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On singularities in the degenerated symplectic geometry

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Abstract

Maximal isotropic varieties of the $\Sigma_{2,0}$ singular symplectic structure are considered. Their versal singularities are classified and the lists of normal forms of small codimensions are given. These normal formas are represented by resttricted classification of singularities of Lagarangian varieties in symplectic manifold with boundary. The links to thermodynamics of the zero-lovel-temperature are discussed.

1. Introduction

The main aim of the applied symplectic geometry is to describe the real states of a system by means of Lagrangian varieties in appropriate cotangent bundle-phase space [1, 3]. In this approach the structural properties of a system under consideration (say phase transitions, bifurcation sets, breaking of the wave fronts, ...) are associated with the structure and generic properties of the corresponding Lagrangian varieties. In early applications of Lagrangian singularities [21, 22, 12] only smooth Lagrangian submanifolds of the phase spaces were used. Although generally successful, this approach showed some shortcomings too. For instance it appeared to be insufficient to describe so called critical phenomena in thermodynamics (since it delivered only the classical values of critical indices [17, 11, 12], not compatible with experimental data).

The first generalisation, to non-smooth Lagrangian varieties, appeared naturally in Melrose's theory of glancing hypersurfaces [15] which was subsequently extended in Arnold's papers (see e. g. [4]) on singularities of systems of rays in the variational obstacle problem. Such generalisations appeared also in the discussion of thermodynamical phase coexistence in [10]. However in an attempt to model properly the critical point of thermodynamics (where possibly some fundamental laws of thermodynamics "break down" [17, 19]) it seems to be quite natural to go further on and admit some singularities of symplectic structures of the phase space as well. The aim of this paper is to make the first step in this direction. To select reasonably an initial form of "singular symplectic structure" it is natural to turn to the (local) classification of germs of 2-forms [13, 18]. There we find that on $\boldsymbol{R}^{2n}$, the simplest classes of germs (at 0) of stable 2-forms are represented by the canonical symplectic form $\omega=\sum_{i=1}^{n} dx_i \wedge dy_i$ and by the 2-form

$$\sigma=x_1 dx_1 \wedge dy_1+\sum_{i=2}^{n} dx_i \wedge dy_i. \tag{1}$$

On this ground the 'singular symplectic structure' σ and the 'singular Lagrangian fibration' $(\boldsymbol{R}^{2n}, \sigma, \pi)$, where $\pi:\ (x, y)\in\boldsymbol{R}^{2n}\longmapsto x\in\boldsymbol{R}^{n}$, are the natural candidates to start with (cf. [20, 16]). As an additional argument supporting this choice we find that the 2-form (1) bas already emerged as the simplest one in the hierarchy of singular symplectic structures in the above mentioned papers of Melrose and Arnold on the variational obstacle problem. We can also foresee potential applications of this structure in thermodynamics: in modelling the above mentioned critical region, in the investigations of open thermodynamical systems and in modelling the absolute zero temperature region. In the latest context let us consider the 1-form of internal energy [8]

$$\theta=\frac{1}{2}\gamma^2 dS-pdV+\sum_{i=2}^{k}\mu_i dN_i, \tag{2}$$

where γ is a parametric temperature [6]. Then 2-form $d\theta$ has stable singularities of type (1) along the hypersurface $\{\gamma^2=T=0\}$ and is non-singular elsewhere and π is the projection of the thermodynamical phase space $\{\gamma, p, \mu_i,\ S,\ -V,\ N_i\}$ onto the space of thermodynamical forces $\{(\gamma, p, \mu_i)\}$, which are natural control parameters for the thermodynamic system in equilibrium [6, 11, 12, 17, 19]. On assuming (2) we obtain a fine link between the thermodynamical postulate of positivity of absolute temperature and the stability of an applicable structure of thermodynamics [6]. In this approach the normal states of equilibrium apart from $\gamma=0$ are described by Lagrangian submanifolds, in agreement with classical theory. Thus in the case of extended phase space with the 1-form of internal energy (2) it is natural to set as an initial goal the classification of local forms of maximal isotropic submanifolds near the singularity hypersurface $\{\gamma=0\}$. This is exactly the starting point of this paper, although formulated in terms of the 2-form $\sigma\overset{\text{def}}{=}d\theta$ rather than the 1-form (2). We end up with an initial classification of maximal isotropic varieties of the singular symplectic structure (1).

The paper is organised as follows. At the beginning of Section 2 the natural equivalences of $(\boldsymbol{R}^{2n}, \sigma)$ (σ-equivalences) are introduced and it is shown that a substantial class of them can be obtained by lowering restricted Lagrangian equivalences of the Lagrangian fibration $(\boldsymbol{R}^{2n}, \omega)$, $\omega = \sum dx_i \wedge dy_i$ (*restricted* means preserving the hypersurface $\{x_1 = 0\}$). Next the isotropic varieties of $(\boldsymbol{R}^{2n}, \sigma, \pi)$, σ-varieties, are introduced formally in terms of generating families. Their classification up to σ-equivalences is shown to be equivalent to a classification of Lagrangian varieties in $(\boldsymbol{R}^{2n}, \omega, \pi)$ up to restricted Lagrangian equivalences. Finally, the case of maximal isotropic submanifolds (σ-manifold) in $(\boldsymbol{R}^{2n}, \sigma, \pi)$ is considered in able to show that isotropic varieties appear in this sort of considerations quite naturally. Representative features of the geometry of σ-manifolds are illustrated by a number of Examples. Section 3 considers classification of Lagrangian varieties up to restricted Lagrangian equivalences. The initial classification list of normal forms of generating families is obtained here. These results are derived in the standard singularity theory fashion, with an essential use of Arnold's classification of boundary singularities. In Section 4 these results are finally utilised to classify maximal isotropic varieties of $(\boldsymbol{R}^{2n}, \sigma)$ and some examples of the simplest normal forms are considered.

2. Maximal isotropic varieties

2.1 σ-equivalences. Let us consider $\boldsymbol{R}^{2n}$ with fixed coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$ and a 2-form $\sigma = x_1 dx_1 \wedge dy_1 + \sum_{i=2}^{n} dx_i \wedge dy_i$. A diffeomorphism $\boldsymbol{R}^{2n} \to \boldsymbol{R}^{2n}$ preserving the 2-form σ and the fibration π: $\boldsymbol{R}^{2n} \to \boldsymbol{R}^n$, $(x, y) \to x$ is called a *σ-equivalence.* (As it has been mentioned already, the σ-equivalences form the natural group of permissible transformations of $(\boldsymbol{R}^{2n}, \sigma, \pi)$ with natural thermodynamic interpretations.)

We shall discuss now natural links between σ-equivalences and Lagrangian equivalences in the theory of Lagrangian singularities [2, 4, 22, 23]. Let $\omega \overset{\mathrm{def}}{=} \sum dx_i \wedge dy_i$ be a symplectic form on $\boldsymbol{R}^{2n}$. We recall that symplectomorphism of $(\boldsymbol{R}^{2n}, \omega)$ preserving fibration π is called a *Lagrangian equivalence* (*L-equivalence*). An L-equivalence preserving the hyperplane $\{x_1 = 0\}$ will be called a *restricted Lagrangian equivalence* (for short: *rL-equivalence*).

The transformation

$$\rho: (x, y) \in \boldsymbol{R}^{2n} \longmapsto \left(\frac{1}{2} x_1^2, x_1, \dots, x_n, y_1, \dots, y_n\right) \in \boldsymbol{R}^{2n} \tag{3}$$

preserves the fibration π and satisfies the condition

$$\rho^* \omega = \sigma. \tag{4}$$

Obviously ρ is not a unique transformation with these properties. For example its composition with any Lagrangian equivalence of $(\boldsymbol{R}^{2n}, \omega, \pi)$ has the same properties.

