{p}\tilde{q})^{1/2}(t, \lambda) \qquad \text{for every } t, \lambda.$$

if ε is sufficiently small, and therefore

$$|r(t, \lambda)|\leqq\frac{1}{4}(pq)^{1/2}(t, \lambda) \qquad \text{for every } t, \lambda.$$

Hence we have

$$(3.6) \qquad 2^{-1}\int_0^\infty(\lambda+1)^{2\kappa}\{p(t, \lambda)d(E_\lambda u, u)+q(t, \lambda)d(E_\lambda v, v)\}$$
$$\leqq\|U\|_{t,\kappa}^2\leqq 2\int_0^\infty(\lambda+1)^{2\kappa}\{p(t, \lambda)d(E_\lambda u, u)+q(t, \lambda)d(E_\lambda v, v)\},$$

for every $U=\begin{pmatrix}u\\ v\end{pmatrix}$. If we take ε small enough to satisfy

$$|(\nu^\circ-\nu^\circ*\rho_\varepsilon)(t, \lambda)|\leqq\frac{1}{2}\nu^\circ(t, \lambda) \qquad \text{for } \lambda\geqq 0,\ -1\leqq t\leqq 1,$$

for $\nu=p, q$, then

$$(3.7) \qquad \frac{1}{2}\nu^\circ(t, \lambda)\leqq\tilde{\nu}(t, \lambda)\leqq 2\nu^\circ(t, \lambda) \qquad \text{for } \lambda\geqq 0,\ -1\leqq t\leqq 1,$$

for $\nu=p, q$. By (3.6), (3.7) and the definitions of p and q, the proof is complete if we show that

(3.8) $$(G(t, \lambda)\leqq)\sup_{\lambda\geqq 0}\|g(\cdot, \lambda)\|_{L^1(-1,1)}<\infty.$$

Let h_1, h_2 and h_3 be functions defined by the right-hand sides of (3.2), (3.3) and (3.4) respectively, with $\tilde{p}$, $\tilde{q}$ and $\tilde{r}$ replaced by p°, q° and r° respectively. We first show that

(3.9) $$\sup_{\lambda\geqq 0}\|h_i(\cdot, \lambda)\|_{L^1(s_1,s_2)}<\infty,\ i=1,\ 2,\ 3.$$

It is trivial that

(3.10) $$\sup_{0\leqq\lambda\leqq 1}\|h_i(\cdot, \lambda)\|_{L^1(-1,1)}<\infty,\ i=1,\ 2,\ 3.$$

So we estimate h_i $(i=1,2,3)$ for $\lambda\geqq 1$. From now on in the proof, we denote by the same c the various constants independent of λ and t. By the definition,

(3.11) $$h_i(t, \lambda)=0 \qquad \text{for } |t|\geqq t_\lambda,\ i=1,\ 2.$$

by (0.4), we have

(3.12) $$\begin{aligned} h_1(t, \lambda)&=(p^{\circ\prime}/p^\circ)(t, \lambda)\\ &=(\phi(t_\lambda)-\phi(-t_\lambda))/\{\phi(t_\lambda)(t+t_\lambda)+\phi(-t_\lambda)(t_\lambda)(t_\lambda-t)\}\\ &\leqq ct_\lambda^{-1}, \qquad \text{for } |t|\leqq t_\lambda. \end{aligned}$$

In the way similar to this, we have

(3.13) $$h_2(t, \lambda)\leqq c \qquad \text{for } |t|\geqq t_\lambda.$$

By (0.4), (0.5) and (3.1),

(3.14) $$\begin{aligned} h_3(t, \lambda)&=4|r^{\circ\prime}(t, \lambda)|\lambda^{-1/2}\\ &\leqq c(t^{-\alpha-2}\lambda^{-1/2})=c(t_\lambda^{\alpha+1}|t|^{-\alpha-1}). \end{aligned}$$

for $|t|>t_\lambda$, and

(3.15) $$\begin{aligned} h_3(t, \lambda)&=4|r^\circ(t, \lambda)-\phi^2(t)\lambda q^\circ(t, \lambda)|\lambda^{-1/2}\\ &\leqq c\lambda^{1/2}(t_\lambda^\alpha+t_\lambda^{-\alpha}|t|^{2\alpha})\leqq c(t_\lambda^{-1}+t_\lambda^{-2\alpha-1}|t|^{2\alpha}), \end{aligned}$$

for $|t|\leqq t_\lambda$. From (3.11)~(3.15), (3.9) follows. Using (3.9), we easily see that (3.8) holds if ε is small enough.

