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Nevanlinna and Smirnov classes on the upper half plane

Nozomu MOCHIZUKI
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1. Introduction and notations

The Nevanlinna and Smirnov classes defined on the unit disk U in $\boldsymbol{C}$ will be denoted by $N(U)$ and $N_*(U)$, respectively. In this paper, we shall define the Nevanlinna class, $N_0(D)$, and the Smirnov class, $N_*(D)$, on $D:=\{z\in\boldsymbol{C}|\mathrm{Im}\, z>0\}$. Yanagihara and Nakamura [10, 8. (3)] posed a problem to introduce the Smirnov class on D; our treatment will be an answer. We let $N_0(D)$ consist of all holomorphic functions f on D such that

$$d(f,0):=\sup_{y>0}\int_{\boldsymbol{R}}\log\ (1+|f(x+iy)|)dx<+\infty,$$

and we let $N_*(D)$ consist of f such that $\log\ (1+|f(z)|)\leqq P[\phi](z)\ (z\in D)$ for some $\phi\in L^1(\boldsymbol{R})$, $\phi\geqq 0$, where the right side means the Poisson integral. $N_0(D)$ is an algebra over $\boldsymbol{C}$ and $N_*(D)$ is its subalgebra. First we prove a factorization theorem for functions in $N_0(D)$, as Krylov [4] does for functions in the class $\mathfrak{N}$. $\mathfrak{N}$ is defined by L^1-boundedness of $\log^+|f(x+iy)|$ and, since $1\in\mathfrak{N}$ and $2\notin\mathfrak{N}$, this is not a vector space. $N(U)$ and $N_*(U)$ have remarkable topological properties, as shown by Shapiro and Shields [6] and Roberts [5]. We shall show that our classes have very similar properties. On the other hand, it will be proved that $N_0(D)$ and $N_*(D)$ cannot be linearly isometric to $N(U)$ and $N_*(U)$, respectively, in contrast to the fact that $H^p(D)$ are linearly isometric to $H^p(U)$ for all p, $0<p\leqq+\infty$.

We denote by σ the normalized Lebesgue measure on T, the unit circle in $\boldsymbol{C}$. Let $\Psi(z)=(z-i)(z+i)^{-1}\ (z\in\bar{D})$. Let ν be a real measure on T. Then there corresponds a finite real measure μ on $\boldsymbol{R}$ such that

$$\int_{\boldsymbol{R}}h(t)d\mu(t)=\int_{T^*}(h\circ\Psi^{-1})(\eta)d\nu(\eta)\quad(h\in C_c(\boldsymbol{R})),$$

where $T^*=T\setminus\{1\}$. Denoting the kernel $(\eta+w)(\eta-w)^{-1}$ by $H(w,\eta)$ $((w,\eta)\in U\times T)$, we can write

$$(1.1)\qquad \frac{1}{i}\int_{\boldsymbol{R}}\frac{1+tz}{t-z}d\mu(t)=\int_{T^*}H(\Psi(z),\eta)d\nu(\eta)$$
$$=\int_{T}H(\Psi(z),\eta)d\nu(\eta)-iaz\quad(z\in D),$$

where $\alpha=-\nu(\{1\})$. We write Poisson integrals as follows:

$$P[\mu](z)=\int_{\boldsymbol{R}}\frac{1}{\pi}\frac{y}{(x-t)^2+y^2}\mathrm{d}\mu(t)\quad(z=x+iy\in D),$$
$$Q[\nu](w)=\int_{T}\frac{1-|w|^2}{|\eta-w|^2}d\nu(\eta)\quad(w\in U).$$

Taking the real parts in (1.1), we have

$$(1.2)\qquad P[\pi(1+t^2)d\mu(t)](z)=Q[\nu](\Psi(z))+\alpha\cdot\mathrm{Im}\,z\quad(z\in D).$$

2. A factorization theorem

Let $f\in N_0(D)$. Then the subharmonic function $\log(1+|f|)$ has the following properties, by [7, Chap. II, Theorem 4.6]:
(A) $\log(1+|f|)$ has the least harmonic majorant $P[\tau]$, where τ is a finite real measure on $\boldsymbol{R}$.
(B) $\|\tau\|\leqq d(f,0)$.
(C) $\sup\{\log(1+|f(z)|)|z\in\bar{D}_\delta\}\leqq A_0\delta^{-1}d(f,0)$, with a constant A_0 independent of f and δ, where $D_\delta=\{z\in\boldsymbol{C}|\mathrm{Im}\,z>\delta\}$, $\delta>0$.
(D) $\log(1+|f(z)|)\to 0$ as $|z|\to+\infty$ $(z\in\bar{D}_\delta)$, for each $\delta>0$.