PROPOSITION 2.1 *For any rL-equivalence Φ of $(\boldsymbol{R}^{2n}, \omega)$ there exists a σ-equivalence ϕ commuting the diagram*

$$\begin{array}{ccc} \boldsymbol{R}^{2n} & \xrightarrow{\Phi} & \boldsymbol{R}^{2n} \\ \uparrow \rho & & \uparrow \rho \\ \boldsymbol{R}^{2n} & \xrightarrow{\phi} & \boldsymbol{R}^{2n} \end{array}.$$

PROOF. For an *rL-equivalence Φ we have $\Phi(x, y) = (X_i(x), Y_i(x, y))$, where $X_1(x) = x_1(a + \alpha(x))$, $0 \neq a \in \boldsymbol{R}$ and $\alpha \in m_x^2$.* A diffeomorphism ϕ commuting diagram (5) and preserving fibration π, can be defined as follows:

$$\phi(x, y) \stackrel{\text{def}}{=} (x_1\sqrt{a + \alpha(\xi)}, X_2(\xi), \dots, X_n(\xi), \\ Y_1(\xi, y), \dots, Y_n(\xi, y))|_{\xi = \left(\frac{1}{2}x_1^2, x_2, \dots, x_n\right)}.$$

For such ϕ we have $\phi^* \sigma = \phi^* \rho^* \omega = \rho^* \Phi^* \omega = \rho^* \omega = \sigma$ (see (4)). Q.E.D.

REMARK 2.2 It is easily seen that transformation

$$\rho' : \boldsymbol{R}^{2n} \to \boldsymbol{R}^{2n}, \ (x, y) \longmapsto (x, x_1 y_1, y_2, \dots, y_n)$$

preserves the fibration π and satisfies (4). This raises the question whether a smooth mapping $h: \boldsymbol{R}^{2n} \to \boldsymbol{R}^{2n}$ such that $h^* \omega = \sigma$ must be equivalent to ρ or ρ' (it can be easily checked that this is the case in the space of two-jets of such mappings).

2.2. σ-varieties. Let $F(\lambda, \xi) \in C^\infty(\boldsymbol{R}^m \times \boldsymbol{R}^n)$, $(\lambda, \xi) \in \boldsymbol{R}^m \times \boldsymbol{R}^n$. We define *$\sigma$-variety, $V_F \in \boldsymbol{R}^{2n}$,* by the following equations

$$y = \frac{\partial F}{\partial \xi}(\lambda, \xi)|_{\xi = \left(\frac{1}{2}x_1^2, x_2, \dots, x_n\right)}, \tag{6}$$

$$0 = \frac{\partial F}{\partial \lambda}(\lambda, \xi)|_{\xi = \left(\frac{1}{2}x_1^2, x_2, \dots, x_n\right)}. \tag{7}$$

The local classification of σ-varieties up to σ-equivalences is the main objective of this paper.

It is convenient to associate with $(V_F, 0)$ a *Lagrangian variety (L-variety) of $(\boldsymbol{R}^{2n}, \omega)$*, $(L_F, 0)$, defined by the equations

$$y=\frac{\partial F}{\partial x}(\lambda, x),$$
$$0=\frac{\partial F}{\partial \lambda}(\lambda, x).$$

(Such L-varieties appeared naturally in Arnold's theory of singularities of systems of rays [4].) Obviously σ-variety $(V_F, 0)$ is a ρ pull-back of L-variety $(L_F, 0)$, i.e. $V_F=S^{-1}(L_F)$.

The germ $(F, 0)$, with F as above, will be called a *generating family* of $(V_F, 0)$ or of $(L_F, 0)$, respectively.

It is well known [2, 22] that if $(F, 0)$ is a Morse family, i.e.

$$\operatorname{rank}\left(\frac{\partial^2 F}{\partial\lambda\partial\lambda}, \frac{\partial^2 F}{\partial\lambda\partial x}\right)\Big|_0=\max=m,$$

then $(L_F, 0)$ is a Lagrangian submanifold of $(\boldsymbol{R}^{2n}, \omega)$. (*Lagrangian submanifold* is defined as an immersed submanifold $\iota: \boldsymbol{R}^n\to\boldsymbol{R}^{2b}$ such that $\iota^*\omega=0$; in such a case the germ $(L, 0)$, $L\overset{\text{def}}{=}\iota(\boldsymbol{R}^n)$, will be called an L-*germ.*) In the generic case, when the generating family F is a polynomial, the corresponding L-variety is stratifable with all strata isotropic (i.e. with vanishing pull-backs of ω on them) and maximal strata Lagrangian [8, 10].

Two generating families $(F_i, 0)$, $F_i(\lambda, x)\in C^\infty(\boldsymbol{R}^k\times\boldsymbol{R}^n)$, $i=1, 2$, are called *equivalent* if there exists a diffeomorphism

$$\Phi:\ (\boldsymbol{R}^k\times\boldsymbol{R}^n, 0)\to(\boldsymbol{R}^k\times\boldsymbol{R}^n, 0),\ (\lambda, x)\longmapsto(\Lambda(\lambda, x), X(x))$$

and a smooth function $f\in C^\infty(\boldsymbol{R}^n)$ such that

$$F_2(\Lambda(\lambda, x), X(x))=F_1(\lambda, x)+f(x) \qquad (8)$$

near $0\in\boldsymbol{R}^k\times\boldsymbol{R}^n$. The equivalence of generating families which preserves the hyperplane $\{x_1=0\}$ will be called *restricted* (*r-equivalence*). For r-equivalences the first coordinate of X, is divisible by x_1 i.e.

$$X_1(x)=x_1(\alpha+\phi(x)), \qquad (9)$$

where $\alpha=\text{const}\neq0$ and $\phi\in m(n)$. By straightforward calculation we obtain:

PROPOSITION 2.3 *Two L-varieties generated by r-equivalent generating families are rL-equivalent.*

REMARK 2.4 For Morse families and L-germs the converse is true. From [2, 23] it follows that any two L-equivalent L-germs have equivalent minimal Morse families (i. e. Morse families $F_i(\lambda, x)$ such that $\partial^2 F_i/\partial\lambda\partial\lambda|_0=0$).

Propositions 2.1 and 2.3 imply.

COROLLARY 2.5 Two σ-varieties generated by r-equivalent generating families are σ-equivalent.

2.3. Special case of σ-manifolds. In this subsection we discuss the interesting particular case when σ-variety is an n-submanifold. The following argument could be viewed as an additional justification for the 'naturality' of the above definition of σ-variety.

An immersed n-dimensional submanifold $M=\iota(\boldsymbol{R}^n)$ of $\boldsymbol{R}^{2n}$, where $\iota: \boldsymbol{R}^n \rightarrow \boldsymbol{R}^{2n}$ is a smooth immersion such that $\iota^*\sigma=0$, will be called a *σ-manifold.* We define the *symmetrisation of M* as follows

$$\mathrm{Sym}(M) \stackrel{\mathrm{def}}{=} \{(\pm x_1, x_2, \ldots, x_n, y)\,;\ (x, y) \in M\}.$$

The property of being a σ-manifold is obviously preserved by σ-equivalences. But symmetrisations of σ-equivalent σ-manifolds are not σ-equivalent in general.

EXAMPLE 2.6 σ-equivalence $(x, y) \longmapsto (x, y+x^3)$ of $(\boldsymbol{R}^2, x dx \wedge dy)$ carries σ-manifold $M_1 \stackrel{\mathrm{def}}{=} \{y=x^2\}$ onto the σ-manifold $M_2 \stackrel{\mathrm{def}}{=} \{y=x^2+x^3\}$. However, their symmetrisations, $(\mathrm{Sym}(M_1), 0)$ and $(\mathrm{Sym}(M_2), 0)$, are not σ-equivalent (see Fig. 1).