REMARK 3.2. Banach space X_t^κ is equivalent to π_t^κ, for each real numbers κ and t with $-1\leqq t\leqq 1$. More precisely, there is a positive constant a_2 $(\geqq 1)$ depending only on the constant C in (0.4) and (0.5) such that for each real number κ, the following inequalities hold for every

$(x, y) \in D_{(1/2)+\kappa} \times D_\kappa$.

(i) For every $t \in [S_1, S_2] \cap [-T, T]$ $(T>0)$,

$$a_2^{-1}(t^\alpha \|x\|^2_{(1/2)+\kappa} + T^{-\alpha}\|y\|^2_\kappa)^{1/2} \leqq \left\|\begin{pmatrix} x \\ y \end{pmatrix}\right\|_{X_t^\kappa} \leqq a_2(T^\alpha \|x\|_{(1/2)+\kappa} + t^{-\alpha}\|y\|^2_\kappa)^{1/2}$$

if $\alpha \geqq 0$, and

$$a_2^{-1}(T^\alpha \|x\|^2_{(1/2)+\kappa} + |t|^{-\alpha}\|y\|^2_\kappa)^{1/2} \leqq \left\|\begin{pmatrix} x \\ y \end{pmatrix}\right\|_{X_t^\kappa} \leqq a_2(|t|^\alpha \|x\|_{(1/2)+\kappa} + T^{-\alpha}\|y\|^2_\kappa)^{1/2}$$

if $\alpha < 0$.

(ii) $$a_2^{-1}(\|x\|^2_{\gamma+\kappa} + \|y\|^2_{\sigma+\kappa})^{1/2} \leqq \left\|\begin{pmatrix} x \\ y \end{pmatrix}\right\|_{X_0^\kappa} \leqq a_2(\|x\|^2_{\kappa+\gamma} + \|y\|^2_{\sigma+\gamma})^{1/2}.$$

We first prove (i) when $\alpha \geqq 0$. When $\alpha < 0$, it is proved similarly. Noting that

$$C^{-1}|t|^\alpha(\lambda+1) \leqq p^\circ(t, \lambda) \leqq CT^\alpha(\lambda+1)$$

for $t \in [S_1, S_2] \cap [-T, T]$, we have

$$C^{-1}|t|^\alpha \|x\|^2_{(1/2)+\kappa} \leqq \int_0^\infty (\lambda+1)^{2\kappa} p^\circ(t, \lambda)\, d(E_\lambda x, x) \leqq C\|x\|^2_{(1/2)+\kappa}$$

for every $x \in D_{(1/2)+\kappa}$ and $t \in [S_1, S_2] \cap [-T, T]$. Noting that

$$C^{-1}T^{-\alpha} \leqq q^\circ(t, \lambda) \leqq Ct^{-\alpha},$$

for $t \in [S_1, S_2] \cap [-T, T]$, we have

$$C^{-1}\|y\|^2_\kappa \leqq \int_0^\infty (\lambda+1)^{2\kappa} q^\circ(t, \lambda)\, d(E_\lambda y, y) \leqq Ct^{-\alpha}\|y\|^2_\kappa,$$

for every $y \in D_\kappa$. Hence, with the aid of Proposition 3. 1, we obtain (i).

Secondly, we prove (ii). From (0.4) and (3.1), it follows that

$$a_2'^{-1}\lambda^{-\alpha/2(\alpha+1)} \leqq \phi(\pm t_\lambda) \leqq a_2'\lambda^{-\alpha/2(\alpha+1)} \quad \text{if } \lambda > 1,$$

with some positive constant a_2'. Using this inequality and the definitions of γ and σ, we get

$$a_2'^{-1}\lambda^{2\gamma} \leqq p^\circ(0, \lambda) = \lambda\{\phi(t_\lambda) + \phi(-t_\lambda)\}/2 \leqq a_2'\lambda^{2\gamma},$$
$$a_2'^{-1}\lambda^{2\sigma} \leqq q^\circ(0, \lambda) = 2/\{\phi(t_\lambda) + \phi(-t_\lambda)\} \leqq a_2'\lambda^{2\sigma},$$

if $\lambda > 1$. By Proposition 3. 1, the above inequalities imply (ii).

Now, we have the following proposition, which is the purpose of this section.

PROPOSITION 3.2. *Assume* (0.1)~(0.6). *If* ε *is sufficiently small, then for each* κ, *the Hilbert spaces* $\{X_t=X_t^\kappa\}$, $\{Y_t=X_t^{(1/2)+\kappa}\}$, $Z=Z^\kappa$ *and the operator* $\{A(t)\}$ *satisfy the assumption of Theorem* A *with* $\Gamma=\{0\}$ *and* $\omega_X\equiv\omega_Y\equiv 0$.

If Proposition 3.2 is assumed, the next proposition follows.