The property (A) implies that $f\circ\Psi^{-1}\in N(U)$, hence the nontangential limit $f^*(x)$ exists for a.e. $x\in\boldsymbol{R}$. Since $\log|(f\circ\Psi^{-1})^*|\in L^1(T)$, we have $\log|f^*|\in L^1(\boldsymbol{R},(1+t^2)^{-1}dt)$, and Fatou's lemma shows that $\log(1+|f^*|)\in L^1(\boldsymbol{R})$. Let $d(f,g)=d(f-g,0)$ for $f,g\in N_0(D)$. Then $(N_0(D),d)$ is seen to be a complete metric space.

THEOREM 2.1. *Let $f\in N_0(D)$, $f\neq 0$. Then f is expressed, uniquely, in the form*

$$(2.1)\qquad f(z)=ae^{i\alpha z}b(z)d(z)g(z)\quad(z\in D),$$

with the following properties.
(i) $a\in\boldsymbol{C}$, $|a|=1$; $\alpha\in\boldsymbol{R}$, $\alpha\geqq 0$.
(ii) *$b(z)$ is the Blaschke product formed from the zeros of f.*
(iii) $d(z)=\exp\left(\frac{1}{\pi i}\int_{\boldsymbol{R}}\frac{1+tz}{t-z}\frac{1}{1+t^2}\log h(t)\,dt\right)$,

where $h(t)\geqq 0$, $\log h\in L^1(\boldsymbol{R},(1+t^2)^{-1}dt)$, and $\log(1+h)\in L^1(\boldsymbol{R})$.

(iv) $$g(z)=\exp\left(\frac{1}{i}\int_{R}\frac{1+tz}{t-z}d\mu(t)\right),$$

where μ is a finite real measure on $\boldsymbol{R}$, singular with respect to Lebesgue measure, and such that

$$\int_{R}(1+t^{2})d\mu^{+}(t)<+\infty.$$

If f is expressed in the form (2.1), *then $f\in N_{0}(D)$.*

PROOF. Suppose $f\in N_{0}(D)$, $f\neq 0$. Then the canonical factorization theorem for $N(U)$ shows that $(f\circ\Psi^{-1})(w)=aB(w)F(w)S_{1}(w)S_{2}(w)^{-1}$ ($w\in U$), where $a\in\boldsymbol{C}$ with $|a|=1$, $B(w)$ is the Blaschke product formed from the zeros of $f\circ\Psi^{-1}$, and

$$F(w)=\exp\left(\int_{T}H(w,\eta)\log|(f\circ\Psi^{-1})^{*}(\eta)|d\sigma(\eta)\right),$$
$$S_{j}(w)=\exp\left(-\int_{T}H(w,\eta)d\nu_{j}(\eta)\right)\quad (j=1,2).$$

Here ν_{j} are positive singular measures on T and, moreover, mutually singular. In the factorization $f(z)=aB(\Psi(z))F(\Psi(z))S_{1}(\Psi(z))S_{2}(\Psi(z))^{-1}$, $b(z):=B(\Psi(z))$ is the Blaschke product with respect to the zeros of f, and the change of variables $\eta=\Psi(t)$ ($t\in\boldsymbol{R}$) shows that $d(z):=F(\Psi(z))$ is of the form (iii). Let μ be the singular measure on $\boldsymbol{R}$ corresponding to $\nu_{2}-\nu_{1}$. Then, by (1.1), we can write $S_{1}(\Psi(z))S_{2}(\Psi(z))^{-1}=g(z)e^{i\alpha z}$, where g is of the form (iv). Since f belongs to the class $\mathfrak{N}$ of Krylov, [4, Theorem XVII] implies that $\alpha\geqq 0$ and that $(1+t^{2})d\mu^{+}(t)$ is a finite measure. Suppose, conversely, that f is expressed in the form (2.1). Then