PROPOSITION 2.7 *Let $(M, 0)$ be a σ-manifold. Then there exist a σ-equivalence $\Phi: (\boldsymbol{R}^{2n}, \sigma) \rightarrow (\boldsymbol{R}^{2n}, \sigma)$ and a Morse family germ $(G, 0)$, $G(\lambda, x) \in C^\infty(\boldsymbol{R}^m \times \boldsymbol{R}^n)$ such that*

$$(\mathrm{Sym}(\Phi(M)), 0) = (V_G, 0),$$

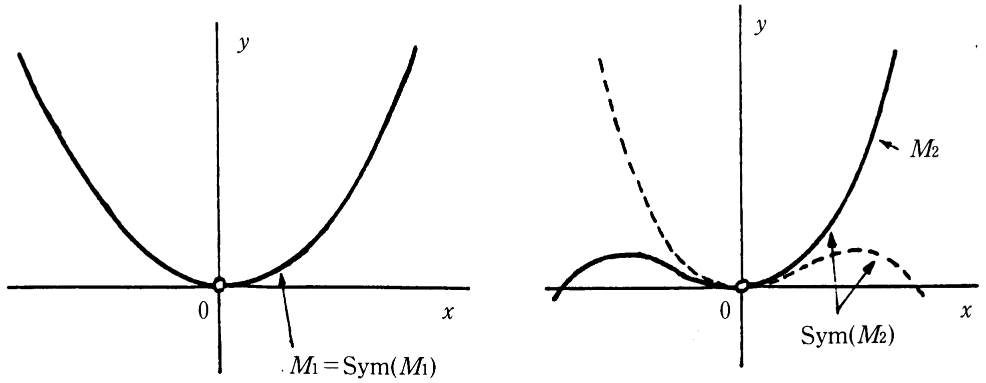

Figure 1. Sketches of two non-σ-equivalent symmetrisations of σ-equivalent σ-manifolds for Example 1.

where $V_G \subset \boldsymbol{R}^{2n}$ *is the* σ*-variety generated by* G (*see eqns*(6) *and* (7)).

PROOF. The proof is divided into few steps.

A. (Sym(M), 0) *in given at least by one of the following systems of equations*:

$$\frac{1}{2}x_1^2=\frac{\partial F}{\partial y_1}(y_1, x_I, y_J)$$
$$y_I=\frac{\partial F}{\partial x_I}(y_1, x_I, y_J) \tag{10}$$
$$-x_J=\frac{\partial F}{\partial y_J}(y_1, x_I, y_J)$$

or

$$x_1y_1=\frac{\partial F}{\partial x_1}(x_1, x_I, y_J)$$
$$y_I=\frac{\partial F}{\partial x_I}(x_1, x_I, y_J) \tag{11}$$
$$-x_J=\frac{\partial F}{\partial y_J}(x_1, x_I, y_J)$$

where F *is a germ of smooth function on* $\boldsymbol{R}^n$, $I \overset{\text{def}}{=} (i_1, \ldots, i_k)$, $J=(j_1, \ldots, j_{n-k-1})$ *and* $I\cup J=\{2, \ldots, n\}$.

PROOF A. A germ $(\iota, 0)$ of the immersion $\iota: \boldsymbol{R}^n \to \boldsymbol{R}^{2n}$, $M=\iota(\boldsymbol{R}^n)$, can be always written at least in one of the following two forms.

$$\iota: (x_I, y_1, y_J)\in \boldsymbol{R}^n \longmapsto (X_1(x_I, y_1, y_J), x_I, X_J(x_I, y_1, y_J), y_1, Y_I(x_I, y_1, y_J), y_J)\in \boldsymbol{R}^{2n}, \tag{12}$$

or

$$\iota: (x_1, x_I, y_J)\in \boldsymbol{R}^n \longmapsto (x_1, x_I, X_1(x_1, x_I, y_J), X_1(x_1, x_I, y_J), Y_I(x_1, x_I, y_J), y_J)\in \boldsymbol{R}^{2n}, \tag{13}$$

where $X_J: \boldsymbol{R}^n \to \boldsymbol{R}^{|J|}$, $Y_I: \boldsymbol{R}^n \to \boldsymbol{R}^{|I|}$ and Y_1, $X_1: \boldsymbol{R}^n \to \boldsymbol{R}$ are smooth germs ($I\cup J=\{2, \ldots, n\}$, $I\cap J=\emptyset$). In the case (12) the requirement $\iota^*\sigma=0$ yields the equations

$$X_1\frac{\partial X_1}{\partial x_i}-\frac{\partial Y_i}{\partial y_1}=0, \tag{14}$$
$$X_1\frac{\partial X_1}{\partial y_j}-\frac{\partial X_j}{\partial y_1}=0, \tag{15}$$
$$\frac{\partial X_i}{\partial x_{i'}}-\frac{\partial Y_{i'}}{\partial x_i}=0, \tag{16}$$

$$\frac{\partial Y_i}{\partial y_j}+\frac{\partial X_j}{\partial x_i}=0, \tag{17}$$
$$\frac{\partial Y_j}{\partial y_{j'}}-\frac{\partial X_{j'}}{\partial y_j}=0, \tag{18}$$

for any $i,\ i'\in I$ and $j,\ j'\in J$. On substituting $\tilde{X}_1=\frac{1}{2}X_1^2$ we find [2, 22] that there exists a smooth germ $F(x_I, y_1, y_J)$ *such that* $\tilde{X}_1=\frac{\partial F}{\partial y_1}$, $Y_I=\frac{\partial F}{\partial x_I}$ and $-X_J=\frac{\partial F}{\partial y_J}$. Representation (10) of $(\mathrm{Sym}(M), 0)$ follows immediately from these equations. Similarly in the case (13) condition $\iota^*\sigma=0$ implies equations (16)-(18) and the following two systems of equations:

$$x_1\frac{\partial Y_1}{\partial x_i}-\frac{\partial Y_i}{\partial x_1}=0,$$
$$x_1\frac{\partial Y_1}{\partial y_j}-\frac{\partial X_j}{\partial x_1}=0,$$

instead of (14) and (15). Inserting $\tilde{Y}_1=x_1Y_1$ we find (cf. [23]) a germ F (x_1, x_I, y_J) such that $\tilde{Y}_1=\frac{\partial F}{\partial x_1}$, $Y_I=\frac{\partial F}{\partial x_I}$ and $-X_J=\frac{\partial F}{\partial y_J}$. These equations yield representation (11) for $(\mathrm{Sym}(M), 0)$.

B. *If* $(\mathrm{Sym}(M), 0)$ *has a representation* (10), *then* $(\mathrm{Sym}(M), 0)=(V_G, 0)$, *where* $G(\lambda, x)\in C^\infty(\boldsymbol{R}^{n-k}\times\boldsymbol{R}^n)$ *is the following Morse family* (*on* $\boldsymbol{R}^n$):

$$G(\lambda_i, \dots, \lambda_{n-k}, x_1, \dots, x_n)\overset{\text{def}}{=}F(\lambda_1, x_{i_1}, \dots, x_{i_k}, \lambda_2, \dots, \lambda_{n-k})$$
$$+\lambda_1x_1+\sum_{\alpha=2}^{n-k}\lambda_\alpha x_{j_{\alpha-1}}.$$

This can be verified easily by straightforward computations.

C. $(M, 0)$ *is always σ-equivalent to a σ-manifold germ with symmetrisation of the formm* (10).

PROOF C. If $(\mathrm{Sym}(M), 0)$ does not allow the representation (10), then ι necessarily has a representation (13) with

$$\frac{\partial Y_1}{\partial x_1}(0)=0. \tag{19}$$

In this case $(\mathrm{Sym}(\Phi M), 0)$, where $\Phi: \boldsymbol{R}^{2n}\to\boldsymbol{R}^{2n}$ is the σ-equivalence $(x, y)\longmapsto(x, x+y)$, has a representation (10) (since any of its representations of the form (12) does not satisfy (19)).

This completes the proof of the Proposition. Q. E. D.

EXAMPLE 2.8 Let us assume $(\boldsymbol{R}^n, \sigma) \overset{\text{def}}{=} (\boldsymbol{R}^2, x dx \wedge dy)$.

(a) Let $M \subset \boldsymbol{R}^2$ be the parabola $\{(t^2, t)\}$. The sets $\mathrm{Sym}(M)$, $L \overset{\text{def}}{=} \rho(M)$ and $L' \overset{\text{def}}{=} \rho'(M)$, are sketched in Fig. 2(a). On the basis of Proposition 2.7 we easily calculate the generating function for L: $F(y) \overset{\text{def}}{=} \frac{1}{5} y^5$.