PROPOSITION 3.3. *In the same situation as in Proposition* 3.2, $A(t)$ *generates the evolution operator* $U(t, s)$ *on* $\{X_t^\kappa\}$ *for each* κ *with the following properties.*

(i) $\|U(t,s)\|_{X_s^\kappa, X_t^\kappa}\leqq 1$ *for* $S_1\leqq s\leqq t\leqq S_2$,

(ii) *For every* $r\neq 0$ *and* $V\in X_r^{(1/2)+\kappa}$, $U(t,s)V$ *is continuous in* X_r^κ *with respect to* (t,s) *in the neighborhood of* (r,r).

(iii) *For every* $U_0\in X_{t_0}^{(1/2)+\kappa}$, $U(\cdot)=U(\cdot, t_0)U_0$ *is a unique solution of* (*E*) *in* Z^κ *in the sense of Definition* 3. *Furthermore, the following hold*;

$$U(\cdot)\in AC([t_0, S_2]; Z^\kappa)\cap AC_{\mathrm{loc}}([t_0, S_2]\backslash[0]; D_{(1/2)+\kappa}\times D_\kappa),$$
$$\frac{d}{dt}U(t)+A(t)U(t)=0 \text{ in } Z^\kappa \text{ a. e. } t \text{ on } (t_0, S_2).$$

PROOF. By Proposition 3.2 and Theorem A, the conclusion except the uniqueness of a solution in (iii) holds. Theorem A guarantees the uniqueness of a solution of (E) in Z with bounded Y_t-norm. In this case, every solution of (E) in Z^κ has a bounded $X_r^{-(1/2)+\kappa}$-norm, since it belongs to $C([t_0, S_2]; Z^\kappa)$ and $\|\cdot\|_{X_t^{-(1/2)+\kappa}}\leqq\|\cdot\|_{Z^\kappa}$. If we take $-1+\kappa$ for κ in Proposition 3.2, then $Z=Z^{-1+\kappa}$ and $Y_t=X_r^{-(1/2)+\kappa}$. Thus the uniqueness holds as a solution in $Z^{-1+\kappa}$ with bounded $X_r^{-(1/2)+\kappa}$-norm.

PROOF OF PROPOSITION 3.2. We prove the case that $\kappa=0$. The other case is proved parallel to this.

(S.1) It is easy to see that

$$\lambda^{2\gamma'}\leqq cp^\circ(t,\lambda)\leqq c'(\lambda+1)^{1-\theta}\lambda^{2\gamma'},$$
$$\lambda^{2\sigma'}\leqq cq^\circ(t,\lambda)\leqq c'(\lambda+1)^{1-\theta}\lambda^{2\sigma'},$$

for some constants $\theta\in(0,1]$ and $c, c'>0$ independent of t and λ. By using Proposition 3.1, these inequalities imply (S.1).

(S.2) Let $t_n\to t$ as $n\to\infty$, and

$$(3.16)\qquad U_n=\begin{pmatrix}u_n\\ v_n\end{pmatrix}\in X_{t_n}\to U=\begin{pmatrix}u\\ v\end{pmatrix} \text{ in } Z \text{ as } n\to\infty,$$

with

(3.17) $\sup_n \|U_n\|_{X_{t_n}} (=M) < \infty.$

Let η be an arbitrary fixed positive number. Then the total variation of $\|E_\lambda(u_n-u)\|^2$ and $\|E_\lambda(v_n-v)\|^2$ on $(-\infty, \eta]$ are dominated by $\|E_\eta(u_n-u)\|^2$ and $\|E_\eta(v_n-v)\|^2$ respectively, which tend to 0 by (3.16). From this and the continuities of the functions p, q, r with respect to t uniformly in $\lambda \leqq \eta$, we have

(3.18) $\int_0^\eta \mu_{t_n,\lambda}(U_n) \to \int_0^\eta \mu_{t,\lambda}(U)$ as n.

On the other hand, by (3.17) we have

$$\int_0^\eta \mu_{t_n,\lambda}(U_n) \leqq M \text{ for every } n.$$

By (3.18), letting $n \to \infty$ in the last inequality yields

$$\int_0^\eta \mu_{t,\lambda}(U) \leqq M.$$

Since this inequality holds for every positive number η, we obtain

$$\int_0^\infty \mu_{t,\lambda}(U) \leqq M \text{ and } U \in X_t.$$

In the same way, we obtain the conclusion for Y_t.