$$|f(z)|=|e^{i\alpha z}||b(z)|\exp(P[\log\, h+\pi(1+t^{2})d\mu(t)](z)).$$

Let ν_{0} be the measure on T concentrated on $\{1\}$ and $\nu_{0}(\{1\})=-\alpha$. Then, by letting $z=\Psi^{-1}(w)$, we have $|\exp(i\alpha z)|=\exp(Q[\nu_{0}](w))$. μ determines a singular measure ν on T^{*}, for which we have $P[\pi(1+t^{2})d\mu(t)](z)=Q[\nu](w)$, by (1.2). Thus,

$$|(f\circ\Psi^{-1})(w)|=|B(w)|\exp(Q[\log(h\circ\Psi^{-1})+\nu+\nu_{0}](w))\quad (w\in U).$$

From $\log^{+}|(f\circ\Psi^{-1})(w)|\leqq Q[\log^{+}(h\circ\Psi^{-1})+\nu^{+}](w)$, we see that $f\circ\Psi^{-1}\in N(U)$ and, letting $y\rightarrow 0^{+}$ in $|f(x+iy)|$, we have $|f^{*}(x)|=h(x)$ for a.e. $x\in\boldsymbol{R}$. Hence, by the canonical factorization of $f\circ\Psi^{-1}$, we get

$$|(f\circ\Psi^{-1})(w)|=|B(w)|\exp(Q[\log(h\circ\Psi^{-1})+\nu_{2}-\nu_{1}](w))\quad (w\in U).$$

These two expressions of $|f\circ\Psi^{-1}|$ show that $\nu^{+}-(\nu^{-}-\nu_{0})=\nu_{2}-\nu_{1}$ and this implies that $\nu_2=\nu^+$. Now $\log(1+|f\circ\Psi^{-1}|)$ has the least harmonic majorant $v'=Q[\log(1+|(f\circ\Psi^{-1})^*|)+\nu_2]$, as shown in [6, Proof of Theorem 3.1]. The function $v:=v'\circ\Psi$ is the least harmonic majorant of $\log(1+|f|)$, and we have

$$v=P[\log(1+|f^*|)+\pi(1+t^2)d\mu^+(t)]. \tag{2.2}$$

This shows that $f\in N_0(D)$. Finally, suppose that f has another factorization of the form (2.1). Then

$$|(f\circ\Psi^{-1})(w)|=|B(w)|\exp(Q[\log(h\circ\Psi^{-1})+\nu'+\nu_0'](w)) \quad (w\in U)$$

for some ν' and ν_0', and it follows that $\nu=\nu'$ and $\nu_0=\nu_0'$.

COROLLARY 2.2. *Let $f\in N_0(D)$, $f\neq 0$. Then*

$$\begin{aligned} d(f,0)&=\lim_{y\to 0^+}\int_R \log(1+|f(x+iy)|)dx \\ &=\int_R \log(1+|f^*(x)|)dx+\int_R \pi(1+x^2)d\mu^+(x). \end{aligned}$$

PROOF. The integral of $\log(1+|f(x+iy)|)$ is increasing as $y\to 0^+$, as seen from the property (D) and [1, Theorem 1]. Thus the first equality holds. In the second equality, the left side does not exceed the right, by (2.2), Finally, the property (B) implies that

$$\|\log(1+|f^*(x)|dx+\pi(1+x^2)d\mu^+(x)\|\leqq d(f,0).$$

COROLLARY 2.3. *Let $f\in N_0(D)$, $f\neq 0$. Then*

$$\lim_{c\to 0} d(cf,0)=\int_R \pi(1+x^2)d\mu^+(x).$$

PROOF. If $c\in \mathbf{C}$, $c\neq 0$, the measure occurring in the factorization (2.1) of cf is the same as that of f, hence Corollary 2.2 shows this.