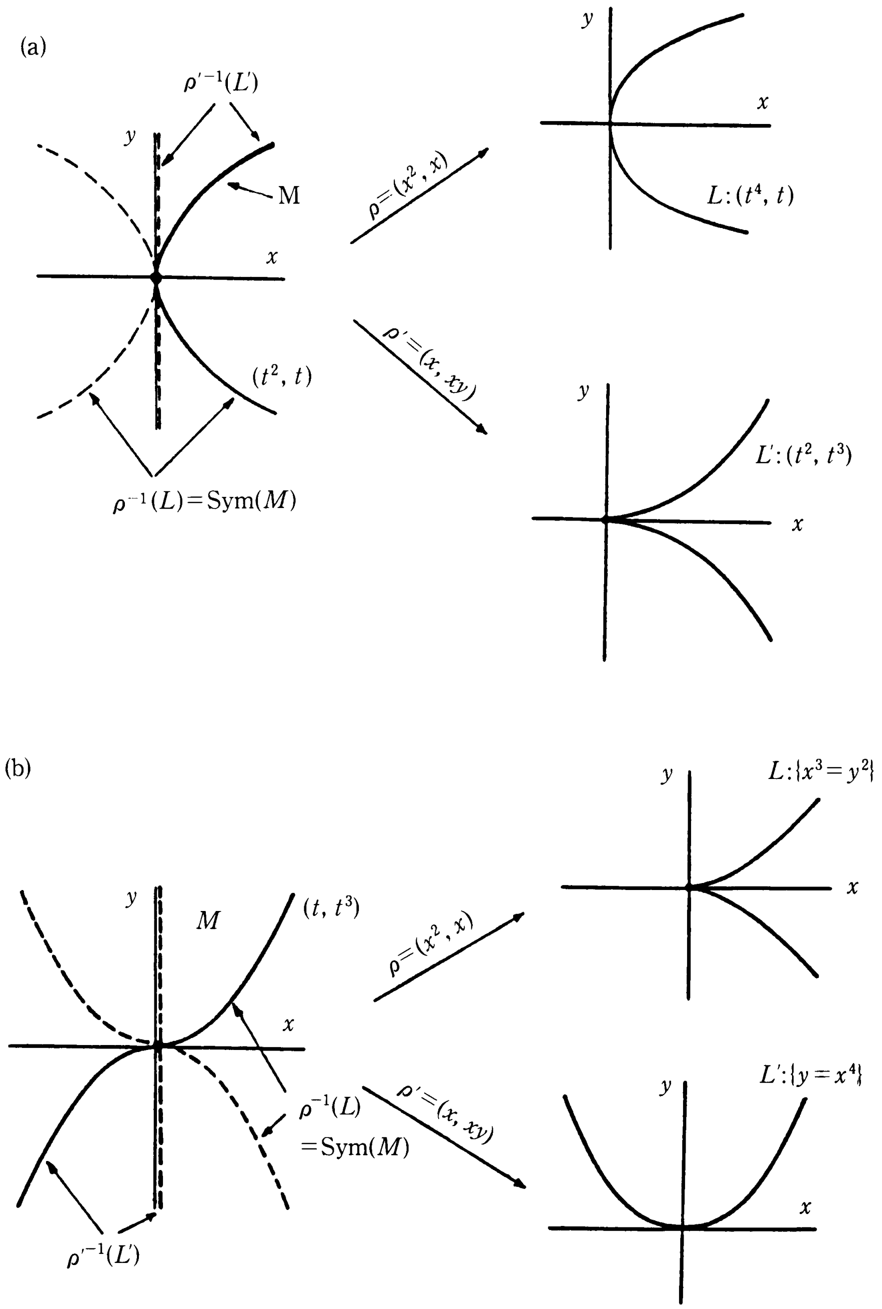

Figure 2. Sketches representing some basic features of geometry of σ-manifolds for Examples 2.8(a) and 2.8(b).

(b) For the germ $(\Sigma, 0)$ of the σ-manifold $\Sigma \overset{\text{def}}{=} \{(x, x^3)\}$ not having the representation (10), the set $L \overset{\text{def}}{=} \rho(\Sigma)$ is a non-smooth semmicubical parabola $x^3 = y^2$ (Fig. 2(b)). If we use ρ' instead of ρ, the set $L' \overset{\text{def}}{=} \rho'(\Sigma)$, becomes the smooth curve $y = x^4$ (Fig. 2(b)). This suggests that ρ' could be used like ρ to describe gems of σ-manifolds in terms of L-germs. However, not all σ-manifold germs are σ-equivalent to ones having smooth representations via ρ'(e.g. $(\Sigma, 0)$, $\Sigma = (y^2, y)$) and also there is a problem with lowering of L-equivalences to σ-equivalences through ρ'.

(c) In the particular case of a σ-manifold $(M, 0)$ satisfying the equation $(M, 0) = (\mathrm{Sym}(M), 0)$ and not having the representation (10) the image $L \overset{\text{def}}{=} \rho(M)$ is always a smooth manifold with boundary. For instance, for $M \overset{\text{def}}{=} \{y = x^2\} \subset \boldsymbol{R}^2$ *we have* $L \overset{\text{def}}{=} \{(x, x) : x \geq 0\}$ and $\rho^{-1}(L) = M$ (Fig. (a)). According to Proposition 2.7 we can deform M by a σ-equivalence to the σ-manifold $\widetilde{M} \longmapsto \{y = x^2 + \lambda x\}$ having the representation (10). In this case the set $\rho(\widetilde{M})$ becomes a smooth curve $\widetilde{L}$ obtained by splitting the half-line L (Fig. 3(b)).

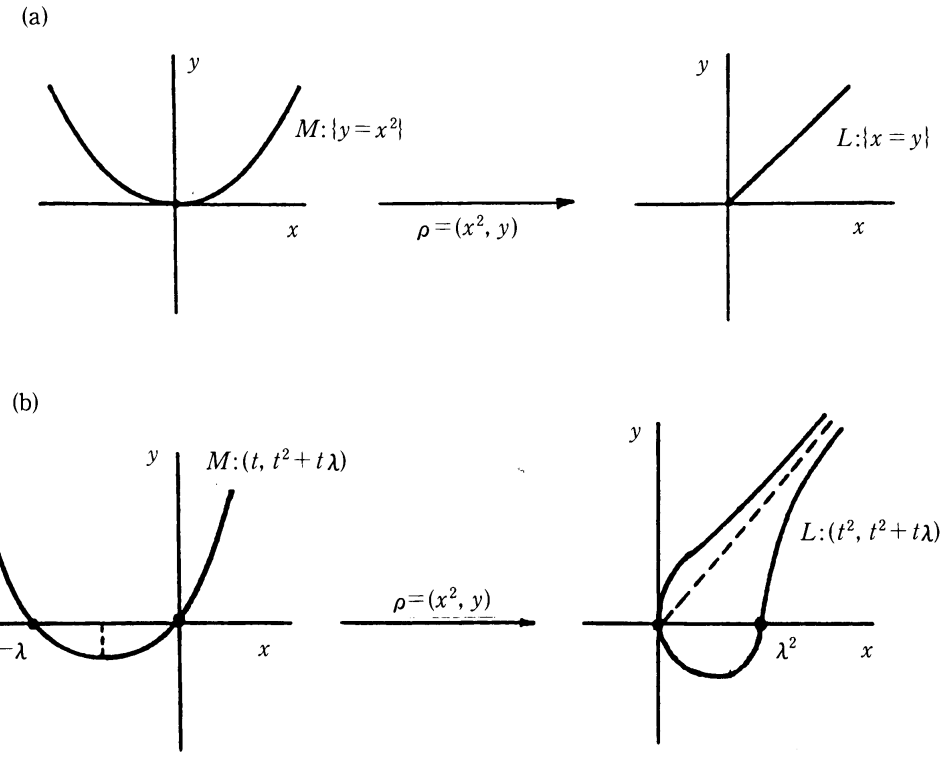

Figure 3. σ-manifolds for Example 2.8(c).