(S. 3) Let $t \neq 0$. We take δ such that $[t-\delta, t+\delta] \not\ni 0$. Then we see that

$$\sup\{|p'(s,\lambda)|/p(t,\lambda),\ |q'(s,\lambda)|/q(t,\lambda),\ |r'(s,\lambda)|/(pq)^{\frac{1}{2}}(t,\lambda),\ |p''(s,\lambda)|/p(t,\lambda),\ |q''(s,\lambda)|/q(t,\lambda),\ |r''(s,\lambda)|/(pq)^{\frac{1}{2}}(t,\lambda);\ s \in [t-\delta, t+\delta] \cap [S_1, S_2],\ \lambda \geqq 0\} < \infty.$$

From this, it follows that (S. 3) holds.

(S. 4) Let ε be an arbitrary fixed number. We take λ^* large enough to satisfy

(3.19) $\lambda^* + 1 > \varepsilon^{-2}.$

Let h be an arbitrary number with

(3.20) $0 < h \leqq t_{\lambda^*},$

where t_{λ^*} is defined by (3.1). We define

$$P=E_{\lambda^*}|_{Y_0}\,;\ Y_0\to Y_h,$$

the restriction of E_{λ^*} on Y_0. We prove that P satisfies the condition of (S. 4). It follows from (3.20) that

$$p(h,\lambda)=p(0,\lambda),\ q(h,\lambda)=q(0,\lambda),\ r(h,\lambda)=r(0,\lambda),$$

for every $\lambda\leqq\lambda^*$. From these relations and (3.19), it follows that

$$\|PU\|_{X_h}=\|PU\|_{X_0}\leqq\|U\|_{X_0},\ \|PU\|_{Y_h}=\|PU\|_{Y_0}\leqq\|U\|_{Y_0},$$
$$\|(I-P)U\|_Z^2\leqq\int_{\lambda^*}^{\infty}\mu_\lambda(U)\leqq(\lambda^*+1)^{-1}\int_{\lambda^*}^{\infty}(\lambda+1)\mu_{\lambda,0}(U)\leqq\varepsilon^2\|U\|_{Y_0}^2,$$

for every $U\in Y_0$. Thus (S. 4) holds.

(A. 1) Let t be an arbitrary fixed number in $[S_1, S_2]\backslash\{0\}$. Using the fact that $\phi^2(t)\Lambda$ is a non-negative self-adjoint operator, we easily see that (A. 1) holds.

(A. 2) We shall prove the condition for X_t. In the same way, we can prove the condition for Y_t. Let $t\neq0$. By the definition of $\|\cdot\|_t$, we have

$$(3.21)\quad (d/dt)\|U\|_{X_t}^2=\int_0^\infty\{p'(t,\lambda)d(E_\lambda u,u)+q'(t,\lambda)d(E_\lambda v,v)+2r'(t,\lambda)dE_\lambda(u,v)\},$$
$$=\int_0^\infty e^{-G(t)}\{(\tilde{p}'-g\tilde{p})(t,\lambda)d(E_\lambda u,u)+(\tilde{q}'-g\tilde{q})(t,\lambda)d(E_\lambda v,v)+2(\tilde{r}'-g\tilde{r})(t,\lambda)d(E_\lambda u,v)\},$$

$$(3.22)\quad (AU,U)_{X_t}=\int_0^\infty e^{-G(t)}[-\tilde{p}(t,\lambda)d(E_\lambda u,v)+\tilde{q}(t,\lambda)\phi^2(t)\lambda d(E_\lambda u,v)+\tilde{r}(t,\lambda)\{-dE_\lambda(v,v)+\phi^2(t)\lambda d(E_\lambda u,u)\}].$$

Comparing each terms which corresponds to $d(E_\lambda u,u)$, $d(E_\lambda v,v)$ and $d(E_\lambda u,v)$ respectively, and noting that

$$(\tilde{p}\tilde{q})^{1/2}d(E_\lambda u,v)\leqq\frac{1}{2}(\tilde{p}(t,\lambda)d(E_\lambda u,u)+\tilde{q}(t,\lambda)d(E_\lambda v,v)),$$

we see that (2.1) holds with $\omega_X=0$ if the following hold;

$$(3.23)\quad \tilde{p}'(t,\lambda)\leqq2\phi^2(t)\lambda\tilde{r}(t,\lambda)+\frac{1}{2}(g\tilde{p})(t,\lambda),$$

$$(3.24)\quad \tilde{q}'(t,\lambda)\leqq-2\tilde{r}(t,\lambda)+\frac{1}{2}(g\tilde{q})(t,\lambda),$$

$$(3.25)\quad 2|\tilde{r}'(t,\lambda)-(g\tilde{r})(t,\lambda)+\tilde{p}(t,\lambda)-\phi^2(t)\lambda\tilde{q}(t,\lambda)|$$

$$\leqq (g(\tilde{p}\tilde{q})^{1/2})(t, \lambda).$$

Thus it suffices to show (3.23)~(3.25). But these are trivial from the definitions of $\tilde{p}$, $\tilde{q}$, $\tilde{r}$ and g. Here we note that from (3.5), (3.25) holds if

$$2|\tilde{r}'(t, \lambda)+\tilde{p}(t, \lambda)-\phi^2(t)\lambda\tilde{q}(t, \lambda)|\leqq\frac{1}{2}(g(\tilde{p}\tilde{q})^{1/2})(t, \lambda).$$

(A. 3) This is trivial.