COROLLARY 2.4. *Let $f\in N_0(D)$, $f\neq 0$. Then the following are mutually equivalent.* (i) $f\in N_*(D)$. (ii) $f\circ\Psi^{-1}\in N_*(U)$. (iii) $\mu\leqq 0$, *in the factorization* (2.1).

$$\text{(iv)} \lim_{y\to 0^+}\int_R \log(1+|f(x+iy)|dx=\int_R \log(1+|f^*(x)|)dx.$$

PROOF. If $f\in N_*(D)$, then $\log(1+|(f\circ\Psi^{-1})(w)|)\leqq Q[\phi\circ\Psi^{-1}](w)$ $(w\in U)$, where $\phi\circ\Psi^{-1}\in L^1(T)$. Hence $f\circ\Psi^{-1}\in N_*(U)$. Suppose $f\circ\Psi^{-1}\in N_*(U)$. Then $\nu_2=0$ in the factorization of $f\circ\Psi^{-1}$, hence $\nu^+=0$, as in the proof of Theorem 2. 1. This implies that $\mu^+=0$. If $\mu\leqq 0$, then $f\in N_*(D)$ by (2.2). Finally, Corollary 2. 2 shows that part (iii) is equivalent to part (iv).

COROLLARY 2.5. *The space $(N_*(D), d)$ is an F-algebra, that is, a complete linear metric space with multiplication continuous. Moreover, $N_*(D)$ is the largest topological linear subspace of $N_0(D)$.*

PROOF. First, we show the completeness. Difine an operator A by $Af=f\circ\Psi^{-1}$ for holomorphic functions f on D. Clearly, $A(N_0(D))\subset N(U)$. Let

$$\rho(g, 0)=\sup_{0<r<1}\int_T \log(1+|g(r\eta)|)d\sigma(\eta) \quad (g\in N(U)).$$

Then, as in Corollary 2.2, we can see that

$$\rho(g, 0)=\int_T \log(1+|g^*(\eta)|)d\sigma(\eta)+\nu_2(T),$$

if $g\neq 0$. Hence, for $f\in N_0(D)$, we have

$$\rho(Af, 0)=\int_{\boldsymbol{R}}\log(1+|f^*(x)|)\pi^{-1}(1+x^2)^{-1}dx+\mu^+(\boldsymbol{R})\leqq d(f, 0).$$

Thus $A: N_0(D)\to N(U)$ is continuous. Since $N_*(D)=N_0(D)\cap A^{-1}(N_*(U))$ by part (ii) of Corollary 2.4, $N_*(D)$ is complete. The second statement is an easy consequence of Corollary 2.3 and part (iii) of Corollary 2.4.

3. Topological properties

For $\bar{f}=f+N_*(D)$, $\bar{g}=g+N_*(D)\in N_0(D)/N_*(D)$, the metric $\bar{d}$ is defined by letting $\bar{d}(\bar{f}, \bar{g})=\inf\{d(f-g, h)|h\in N_*(D)\}$.

THEOREM 3.1. (i) *$N_0(D)$ is disconnected.* (ii) *Every finite dimensional linear subspace of $N_0(D)/N_*(D)$ has the discrete topology.* (iii) *$N_0(D)/N_*(D)$ is not discrete.*

PROOF. (i) For $x\in\boldsymbol{R}$, we define a functional λ_x by

$$\lambda_x(f)=\limsup_{y\to 0^+} y\log^+|f(x+iy)| \quad (f\in N_0(D)).$$

The existence of $\lambda_x(f)$ is guaranteed by the property (C). λ_x is subadditive and continuous. Moreover, if $\lambda_x(f)<\lambda_x(g)$, then $\lambda_x(f+g)=\lambda_x(g)$. From these properties it can be proved that f and g belong to different components of $N_0(D)$, whenever $\lambda_x(f)\neq\lambda_x(g)$ for some $x\in\boldsymbol{R}$, just as in [6, Theorem 2.1]. Now, for $\alpha>0$, we define $f_\alpha\in N_0(D)$ as follows:

$$f_\alpha(z)=\exp\left(\frac{1}{\pi i}\int_{\boldsymbol{R}}\frac{1+tz}{t-z}\frac{1}{1+t^2}\log\frac{1}{1+t^2}dt\right)\exp\left(\frac{i\alpha}{z}\right) \quad (z\in D),$$

where $i\alpha z^{-1}$ is the function defined by the measure μ_α which is concentrated on $\{0\}$ with $\mu_\alpha(\{0\})=\alpha$. Since