3. A classification of Lagrangian varieties

We recall [2, 7] that a generating family $(F(\lambda, x), 0)$, $(\lambda, x)\in \boldsymbol{R}^k\times \boldsymbol{R}^n$, is *versal* if any other generating family $(F'(\lambda, x'), 0)$, $(\lambda, x)\in \boldsymbol{R}^k\times \boldsymbol{R}^{n'}$, such that $F'|_{x'=0}=F|_{x=0}$ is induced from F, i.e. if there exists a mapping

$$(\lambda, x')\in \boldsymbol{R}^k\times \boldsymbol{R}^{n'}\longmapsto (\Lambda(\lambda, x'), X(x'))\in \boldsymbol{R}^k\times \boldsymbol{R}^n \tag{20}$$

and a function $f: \boldsymbol{R}^{n'}\rightarrow \boldsymbol{R}$ such that

$$F'(\lambda, x')=F(\Lambda(\lambda, x'), X(x'))+f(x').$$

(Classifications of versal families can be found in [2, 14, 17]).

For the purposes of this paper it seems natural to consider *restricted versality* by imposing on the inducing mappings (20) a requirement of preservation of distinguished hyperplanes, i.e. in the case of hyperplanes $\{x_1=0\}$ and $\{x_1'=0\}$, by assuming $X(\{x_1'=0\})\subset\{x_1=0\}$. This requirement means that x_1, the first coordinate of X, is of the form (9). The following result reduces the restricted versality to ordinary versality.

PROPOSITION 3.1 *A family $(F(\lambda, x), 0)$ is restricted versal if and only if the family $(F(\lambda, x)|_{x_1=0}, 0)$ is versal.*

PROOF. $\Leftarrow$. Assume $(F(\lambda, x)|_{x_1=0}, 0)$, $(\lambda, x)\in \boldsymbol{R}^k\times \boldsymbol{R}^n$, is a versal family and $(F'(\lambda, x'), 0)$, $(\lambda, x')\in \boldsymbol{R}^k\times \boldsymbol{R}^m$ is such that $F'(\lambda, 0)=F(\lambda, 0)$. Then $(\lambda, x')\longmapsto (\Lambda(\lambda, x'), 0, X_2(\lambda, x'), \ldots, X_n(\lambda, x'))$ is the demanded morphism.

$\Rightarrow$. Following the standard lines of versality theory [5, 20] for restricted versality we obtain the following necessary condition:

$$\left\langle\frac{\partial F}{\partial x}\right\rangle_{\mathscr{E}_{\lambda x}}+\left\langle x_1\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}, \ldots, \frac{\partial F}{\partial x_n}, 1\right\rangle_{\mathscr{E}_x}=\mathscr{E}_{\lambda x}.$$

Factorising by $m_x\mathscr{E}_{\lambda x}$ we get the following condition of infinitesimal versality for $F|_{x_1=0}$:

$$\left\langle\frac{\partial F}{\partial \lambda}\Big|_{x=0}\right\rangle_{\mathscr{E}_\lambda}+\left\langle\frac{\partial F}{\partial x_2}\Big|_{x=0}, \ldots, \frac{\partial F}{\partial x_n}\Big|_{x=0}, 1\right\rangle_{\boldsymbol{R}}=\mathscr{E}_\lambda.$$

As is well known this condition implies versality of $F|_{x_1=0}$ [2, 5, 13]. Q.E.D.

In the case when the vector space $\mathscr{E}_\lambda/\left\langle\frac{\partial F}{\partial \lambda}(\lambda, x)|_{x=0}\right\rangle_{\mathscr{E}_\lambda}$ has a finite number of generators, say $\{e_1(\lambda), \ldots, e_m(\lambda), 1\}$, we have the decomposition

$$F(\lambda, x)=F(\Lambda(\lambda, x), 0)+\sum_{i=1}^{m} e_i\circ\Lambda(\lambda, x)u_i(x)+f(x)$$

for some smooth $u=(u_1, \dots, u_m): \boldsymbol{R}^n\to\boldsymbol{R}^m$ and $f: \boldsymbol{R}^n\to\boldsymbol{R}$ [5, 20], where $\Lambda: \boldsymbol{R}^k\times\boldsymbol{R}^n\to\boldsymbol{R}^k$, $\Lambda|_{\boldsymbol{R}^k\times\{0\}}=id_{\boldsymbol{R}^k}$. From Proposition 3.1 we find that any other r-equivalent family $(F', 0)$ has the form

$$F'(\lambda, x)=F(\Lambda(\lambda, x), 0)+\sum_{i=1}^{m} e_i(\Lambda(\lambda, x))u_i'(x)+f(x),$$

where $\Lambda|_{\boldsymbol{R}^k\times\{0\}}$ is a diffeomorphism of $(\boldsymbol{R}^k, 0)$ and u' commutes the following diagram

$$\begin{array}{ccc} (\boldsymbol{R}^n, \{x_1=0\}, 0) & \xrightarrow{u} & (\boldsymbol{R}^m, 0) \\ \downarrow \phi & \nearrow u' & \\ (\boldsymbol{R}^n, \{x_1=0\}, 0) & & \end{array} \tag{21}$$

Here ϕ is a diffeomorphism preserving the hyperplane $\{x_1=0\}$. It is apparent that r-equivalence classes of generating families $(F(\lambda, x), 0)$ are parametrised by singularities of $F|_{x=0}$ and equivalence classes of mappings u in the sense of diagram (21) (we call them $\mathscr{A}_r$*-equivalences*). In this context it is natural to introduce the following characteristics of F: (*i*) *codimension of* $(F, 0)$, codim $F \overset{\text{def}}{=} \dim \left(\mathscr{E}_\lambda / \left\langle \frac{\partial F}{\partial \lambda}(\lambda, x)|_{x=0} \right\rangle_{\mathscr{E}_\lambda}\right)$ and (*ii*) *corank of* $F=m-rank\left(\frac{\partial \tilde{u}}{\partial x}\right)\Big|_{x=0}$, where $\tilde{u}: \boldsymbol{R}^n\to\boldsymbol{R}^m$ is assumed to be such that F is induced via a pull-back $(\tilde{\Lambda}, \tilde{u})$ from an universal unfolding $\tilde{F}$ of $F|_{x=0}$. It is easily seen that these two characteristics are invariants of r-equivalences.

REMARK 3.2 The above equivalence of generating families can be expressed also in a more general way. We call two generating families F and F', on $\boldsymbol{R}^k\times\boldsymbol{R}^n$, *equivalent* (also *pull back equivalent*) if they commute the following diagram

$$\begin{array}{ccccc} (\boldsymbol{R}^k\times\boldsymbol{R}^n, \{x_1=0\}, 0) & \searrow F & & \searrow \tilde{\Psi} & \\ \uparrow \Phi & & \boldsymbol{R} & \xleftarrow{\tilde{F}} & (\boldsymbol{R}^k\times\boldsymbol{R}^m, 0), \\ (\boldsymbol{R}^k\times\boldsymbol{R}^n, \{x_1=0\}, 0) & \nearrow F' & & \nearrow \Psi' & \end{array}$$

where $\tilde{F}$ is a universal unfolding of the germ $F|_{x=0}=F'|_{x=0}$ from which F and F' are induced by pull backs Ψ and Ψ', respectively, and Φ is a standard r-equivalence. The equivalence so defined is suitable for providing a classification list of normal forms of generating families and, at the end, of normal forms of σ-varieties.

Now using Arnold's classification methods [4] we obtain lists of normal forms for some simplest r-equivalence classes. At first we consider the case of codim=1. The cases of codim=2 and 3 will be considered subsequently in the remaining part of this section.