(A. 4) From Proposition 3. 1 with $\kappa=0$, we have

$$\|A(t)\binom{u}{v}\|_{X_t}^2=\left\|\begin{pmatrix}-v\\ \phi^2(t)\Lambda u\end{pmatrix}\right\|_{X_t}^2$$
$$\leqq a_1\{\int_0^\infty p^\circ(t, \lambda)dE_\lambda\|v\|^2+\int_0^\infty q^\circ(t, \lambda)\phi^4(t)\lambda^2 dE_\lambda\|u\|^2\}.$$

From Proposition 3. 1 with $\kappa=1/2$, we have

$$\left\|\binom{u}{v}\right\|_{Y_t}^2\leqq a_1^{-1}\int_0^\infty(\lambda+1)\{p^\circ(t, \lambda)dE_\lambda\|u\|^2+q^\circ(t, \lambda)dE_\lambda\|v\|^2\}.$$

Hence, for (A. 4), it suffices to prove the following inequalities.

(3.26) $\phi^4(t)\lambda^2 q^\circ(t, \lambda)\leqq\xi^2(t)(\lambda+1)p^\circ(t, \lambda)$,

(3.27) $p^\circ(t, \lambda)\leqq\xi^2(t)(\lambda+1)q^\circ(t, \lambda)$,

for every $t\in[S_1, S_2]$ and $\lambda\in[0, \infty)$. By the definition of p° and q°, inequalities (3.26) and (3.27) are satisfied if the following hold:

(3.28)	$\phi(t)+1\leqq\xi(t)$	for $0\leqq\lambda\leqq1$, $S_1\leqq t\leqq S_2$,
(3.29)	$\phi^2(t)/\phi(\pm t_\lambda)+\phi(\pm t_\lambda)\leqq\xi(t)$	for $\lambda\leqq1$, $\|t\|\leqq t_\lambda$,
(3.30)	$\phi(t)\leqq\xi(t)$	for $\lambda\leqq1$, $\|t\|\leqq t_\lambda$.

From Assumption (0.4), we easily see that these hold by taking $\xi(t)=c(|t|^{2\alpha}+1)$ for sufficiently large constant c. Since $2\alpha>-\nu-1=-1$, ξ is integrable, and the proof of Proposition 3. 2 is complete.

§ 4. Proof of theorem 1

As is noted in § 1, we have only to prove Theorem 1 except (1.9) in case that $\alpha>-1/2$ and $\nu=0$. We assume that $\eta=0$. When $\eta\neq0$, it is proved parallel to this. We asume $[S_1, S_2]=[-1, 1]$ without loss of generality. X_t^κ, Z^κ and $A(t)$ denote the Hilbert spaces and the operator defined in § 3. $U(t, s)$ denotes the evolution operator given by Proposition 3. 3. By putting $u'=v$, (WE) is equivalent to the following equation in Z^0;

$$\left.\begin{array}{l}\frac{d}{dt}U(t)+A(t)U(t)+B(t)U(t)=\tilde{F}(t) \ \textit{for} \ t_0<t<1, \\ U(t_0)=\begin{pmatrix}u_0\\u_1\end{pmatrix} \ (=U_0),\end{array}\right\} \text{(EE)}$$

where

$$U(t)=\begin{pmatrix}u(t)\\v(t)\end{pmatrix} \ (\in X_t^{1/2}), \ \tilde{F}(t)=\begin{pmatrix}0\\f(t)\end{pmatrix} \ (\in X_t^{1/2}),$$

$$B(t)=\begin{pmatrix}0 & 0\\ \Xi(t) & \psi(t)I\end{pmatrix} \ \text{(the bounded operator on } X_t^{1/2} \text{ for a. e. } t).$$

We shall prove Theorem 1 in the following steps: estimates of operators $B(t)$ and $\tilde{F}(t)$, existence of a solution, estimates of the solution, uniqueness, estimates of the solution under the additional assumption.