$$P[-\log(1+t^2)](iy)+\alpha y^{-1}=-\frac{1}{\pi}\int_R\frac{\log(1+y^2s^2)}{1+s^2}ds+\frac{\alpha}{y}\geqq 0\quad(0<y\leqq\delta)$$

for some $\delta>0$, we can see that $\lambda_0(f_\alpha)=\alpha$. Thus the functions $f_\alpha\,(\alpha>0)$ belong to different components of $N_0(D)$. (ii) Corollaries 2.2 and 2.3 can be used to prove this, in precisely the same way as in [6, Corollary 2, p. 926]. (iii) Put $g_\alpha=\alpha f_\alpha$, f_α being the function defined above. Then $\bar{g}_\alpha\neq\bar{0}$. By Corollary 2.2, we have

$$0<\bar{d}(\bar{g}_\alpha,\bar{0})\leqq d(g_\alpha,0)=\int_R\log\left(1+\frac{\alpha}{1+x^2}\right)dx+\pi\alpha,$$

hence $\bar{g}_\alpha\to\bar{0}$ as $\alpha\to 0$.

4. The component of the origin in $N_0(D)$

The component of the origin can be determined in the same way as in [5]. Let $f\in N_0(D)$, $f\neq 0$. Then, by (2.1), f is expressed in the form $f(z)=f_0(z)g_{\mu^+}(z)$ $(z\in D)$, where

$$f_0(z)=ae^{i\alpha z}b(z)d(z)\exp\left(-\frac{1}{i}\int_R\frac{1+tz}{t-z}d\mu^-(t)\right),$$
$$g_{\mu^+}(z)=\exp\left(\frac{1}{i}\int_R\frac{1+tz}{t-z}d\mu^+(t)\right).$$

Note that $f_0\in N_*(D)$. Let ν be the measure on T^* determined by μ, and $\nu_2-\nu_1$ be the measure occurring in the factorization of Af, where $Af=f\circ\Psi^{-1}$, as in the proof of Corollary 2.5. Then $\nu_2=\nu^+$, $\nu_1=\nu^--\nu_0$, from the proof of Theorem 2.1, and we have $(Af)(w)=(Af_0)(w)S_{\nu^+}(w)^{-1}$ $(w\in U)$, where

$$(Af_0)(w)=aB(w)F(w)\exp\left(-\int_T H(w,\eta)d(\nu^--\nu_0)(\eta)\right),$$
$$S_{\nu^+}(w)=\exp\left(-\int_T H(w,\eta)d\nu^+(\eta)\right).$$

We denote by K_0 the set of functions in $N_0(D)$ for which μ^+ are continuous measures, where we let $0\in K_0$.

THEOREM 4.1. *K_0 is the component of the origin in $N_0(D)$.*

PROOF. The component of the origin in $N(U)$, K, consists of the origin and of functions in $N(U)$ such that ν_2 are continuous. Let K_0' be the component of the origin in $N_0(D)$. Then, since $A(K_0')\subset K$, the correspondence of measures shows that $K_0'\subset K_0$. Therefore, it is sufficient to prove that K_0 is connected. The proofs of Lemma 3.2 and Theorem 3.3 of [5] are valid in the setting of $N_0(D)$, because $K_0=N_0(D)\cap A^{-1}(K)$, a complete metric space. Hence we need only show that every open ball in K_0 is ε-chainable, for any $\varepsilon>0$. Let B be the open ball in K_0, centered at 0 and of radius r. Take $f\in B$, $f\neq 0$. Then $f=f_0 g_{\mu^+}$. Since the measure $\pi(1+x^2)d\mu^+(x)$ is finite and continuous on $\boldsymbol{R}$, we can choose open intervals I_i and closed intervals $J_i\,(1\leqq i\leqq n)$, where $I_1=\{x\in\boldsymbol{R}\,|\,|x|>M\}$ and $J_1=\{x\in\boldsymbol{R}\,|\,|x|\geqq M+1\}$ for some $M>0$, such that $J_i\subset I_i\,(1\leqq i\leqq n)$, $\bigcup_{i=1}^{n}J_i=\boldsymbol{R}$, and