PROPOSITION 3.3 *The list of simple normal forms of r-equivalence classes of generating families* $F(\lambda, x)$, $(\lambda, x)\in \boldsymbol{R}\times\boldsymbol{R}^n$ *of codimension* 1 *is the following*:

$$A_2A_0^0:\ \lambda^3+x_2\lambda,$$
$$A_2A_k^0:\ \lambda^3+(\pm x_2^{k+1}\pm x_1+q)\lambda,\ k\geq 1,$$
$$A_2D_k^0:\ \lambda^3+(x_2x_3^2\pm x_2^{k-1}\pm x_1+q)\lambda,\ k\geq 4,$$
$$A_2E_6^0:\ \lambda^3+(x_2^3\pm x_3^4\pm x_1+q)\lambda,$$
$$A_2E_7^0:\ \lambda^3+(x_2^3+x_2x_3^3\pm x_1+q)\lambda,$$
$$A_2E_8^0:\ \lambda^3+(x_2^3+x_3^5\pm x_1+q)\lambda,$$
$$A_2B_k^1:\ \lambda^3+(\pm x_1^k+x_2^2+q)\lambda,\ k\geq 2,$$
$$A_2C_k^1:\ \lambda^3+(x_1x_2\pm x_2^k+q)\lambda,\ k\geq 2,$$
$$A_2F_4^1:\ \lambda^3+(\pm x_1^2+x_2^3+q)\lambda,$$

where q is a non-degenerate quadratic form of the remaining variables.

PROOF. Up to an r-equivalence we have

$$F(\lambda, x)=\lambda^3+\lambda u(x),$$

where u: $\boldsymbol{R}^n\rightarrow\boldsymbol{R}$. Using the list of simple normal forms of singularities of u on the manifold $\{x_1\geq 0\}\subset\boldsymbol{R}^n$ *with boundary* $\{x_1=0\}$ [2, Sec. 17.4] we obtain the above classification. Q. E. D.

REMARK 3.4 (*i*) In the above list $A_2A_0^0$ is the only restricted versal family.

(*ii*) Families $A_2A_k^0$, $A_2D_k^0$ and $A_2E_i^0$ are Morse families while $A_2B_k^1$, $A_2C^1{}_k$, $A_2F_4^1$ are not (and provide L-varieties which are not manifolds).

(*iii*) Generating families $(\tilde{F}(\lambda, x), 0)$, $(\lambda, x)\in\boldsymbol{R}^k\times\boldsymbol{R}^n$, $k\geq 2$ with $\tilde{F}|_{x=0}$ having singularity A_2 have simple normal forms $F(\lambda_1, x)+Q((\lambda_2, \dots, \lambda_k)$, where F has a one of the normal forms in the Proposition 3.3 and Q is a non-degenerate quadratic form. Obviously $\tilde{F}$ and F generate the same L-variety.

LEMMA 3.5 *In the spaces of mappings* $u=(u_i):(\boldsymbol{R}^n,0)\to(\boldsymbol{R}^m,0)$ *of rank* m *and* $m-1$, *respectively, the simplest singularities can be reduced by* $\mathscr{A}_r$*-equivalences to one of the following normal forms.*

(i) $\operatorname{rank}\left(\frac{\partial u}{\partial x}\right)\Big|_0=m.$

$A_0^0:\ u(x)=(x_2, x_3, \dots, x_{m+1}),$

or

$u=(u_i(x))=(x_2, \dots, x_j, u_j, x_{j+1}, \dots, x_m)$

where $j\in\{1,\dots,m\}$ *and* u_j *has one of the following forms*:

$$A_{k,j}^0:\ u_j=x_1\pm x_{m+1}^{k+1}\pm x_{m+1}^2+\textstyle\sum_{i=0}^{k-1} x_{m+1}^i\phi_i+q,$$
$$D_{k,j}^0:\ u_j=x_1+x_{m+1}x_{m+2}^2\pm x_{m+1}^{k-1}+\textstyle\sum_{i=0}^{k-2} x_{m+1}^i\phi_i+x_{m+2}\phi_{k-1}+q,$$
$$E_{6,j}^0:\ u_j=x_1+x_{m+1}^3\pm x_{m+2}^4+\phi_0+x_{m+1}\phi_1+x_{m+2}\phi_2+x_{m+1}x_{m+2}\phi_3+x_{m+2}^2\phi_4 +x_{m+1}x_{m+2}^2\phi_5+q,$$
$$E_{7,j}^0:\ u_j=x_1+x_{m+1}^3+x_{m+1}x_{m+1}x_{m+2}^3+\phi_0+x_{m+1}\phi_1+x_{m+2}\phi_2+x_{m+1}^2\phi_3 +x_{m+1}x_{m+2}\phi_4+x_{m+2}^2\phi_5+x_{m+1}^2x_{m+2}\phi_6+q,$$
$$E_{8,j}^0:\ u_j=x_1+x_{m+1}^3+x_{m+2}^5+\phi_0+x_{m+1}\phi_1+x_{m+2}\phi_2+x_{m+1}x_{m+2}\phi_3+x_{m+2}^2\phi_4 +x_{m+1}x_{m+2}^2\phi_5+x_{m+2}^3\phi_6+x_{m+1}x_{m+2}^2\phi_7+q,$$

where the ϕ_i*'s are smooth functions of* $x_2, \dots, x_m$ *and* q *is a non-degenerate quadratic form of the variables* $x_{m+3}, \dots, x_n$..

(ii) $\operatorname{rank}\left(\frac{\partial u}{\partial x}\right)\Big|_0=m-1.$

(ii.a) *For any* $j\in\{1,\dots,m\}$,

$u_i=x_i$ *for* $1\le i\ne j\le m$

and u_j *is one of the following forms*:

$$B_{k,j}^1:\ u_j=\pm x_1^k\pm x_{m+1}^2+\textstyle\sum_{i=0}^{k-1} x_1^i\phi_i+q,$$
$$C_{k,j}^1:\ u_j=x_1x_{m+1}\pm x_{m+1}^k+\textstyle\sum_{i=0}^{k-1} x_{m+1}^i\phi_i+q,$$
$$F_{4,j}^1:\ u_j=\pm x_1^2+x_{m+1}^3+\phi_0+x_1\phi_1+x_{m+1}\phi_2+x_1x_{m+1}\phi_3+q,$$

where $\phi_i=\phi_i(x_2,\dots,x_m)$ *and* q *is a non-degenerate quadratic form of the variables* $x_{m+2}, \dots, x_n$.

(ii.b) *For any* j, l, $1\le i\ne l\le m$,

$u_i=x_{i+1},\ u_{i'}=x_{i'},\ u_{i''}=x_{i''-1}$

for $1\le i<\min(j, l)<i'<\max(j, l)<i''\le n$,

$u_l\in \mathrm{m}_{x_1\cdots x_{m-1}}+\mathrm{m}_x^2$

and u_j *has one of the following forms*:

$$A^1_{k,jl}:\ u_j=x_1\pm x_m^{k+1}\pm x^2_{m+1}+\textstyle\sum_{i=0}^{k-1} x^i_m\phi_i+q,$$
$$D^1_{k,jl}:\ u_j=x_1+x_mx^2_{m+1}\pm x_m^{k-1}+\textstyle\sum_{i=0}^{k-2} x^i_m\phi_i+x_{m+1}\phi_{k-1}+q,$$
$$E^1_{6,jl}:\ u_j=x_1+x^3_m\pm x^4_{m+1}+\phi_0+x_m\phi_1+x_mx_{m+1}\phi_2+x^2_{m+1}\phi_3+x_mx^2_{m+1}\phi_4+q,$$
$$E^1_{7,jl}:\ u_j=x_1+x^3_m+x_mx^5_{m+1}+\phi_0+x_m\phi_1+x_{m+1}\phi_2+x^2_m\phi_3+x_mx_{m+1}\phi_4 +x^2_{m+1}\phi_5+x^2_mx_{m+1}\phi_6+q,$$
$$E^1_{8,jl}:\ u_j=x_1+x^3_m+x^5_{m+1}+\phi_0+x_m\phi_1+x_{m+1}\phi_2+x_mx_{m+1}\phi_3+x^2_{m+1}\phi_4 +x_mx^2_{m+1}\phi_5+x^3_{m+1}\phi_6+x_mx^2_{m+1}\phi_7+q,$$

where $\phi_i=\phi_i(x_2, \dots, x_{m-1})$ *and* q *is a non-degenerate quadratic form of the variables* $x_{m+2}, \dots, x_n$.

PROOF. Diffeomorphic changes of coordinates $X: \boldsymbol{R}^n\to\boldsymbol{R}^n$ preserving the hyperplane $\{x_1=0\}$ (we shall call them *permissible*) are of the form

$$X: x\to(x_+\widetilde{X}_1(x), X_2(x), \dots, X_n(x))$$

This class includes the transformation

$$x\to(x_1, x_{i_2}, \dots, x_{i_n}), \tag{22}$$

where $(i_2, \dots, i_n)$ is a permutation of indices $(2, \dots, n)$.