≪Estimates of $\|B(t)\|_{X_t^{1/2},X_t^{1/2}}$ and $\|\tilde{F}(t)\|_{X_t^{1/2}}$≫ If $\alpha\geqq 0$, (i) of Remark 3.2 with $\varkappa=1/2$ and (1.3) with $\eta=0$ yield

$$\left\|\begin{pmatrix}0\\ \Xi(t)x\end{pmatrix}\right\|_{X_t^{1/2}}=a_2t^{-\alpha/2}\|\Xi(t)x\|_{1/2}\leqq a_2t^{-\alpha/2}b(t)\|x\|_1$$
$$\leqq a_2^2t^{-\alpha}b(t)\left\|\begin{pmatrix}x\\y\end{pmatrix}\right\|_{X_t^{1/2}},$$

for every $(x, y)\in D_1\times D_{1/2}$. From this and (3.6), it follows that

$$\left\|B(t)\begin{pmatrix}x\\y\end{pmatrix}\right\|_{X_t^{1/2}}\leqq\left\|\begin{pmatrix}0\\ \Xi(t)x+\psi(t)y\end{pmatrix}\right\|_{X_t^{1/2}}$$
$$\leqq(a_2^2|t|^{-\alpha}b(t)+2|\psi(t)|)\left\|\begin{pmatrix}x\\y\end{pmatrix}\right\|_{X_t^{1/2}},$$

for $(x, y)\in D_1\times D_{1/2}$, which implies that

$$\text{(4.1)} \qquad \|B(t)\|_{X_t^{1/2},X_t^{1/2}}\leqq a_2^2|t|^{-\alpha}b(t)+2|\psi(t)| \qquad \text{if } \alpha\geqq 0.$$

we similarly obtain

$$\text{(4.2)} \qquad \|B(t)\|_{X_t^{1/2},X_t^{1/2}}\leqq a_2^2b(t)+2|\psi(t)| \qquad \text{if } \alpha<0.$$

By (i) of Remark 3.2 with $\varkappa=1/2$, we have

$$\text{(4.3)} \qquad \|\tilde{F}(t)\|_{X_t^{1/2}}\begin{cases}\leqq a_2|t|^{-\alpha/2}\|f(t)\|_{1/2} & \text{if } \alpha\geqq 0,\\ \leqq a_2\|f(t)\|_{1/2} & \text{if } \alpha<0.\end{cases}$$

≪Existence of a solution≫ We define T^* and R as follows.

(4.4) $T^*(\leqq 1)$ is the supremum of S satisfying

$$\int_{t_0}^{S} \varsigma(t)\,dt \leqq 1/4,$$

where

$$\varsigma(t) \begin{cases} = a_2^2|t|^{-\alpha} b(t) + 2|\psi(t)| & \text{if } \alpha \geqq 0, \\ = a_2^2 b(t) + 2|\psi(t)| & \text{if } \alpha < 0. \end{cases} \tag{4.5}$$

$$R = 2\left(\|U_0\|_{X_t^{1/2}} + \int_{t_0}^{1} \|\tilde{F}(s)\|_{X_t^{1/2}} ds\right). \tag{4.6}$$

We note that $T^* > t_0$ by assumptions (0.6), (1.4) and (1.5) with $\nu = 0$. We set

$$\begin{aligned} G_{T^*} = \{ & V \in C([t_0, T^*]; Z^0); \\ & V(\cdot) \in AC_{\mathrm{loc}}([t_0, T^*] \backslash \{0\}; D_{1/2} \times H), \\ & V(t) \in X_t^{1/2} \text{ for } t_0 \leqq t \leqq T^*, \\ & \|V(t)\|_{X_t^{1/2}} \leqq R \}. \end{aligned} \tag{4.7}$$

We define Banach space χ by

$$\chi = \{ V \in C([t_0, T^*]; Z); \sup_{t_0 \leqq t \leqq T^*} \|V(t)\|_{X_t^{1/2}} < \infty),$$

with norm $\sup_{t_0 \leqq t \leqq T^*} \|V(t)\|_{X_t^{1/2}}$. Then G_{T^*} becomes a bounded closed convex subset of χ.

For an arbitrary $W = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ in G_{T^*}, we consider the equation:

$$\left.\begin{aligned} & \frac{d}{dt} U(t) + A(t) U(t) = -B(t) W(t) + \tilde{F}(t) \text{ on } (t_0, T^*) \\ & U(t_0) = U_0, \end{aligned}\right\} (\mathrm{EE})_W$$