$$\int_{I_i}\pi(1+x^2)d\mu^+(x)<\varepsilon/2 \quad (1\leqq i\leqq n).$$

Let μ_i denote the measure μ^+ restricted to I_i. Then μ^+ corresponds to ν^+ and μ_i corresponds to ν_i, where ν_i is the measure ν^+ restricted to $\Psi(I_i)$. Hence we have $A(g_{\mu_i-\mu^+})=S_{\nu^+-\nu_i}$. By the corona theorem, we can take $s'_i\in H^\infty(U)$ $(1\leqq i\leqq n)$ such that $\sum_{i=1}^{n}s'_iS_{\nu^+-\nu_i}=1$. Letting $s_i=A^{-1}s'_i$, we get $\sum_{i=1}^{n}s_ig_{\mu_i-\mu^+}=1$ on D, hence $\sum_{i=1}^{n}s_if_0g_{\mu_i-\mu^+}=f_0$. Since $\mu_i-\mu^+\leqq 0$ and $s_i\in H^\infty(D)$, we see that $s_if_0g_{\mu_i-\mu^+}\in N_*(D)$. Let L be the linear subspace of $N_*(D)$ generated by $\{s_if_0g_{\mu_i-\mu^+}|1\leqq i\leqq n\}$ and let

$$B_0=\{g\in L|d(g,0)<r-\int_{\boldsymbol{R}}\pi(1+x^2)d\mu^+(x)\}.$$

Then $f_0\in B_0$. There exist functions $K_j\,(1\leqq j\leqq m)$ such that $K_j=\varepsilon_js_if_0g_{\mu_i-\mu^+}$, $\varepsilon_j\in\boldsymbol{C}$, for some i and such that
(a) $d(K_j,0)<\varepsilon/2\,(1\leqq i\leqq m)$, (b) $K_1+\cdots+K_p\in B_0\,(1\leqq p\leqq m)$, and (c) $K_1+\cdots+K_m=f_0$. If we put $\tau_j=\mu_i$ and $f_j=\varepsilon_js_if_0$, where $K_j=\varepsilon_js_if_0g_{\mu_i-\mu^+}$, then $K_j=f_jg_{\mu_i-\mu^+}$ and $d(K_j,0)=d(f_j,0)$. Now we can see that

(i) $\sum_{j=1}^{m}f_jg_{\tau_j}=f_0g_{\mu^+}=f$,

(ii) $d\left(\sum_{j=1}^{p}f_jg_{\tau_j},0\right)\leqq d\left(\sum_{j=1}^{p}K_j,0\right)+\int_{\boldsymbol{R}}\pi(1+x^2)d\mu^+(x)<r$, and

(iii) $d(f_jg_{\tau_j},0)\leqq d(f_j,0)+\int_{\boldsymbol{R}}\pi(1+x^2)d\mu_i(x)<\varepsilon$.

Since $\sum_{j=1}^{p}K_j\in N_*(D)$ and since μ^+ is continuous, the minimum propery of the Jordan decomposition of measures implies that $\sum_{j=1}^{p}f_jg_{\tau_j}\in K_0$. Hence $\sum_{j=1}^{p}f_jg_{\tau_j}\in B$. Thus we have an ε-chain $\{\sum_{j=1}^{p}f_jg_{\tau_j}|1\leqq p\leqq m\}$ from 0 to f.

5. Isometries

We prove two theorems. For $0<p<+\infty$, $H^p(D)$ is defined by L^p-boundedness of $f(x+iy)$, where f is holomorphic on D.

THEOREM 5.1. (i) *There does not exist a linear isometry of* $N(U)$ *onto* $N_0(D)$. (ii) *There does not exist a linear isometry of* $N_*(U)$ *onto* $N_*(D)$.

PROOF. Suppose $f\in H^p(D)$, $0<p\leqq 1$. Then $|f|^{p/2}\leqq P[\phi]$ for some $\phi\in L^2(\boldsymbol{R})$. Hence $f\in N_*(D)$, by $\log(1+x)\leqq p^{-1}x^p$ $(x\geqq 0)$; thus $H^p(D)\subset N_*(D)$ $(0<p\leqq 1)$. Now we prove part (i), part (ii) being very similar. Suppose that an operator A is a linear isometry of $N(U)$ onto $N_0(D)$. We show that A transforms $H^1(U)$ onto $H^1(D)$ as an H^1-isometry, following [8, Theorem 2.1]. Take a sequence $\{y_j\}$ such that $y_1>y_2>\cdots$, $y_j\to 0$ as $j\to\infty$ and, for $g\in H^1(U)$, put

$$a_{nj}=\int_{\boldsymbol{R}}\log\left(1+\frac{1}{n}|(Ag)(x+iy_j)|\right)^n dx \quad (n, j=1, 2, \cdots).$$