Now we consider four different classes of smooth transformations $u: \boldsymbol{R}^n\to\boldsymbol{R}^m$. The idea of the proof is to simplify, at first, as much as possible the form of the mapping u by permissible changes of coordinates and then to specify the forms of remaining functional coefficients with the help of the theory of universal unfoldings.

(*i. a*) $\operatorname{rank}\left(\frac{\partial u}{\partial x}\right)\Big|_{x=0}=m$ and $\operatorname{rank}\left(\frac{\partial u}{\partial(x_2, \dots, x_n)}\right)\Big|_{x=0}=m.$

Applying an appropriate transformation of coordinates (22) we can achieve that $\operatorname{rank}(\partial u/\partial(x_2, \dots, x_{m+1}))|_{x=0}=m$. Now in coordinates $x'_i=u_{i-1}(x)$ for $i=2, \dots, m+1$ and $x'_i=x_i$, otherwise, u has the form (A^0_0):

$$u(x')=(x'_2, \dots, x'_{m+1}).$$

(*i, b*) $\operatorname{rank}(\partial u/\partial x)|_{x=0}=m$ and $\operatorname{rank}(\partial u/\partial(x_2, \dots, x_n))|_{x=0}=m-1$.

After a suitable permutation of coordinates $x_2, \dots, x_n$ we have

$$\operatorname{rank}\left(\frac{\partial(u_1, \dots, u_{j-1}, u_{j+1}, \dots, u_m)}{\partial(x_2, \dots, x_m)}\right)\Big|_0=m-1$$

and

$$u_j=x_1(a_1+\alpha(x))+\beta(x_2, \dots, x_n),$$

for certain $j\in\{1, \dots, n\}$, $0\neq a_1\in\boldsymbol{R}$, $\alpha\in\mathrm{m}_x$ and $\beta\in\mathrm{m}_{x_2\cdots x_n}$. In coordinates $x'_1=x_1(a_1+\alpha(x))$, $x'_k=u_{k-1}$ for $k=2, \dots, j$ and $x'_k=u_k$ for $k=j+1, \dots, m$ transformation u takes the form

$$\begin{aligned}&u_k=x'_{k+1} \text{ for } k=i, \dots, j-1,\\ &u_j=x'_1+a_2x'_2+\dots+a_mx'_m+\beta'(x'_2, \dots, x'_m, x_{m+1}, \dots, x'_n),\\ &u_k=x'_k \text{ for } k=j+1, \dots, m,\end{aligned} \tag{22}$$

where $\beta'\in m^2_{x'_2\cdots x'_n}$. We can view β' as a family of functions of $x'_{m+1}, \dots, x'_n$ parametrised by $x'_2, \dots, x'_m$. In the simplest cases, by a permissible changes of coordinates not affecting $x'_1, \dots, x'_m$ we can obtain β' as a pull-back from standard universal unfoldings [2]. E. g. assuming that $\beta'|_{x'_2=,\dots,x'_m=0}$ has singularity (A_k) and after a suitable change of coordinates

$$x'\longmapsto\tilde{x}=(x'_1, \dots, x'_m, \psi_{m+1}(x'_{m+1}, \dots, x'_n), \dots, \psi_n(x_{m+1}, \dots, x'_n)),$$

we have

$$\beta'=\tilde{x}^{k+1}_{m+1}+\sum_{i=0}^{k-1}x^i_{m+1}\phi_i+q,$$

where $\phi_i\in\mathrm{m}_{x'_2\cdots x'_m}$ and $q=q(x'_{m+2}, \dots, x'_n)$ is a non-degenerate quadratic form. This provides the normal forms A^0_k for u_i (note that the linear term in u_j was included in ϕ_0). Analogously we obtain forms D^0_k, E^0_6, E^0_7 and E^0_8.

(*ii. a*) $\operatorname{rank}\left(\frac{\partial u}{\partial x}\right)\Big|_{x=0}=m-1$ and $\operatorname{rank}\left(\frac{\partial u}{\partial(x_2, \dots, x_n)}\right)\Big|_{x=0}=m-1.$

Analogous to the previous case we find at first, that up to a suitable permissible change of coordinates we have

$$\begin{aligned}&u_1=x_2, \dots, u_{j-1}=x_j,\\ &u_j=a_2x_2+\dots+a_mx_m+\beta(x_1, x_2, \dots, x_m, x_{m+1}, \dots, x_n),\\ &u_{j+1}=x_{j+1}, \dots, u_m=x_m,\end{aligned}$$

where $\beta\in\mathrm{m}^2_x$. We can treat β as an unfolding of a boundary singularity $\tilde{\beta}=\beta|_{x_2=0,\dots,x_m=0}$, with respect to unfolding parameters $x_2, \dots, x_m$. The simplest normal forms of $\tilde{\beta}\in\mathrm{m}^2(x_1, x_{m+1}, \dots, x_n)$, B_k, C_k, and F_4 can be found in [4]. Forms B^1_k, C^1_k and F^1_4 are obtained as unfoldings of these normal forms (and inclusion of the linear term in u_j into ϕ_0).

(*ii. b*) It remains to consider the case:

$$\operatorname{rank}\left(\frac{\partial u}{\partial x}\right)=m-1 \text{ and } \operatorname{rank}\left(\frac{\partial u_1}{\partial(x_2, \dots, x_n)}\right)=m-2$$

at $x=0$. As previously we have, up to a permissible change of coordinates

$$u_i=x_{i+1},\ u_{i'}=x_{i'} \text{ and } u_{i''}=x_{i''-1},$$
$$u_j=a_1x_1+a_2x_2+\ldots+a_mx_m+\alpha(x_2, \ldots, x_m, x_{m+1}, \ldots, x_n),$$
$$u_l=b_1x_1+b_2x_2+\ldots+a_mx_m+\beta(x_2, \ldots, x_m, x_{m+1}, \ldots, x_n),$$

for certain j, k, $1\leq j\neq k\leq n$, α, $\beta\in m_x^2$ and all i, i', i'' such that $1\leq i<\min(j, l)<i'<\max(j, l)<i''<n$. Using a permissible change of coordinates we can simplify one of the functions, say u_j while the form of the other one must remain 'arbitrary'. Using Arnold's list [2] we obtain normal forms A_{kjl}^1, D_{kjl}^1 and E_{ijl}^1 (by virtually specifying u_j: note that $\left.\frac{\partial u_j}{\partial x_1}\right|_0\neq 0$). Q. E. D.

On the basis of Proposition 3.1 and Lemma 3.5 we extend the classification of generating families in Proposition 3.3 to the case of codimension 2 and 3. It is convenient to define the *corank of a generating family* $F(\lambda, x)$, as the corank (at 0) of a pull-back $(\lambda, x)\longmapsto(\Lambda(\lambda, x), X(x))$ inducing F from a universal unfolding of $F|_{x=0}$. Obviously it is an invariant of the r-equivalence class of F.

PROPOSITION 3.6 *Normal forms of corank* 0 *and* 1 *of r-equivalence classes of generating families of codimension* 2 *and* 3 *are listed in Table* 3.

4. Mormal forms of σ-varieties

On the basis of Corollary 2.5 and of the results of Section 3 we obtain the following Theorem.

THEOREM 4.1 *Initial classification of generic σ-varieties is provided by the classification list of generating families in Propositions* 3.3 *and* 3.6.

EXAMPLE 4.2 Restricted versal generating families are of type $A_kA_0^0$, $k\geq 2$, only. Their normal forms are as follows

$$F(\lambda, x)=\lambda^{k+1}+\lambda^{k-1}x_2+\lambda^{k-2}x_3+\ldots+\lambda x_k.$$

In this case the corresponding σ-varieties are given by the equations

$$y_1=0,$$
$$y_i=\lambda^{k-i+1},\ i=2, \ldots, k$$
$$y_j=0,\ j=k+1, \ldots, n$$
$$0=(k+1)\lambda^k+\sum_{i=l}^{k}(k-i+1)\lambda^{k-i}x_i.$$

On Fig. 4 we illustrate the σ-variety for the case $A_2A_0^0$; which is the only possible restrictly versal family for $n=2$. For $n=3$ (and $k\leq 3$) we obtain additionally the cylinder of cusp-surfaces along the axis x_1.