We show that the Hilbert spaces $\{X_t = X_t^0\}$, $\{Y_t\} = \{X_t^{1/2}\}$, $Z = Z^0$, the operator $\{A(t)\}$, and the function $F(\cdot) = -B(\cdot) W(\cdot) + \tilde{F}(\cdot)$ satisfy the assumption of Theorem 2. It is trivial that $D(A(t)) = Y_t$. Thus by Proposition 3.2, the assumption of Theorem 2 other than (i) and (ii) are satisfied. The X_1-measurability of $-B(\cdot) W(\cdot) + \tilde{F}(\cdot)$ on $(-1, 1)$ follows from the assumptions (H1) and (H2), the denseness of D_1 in $D_{1/2}$ and the local continuity of $W : [t_0, T^*] \backslash \{0\} \to D_{1/2} \times H$ $(\sim X_1)$. Therefore by

Remark 2.2 with $i=1$, assumption (i) of Theorem 2 is satisfied. Assumption (ii) follows from (4.1)~(4.3), (1.4)~(1.7) and (4.7) with $V=W$. Hence we can apply Theorem 2 to $(\mathrm{EE})_W$ and obtain a unique solution U with form;

$$(4.8) \qquad U(t)=U(t,t_0)U_0+\int_{t_0}^{t}U(t,s)(\tilde{F}(s)-B(s)W(s))ds,$$

for $t_0\leqq t\leqq T^*$. By Theorem 2 and Remark 2.1, U satisfies the conditions for belonging to G_{T^*} except (4.7). We prove (4.7). By using (i) of Theorem A, (4.1) and (4.2), (4.8) yields

$$(4.9) \qquad \|U(t)\|_{X_t^{1/2}}\leqq\|U_0\|_{X_0^{1/2}}+\int_{t_0}^{t}(\|\tilde{F}(s)\|_{X_s^{1/2}}+\varsigma(s)\|W(s)\|_{X_s^{1/2}})ds,$$

where ς is defined by (4.5). We get (4.7) with $V=U$ from (4.9), (4.6), (4.4) and (4.7) with $V=W$.

By the above, we can define a mapping Φ from D_{T^*} into D_{T^*} by

$$\Phi:\ W\rightarrow U\text{; a solution of } (\mathrm{EE})_W.$$

We show that Φ is a contraction mapping on D_{T^*}. Let W_1 and W_2 be arbitrary elements of D_{T^*}, and put $W=W_1-W_2$. From (4.8), it follows that

$$\Phi W_1(t)-\Phi W_2(t)=-\int_{t_0}^{t}U(t,s)B(s)W(s)ds.$$

Thus using (i) of Theorem 3.1, (4.1), (4.2) and (4.4), we have

$$\begin{aligned}\|(\Phi W_1-\Phi W_2)(t)\|_{X_t^{1/2}}&\leqq\int_{t_0}^{t}\varsigma(s)ds\sup_{t_0\leqq t\leqq T^*}\|W(s)\|_{X_s^{1/2}}\\&\leqq\frac{1}{2}\sup_{t_0\leqq t\leqq T^*}\|W(s)\|_{X_s^{1/2}}.\end{aligned}$$

Hence we get

$$\sup_{t_0\leqq t\leqq T^*}\|(\Phi W_1-\Phi W_2)(t)\|_{X_t^{1/2}}\leqq\frac{1}{2}\sup_{t_0\leqq t\leqq T^*}\|(W_1-W_2)(t)\|_{X_t^{1/2}},$$

which means that Φ is a contraction mapping in D_{T^*}. Hence by the contraction mapping theorem, Φ has a fixed point U, which is a solution of (EE) on $[t_0, T^*]$.

Next, starting from T^*, we extend a solution to $T^{**}(>T^*)$ in the same way. By definition (4.4) and the integrability of ς, we arrive at 1 in finite steps. Thus we have obtained a solution $U=\binom{u}{v}$ of (EE), belonging to $AC([t_0, 1]; Z) \cap AC_{loc}([t_0, 1]\backslash\{0\}; D_{1/2}\times H)$ and having bounded $X_t^{1/2}$-norm. It is easy to see that u becomes a solution of (WE) in the sense stated in the assertion of the theorem.

≪Estimate of the solution $u(t)$≫ Using (4.9) with $W=U$ and Gronwall's lemma finite times, we have

$$\|U(t)\|_{X_t^{1/2}} \leqq (\|U_0\|_{X_0^{1/2}} + \int_{t_0}^{t} \|\tilde{F}(s)\|_{X_s^{1/2}} ds) \exp \int_{t_0}^{t} \varsigma(s)\, ds \leqq M, \tag{4.10}$$

for $t_0 \leqq t \leqq 1$, with some positive constant M. Thus, we obtain (1.8) by noting that

the left-hand side of $(1.8) = \|U(t)\|_{Z^{1/2}} \leqq \|U(t)\|_{X_t^{1/2}}$ for $t_0 \leqq t \leqq 1$.