Then $\{a_{nj}\}$ is increasing in both n and j and, since $nd(n^{-1}Ag, 0)=n\rho(n^{-1}g, 0)$ for each n, we have

$$\lim_{j\to\infty}a_{nj}=\int_T\log\left(1+\frac{1}{n}|g^*(\eta)|\right)^n d\sigma(\eta) \quad (n=1, 2, \cdots).$$

Here the integrand tends to $|g^*(\eta)|$ increasingly as $n\to\infty$, hence $\lim_{n\to\infty}\lim_{j\to\infty}a_{nj}=\|g\|_{H^1(U)}$. We can interchange the limits, and the integral of $|(Ag)(x+iy)|$ with respect to dx is increasing as $y\to 0^+$, as seen from the property (D) and [1, Theorem 1]. We have thus $\|Ag\|_{H^1(D)}=\|g\|_{H^1(U)}$. The same argument for A^{-1} shows that A transforms $H^1(U)$ onto $H^1(D)$. Next we define A_1 by

$$(5.1)\qquad (A_1f)(w)=\pi f(\Psi^{-1}(w))\left(\frac{2i}{1-w}\right)^2 \quad (w\in U)$$

for $f\in H^1(D)$. Since A_1 is a linear isometry of $H^1(D)$ onto $H^1(U)$, $A_1\circ A$ becomes a linear isometry of $H^1(U)$ onto $H^1(U)$. Hence, by [2, Theorem 2], we can write $A_1\circ A$ in the form

$$((A_1\circ A)g)(w)=c\frac{1-|a|^2}{(1-\bar{a}w)^2}g(\varphi(w)) \quad (w\in U)$$

for $g\in H^1(U)$, where $c\in\boldsymbol{C}$ with $|c|=1$ and φ a conformal map of U onto U with $\varphi(a)=0$. Letting $f=Ag$ in (5.1) and $w=\Psi(z)$, we obtain

$$(Ag)(z)=\frac{c}{\pi}\frac{1-|a|^2}{(1-\bar{a}\psi(z))^2}\frac{1}{(z+i)^2}g((\psi\circ\Psi)(z)) \quad (z\in D).$$

Put $g=1$ and let $y\to 0^+$ in $z=x+iy$. Then

$$\begin{aligned}|(A1)(x)|&=\frac{1-|a|^2}{\pi}\frac{1}{|1-\bar{a}\Psi(x)|^2}\frac{1}{|x+i|^2}\\ &\leqq\frac{2}{\pi(1-|a|)}\frac{1}{x^2+1} \quad (x\in\boldsymbol{R}).\end{aligned}$$

Hence, for $0\leqq t\leqq 4^{-1}\pi(1-|a|)$, we have $t|(A1)(x)|\leqq(2(x^2+1))^{-1}$ $(x\in\boldsymbol{R})$. By $\rho(t, 0)=d(At, 0)$ and by the dominated convergence theorem, we obtain

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}t^n=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}\left(\int_{\boldsymbol{R}}|(A1)(x)|^n dx\right)t^n.$$

This must imply that

$$(5.2) \qquad \int_{\boldsymbol{R}}|(A1)(x)|^n dx=1 \quad (n=1, 2, \cdots).$$

But, since $|(A1)(x)|$ is continuous on $\boldsymbol{R}$, it must follow that $|(A1)(x)|\leqq 1$ $(x\in\boldsymbol{R})$. Hence (5.2) is impossible.

REMARK. We note that

$$H^p(D)\not\subset N_0(D) \quad (1<p\leqq+\infty).$$

For $1<p<+\infty$, take α such that $p^{-1}<\alpha<1$. Then $(z+i)^{-\alpha}\in H^p(D)$. But $(z+i)^{-\alpha}\notin N_0(D)$, by $\log(1+x)\geqq 2^{-1}x$ $(0\leqq x\leqq 1)$. The usual Nevanlinna class consists of holomorphic functions f on D such that $\log^+|f|$ have harmonic majorants ([3, p. 69]). Compared with this, $N_0(D)$ is considerably small.