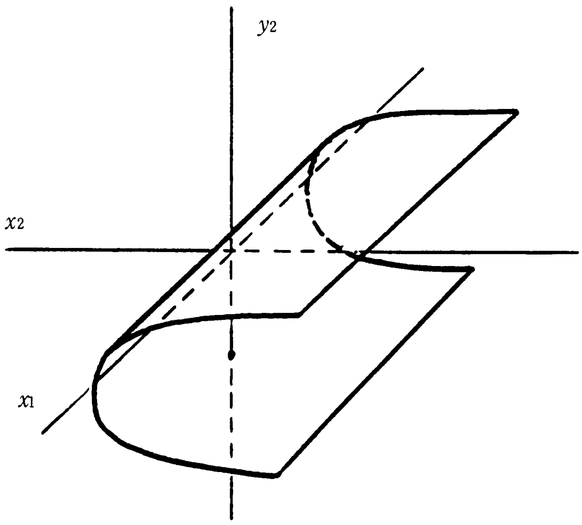

Figure 4. σ-variety for the case $A_2A_0^0$ (Example 4. 2).

EXAMPLE 4. 3 Singularities $A_2A_k^0$, $A_2D_k^0$, $A_2E_k^0$ (see Proposition 3. 3) provide the singular σ-varieties. The simplest, cone-like σ-variety for $A_2A_1^0$ singularity is illustrated in Fig. 5.

$$y_2=\pm 2x_2y_1,$$
$$0=3y_1^2\pm x_2^2\pm\frac{1}{2}x_1^2.$$

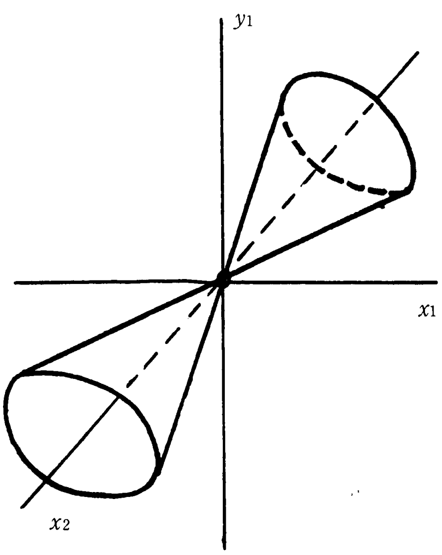

Figure 5. σ-variety for $A_2A_1^0$ singularity for Example 4. 3.

EXAMPLE 4.4 The types $A_2B^1_k$, $A_2C^1_k$, $A_2F^1_4$ of σ-varieties are provided by generating families which are not Morse. As an example we write down, explicitly, the equations of the normal forms of the σ-varieties corresponding to a singularity of type $A_2F^1_4$.

Table 1: Initial list of normal forms for r-equivalence classes of generating families

codim	corank	Type	Normal forms $F=F(\lambda, x)$	Conditions $1\leq j\neq l\leq$codim $u=(u_i(x_1, \dots, x_n))$
	0	$A_3A^0_0$	$\lambda^4+\lambda^2x_2+\lambda x_3$	
		$A_3A^0_{k,j}$		$(u_1, u_2)\in A^0_{k,j}$, $k\geq 1$
	0	$A_3D^0_{k,j}$	$\lambda^4+\lambda^2u_1+\lambda u_2$	$(u_1, u_2)\in D^0_{k,j}$, $k\geq 4$
		$A_3E^0_{k,j}$		$(u_1, u_2)\in E^0_{k,j}$, $k=6, 7, 8$
2		$A_3A^1_{k,j}$		$(u_1, u_2)\in A^1_{k,j}$, $k\geq 1$
		$A_3D^1_{k,j}$		$(u_1, u_2)\in A^1_{k,j}$, $k\geq 4$
	1	$A_3E^1_{k,j}$	$\lambda^4+\lambda^2u_1+\lambda u_2$	$(u_1, u_2)\in E^1_{k,j}$, $k=6, 7, 8$
		$A_3B^1_{k,j}$		$(u_1, u_2)\in B^1_{k,j}$, $k\geq 2$
		$A_3C^1_{k,j}$		$(u_1, u_2)\in C^1_{k,j}$, $k\geq 2$
		$A_3F^1_{4,j}$		$(u_1, u_2)\in F^1_{4,j}$
		$A_4A^0_{k,j}$		$(u_1, u_2, u_3)\in A^0_{k,j}$, $k\geq 0$
		$A_4D^0_{k,j}$	$\lambda^5+\lambda^3u_1+\lambda^2u_2+\lambda u_3$	$(u_1, u_2, u_3)\in D^0_{k,j}$, $k\geq 4$
	0	$A_4E^0_{k,j}$		$(u_1, u_2, u_3)\in E^0_{k,j}$, $k=6, 7, 8$
		$D^\pm_4A^0_{k,j}$		$(u_1, u_2, u_3)\in A^0_{k,j}$, $k\geq 0$
		$D^\pm_4D^0_{k,j}$	$\lambda_1^2\lambda_2+\lambda_2^3+\lambda_2^2u_1+\lambda_1u_2+\lambda_2u_3$	$(u_1, u_2, u_3)\in D^0_{k,j}$, $k\geq 4$
		$D^\pm_4E^0_{k,j}$		$(u_1, u_2, u_3)\in E^0_{k,j}$, $k=6, 7, 8$
		$A_4B^1_{k,j}$		$(u_1, u_2, u_3)\in B^1_{k,j}$, $k\geq 2$
		$A_4C^1_{k,j}$	$\lambda^5+\lambda^3u_1+\lambda^2u_2+\lambda u_3$	$(u_1, u_2, u_3)\in C^1_{k,j}$, $k\geq 2$
		$A_4F^1_{4,j}$		$(u_1, u_2, u_3)\in F^1_{4,j}$
3		$D^\pm_4B^1_{k,j}$		$(u_1, u_2, u_3)\in B^1_{k,j}$, $k\geq 2$
		$D^\pm_4C^1_{k,j}$	$\lambda_1^2\lambda_2+\lambda_2^3+\lambda_2^2u_1+\lambda_1u_2+\lambda_2u_3$	$(u_1, u_2, u_3)\in C^1_{k,j}$, $k\geq 2$
		$D^\pm_4F^1_{4,j}$		$(u_1, u_2, u_3)\in F^1_{4,j}$
	0	$A_4A^1_{k,j}$		$(u_1, u_2, u_3)\in A^1_{k,j}$, $k\geq 1$
		$A_4D^1_{k,j}$	$\lambda^5+\lambda^3u_1+\lambda^2u_2+\lambda u_3$	$(u_1, u_2, u_3)\in D^1_{k,j}$, $k\geq 4$
		$A_4E^1_{k,j}$		$(u_1, u_2, u_3)\in E^1_{k,j}$, $k=6, 7, 8$
		$D^\pm_4A^1_{k,jl}$		$(u_1, u_2, u_3)\in A^1_{k,jl}$, $k\geq 1$
		$D^\pm_4D^1_{k,jl}$	$\lambda_1^2\lambda_2+\lambda_2^3+\lambda_2^2u_1+\lambda_1u_2+\lambda_2u_3$	$(u_1, u_2, u_3)\in D^1_{k,jl}$, $k\geq 4$
		$D^\pm_4E^1_{k,jl}$		$(u_1, u_2, u_3)\in E^1_{k,jl}$, $k=6, 7, 8$

$$y_1x_4=\pm y_4x_1^2$$
$$y_2x_4=\pm 3y_4x_2^2,$$
$$y_ix_4=\pm y_4x_i,\ 3\leq i\leq n-1,$$
$$y_4^2=\frac{1}{3}x_4^2(x_2^3+q\pm\frac{1}{4}x_1^4).$$

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