≪Uniqueness≫ Let u and $\tilde{u}$ be solutions of (WE), and put $w = u - \tilde{u}$, $W = \begin{pmatrix} w \\ w' \end{pmatrix}$. Then W is a solution of the following equation for V:

$$\left.\begin{aligned} &\frac{d}{dt} V(t) + A(t)V(t) = -B(t)W(t) \ \ in\ Z\ a.\,e.\ on\ (t_0, 1), \\ &V(t_0) = 0. \end{aligned}\right\} \text{(E)}$$

By using that $w \in C([t_0, 1]; D_{\gamma'+(1/2)})$ and that D_1 is dense in $D_{\gamma'+(1/2)}$, (H1) implies the measurability of $\Xi(\cdot)w(\cdot)$ in $D_{\gamma'}$. By this and (1.3)' in (H1), $B(\cdot)W(\cdot)$ satisfies the condition of $F(\cdot)$ in Theorem 2 with $Z = Z^{\gamma'-\delta}$. Hence, by the same argument as in (4.9), we have

$$\|W(t)\|_{X_t^{\gamma'+(1/2)}} \leqq \int_0^t \varsigma(s) \|w(s)\|_{X_t^{\gamma'+(1/2)}} ds. \tag{4.11}$$

Since ς is integrable, (4.11) means $W \equiv 0$.

≪Estimate of the solution under the additional assumption≫ Last we show that u satisfies (1.10), under additional assumption. Let $[b_1, b_2]$ be an arbitrary closed interval in $[t_0, 1]\backslash\{0\}$. We consider the following equation for v:

$$\left.\begin{aligned} &v''(t) + \phi^2(t)\Lambda v(t) + \psi(t)u'(t) + \Xi(t)u(t) = 1(t) \ \ on\ (t_0, T) \\ &v(b_1) = u(b_1),\ v'(b_1) = u'(b_1). \end{aligned}\right\} \text{(WE)}'$$

Since $(u(b_1), u'(b_1))$ belongs to $D_1 \times D_{1/2}$, it is well-known that under the assumptions on ϕ, ψ, Ξ and f, (WE)' has a solution v in

$$\bigcap_{i=0}^{2} C^i([b_1, b_2]; D_{(2-i)/2}). \tag{*}$$

The uniqueness of the solution assures $v=u$ on $[b_1, b_2]$. Hence u belongs to the function space (*). Since $[b_1, b_2]$ is an arbitrary closed interval in $[t_0, 1]\backslash\{0\}$, the above implies (1.10).

References

[1] M. L. BERNARDI, Second order abstract differential equations with singular coefficients, Ann. Mat. Pure Appl., 130 (1982), 257-286.

[2] R. W. CARROLL & R. E. SHOWALTER, Singular or degenerate Cauchy problems, Academic Press (1976).

[3] G. COPPOLETTA, Abstract singular evolution equations of "hyperbolic" type, J. Funct. Anal., 50 (1983), 50-66.

[4] M. IMAI, Cauchy problem for the operator of Tricomi's type, Hokkaido Math. J. 8 (1979), 126-143.

[5] V. Ya. IVRII, Sufficient conditions for regular and completely regular hyperbolicity, Trud. Mosk. Mat. Obsc. 33 (1976), 1-65.

[6] A. KUBO, On the mixed problems for weakly hyperbolic equations of second order, Comm. Partial Differntial Equations, 9 (1984), 889-917.

[7] C. LACOMBLEZ, Sur une équation d'évolution du second ordre en t à coefficients dégénérés ou singuliers, C. R. Acad. Sci. Paris, 274 (1974), 241-244.

[8] O. A. OLEINIK, On the Cauchy problem for weakly hyperbolic equations, Comm. Pure Appl. Math., 23 (1970), 569-586.

[9] M. H. PROTTER, The Cauchy problem for a hyperbolic second order equation with data on the parabolic line, Canadian J. Math. 6 (1954), 542-553.

[10] R. SAKAMOTO, Mixed problem for degenerate hyperbolic equations, J. Math. Kyoto Univ. 23 (1983), 563-597.

[11] F. SEGALA, Parametrices for a class of differential operators with multiple characteristics, Ann. Mat. Pure Appl. 146 (1987), 311-336.

[12] M. TANIGUCHI, A systematization of the L^2-well-posed mixed problem and its applications to the mixed problem for weakly hyperbolic equations of second order, Funkcialaj Ekvac. 31 (1988), 249-296.

[13] T. SHIGETA (=T. Yamazaki), Linear evolution equations and a mixed problem for singular or degenerate wave equations, Comm. Partial Differential Equations, 12 (1987), 701-776.

[14] T. YAMAZAKI, On local solutions of some quasilinear degenerate hyperbolic equations, Funkcialaj Ekvac., 31 (1988), 439-457.

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