Our final result shows that every linear isometry of $N_*(D)$ onto $N_*(D)$ is induced by a translation along the real axis. We notice that, in the case of U, a linear isometry A of $N_*(U)$ onto $N_*(U)$ is of the form

$$(Af)(w)=af(bw) \quad (w\in U) \text{ for } f\in N_*(U),$$

where $a, b\in C$ with $|a|=|b|=1$ ([8, Corollary 2.3]); namely, A is induced by a rotation.

THEOREM 5.2. *Let A be a linear isometry of $N_*(D)$ outo $N_*(D)$. Then there exist $c\in\boldsymbol{C}$, $|c|=1$, and $\alpha\in\boldsymbol{R}$ such that*

$$(Af)(z)=cf(z+\alpha) \quad (z\in D) \text{ for } f\in N_*(D).$$

LEMMA 5.3. *Let A be a linear isometry of $H^p(D)$ onto $H^p(D)$, $0<p$*

$<+\infty$, $p\neq 2$. *Then A is written in the form*

(5.3) $(Af)(z)=$

$$c(\psi'(\Psi(z)))^{1/p}\left(\frac{1}{z+i}\right)^{2/p}\left(\frac{2i}{1-(\psi\circ\Psi)(z)}\right)^{2/p} f((\Psi^{-1}\circ\psi\circ\Psi)(z)) \quad (z\in D)$$

for $f\in H^p(D)$, where $c\in \boldsymbol{C}$, $|c|=1$, ψ a conformal map of U onto U, and ψ' the derivative of ψ. If we put $\phi=\Psi^{-1}\circ\psi\circ\Psi$, then

(5.4) $(Af)(z)=c(\phi'(z))^{1/p}f(\phi(z)) \quad (z\in D)$.

PROOF. Define A_p by

(5.5) $(A_p h)(w)=\pi^{1/p}h(\Psi^{-1}(w))\left(\frac{2i}{1-w}\right)^{2/p} \quad (w\in U)$

for $h\in H^p(D)$. Then A_p is a linear isometry of $H^p(D)$ onto $H^p(U)$. Since $A_p\circ A\circ A_p^{-1}$ is a linear isometry of $H^p(U)$ onto $H^p(U)$, there exist $c\in \boldsymbol{C}$, $|c|=1$, and a conformal map ψ of U onto U such that

(5.6) $((A_p\circ A\circ A_p^{-1})g)(w)=c(\psi'(w))^{1/p}g(\psi(w)) \quad (w\in U)$

for $g\in H^p(U)$. Take $f\in H^p(D)$ and let $A_p f=g$. Then $(A_p(Af))(w)=((A_p\circ A\circ A_p^{-1})g)(w)$. By (5.5) and (5.6), we get

$$(Af)(\Psi^{-1}(w))=c\pi^{-1/p}(\psi'(w))^{1/p}\left(\frac{2i}{1-w}\right)^{-2/p}g(\psi(w)).$$

By (5.5) again and by letting $w=\Psi(z)$, we obtain (5.3).

LEMMA 5.4. *Let V be the family of holomorphic functions f on D such that $|f(z)||z+i|^2$ are bounded. Then V is a linear subspace of $H^p(D)$ $(1\leqq p<+\infty)$, for which the following hold*: (i) *V is dense in $H^p(D)$.* (ii) *V is dense in $N_*(D)$.*

PROOF. We prove part (ii). Let $f\in N_*(D)$. For $s>0$, we define f_s by $f_s(z)=f(z+is)$ $(z\in D)$. Clearly, $f_s\in N_0(D)$, and f_s satisfies part (iv) of Corollary 2.4. Hence $f_s\in N_*(D)$. By using a generalized form of the dominated convergence theorem, as in [9, Theorem 2], we can conclude that $d(f_s, f)\to 0$ as $s\to 0^+$. Now there exists a sequence $\{g_j\}$ of continuous functions on $\bar{D}$ which are holomorphic on D and such that $|g_j(z)|\leqq 1$ $(z\in \bar{D})$, $|g_j(z)|\,|z+i|^2\to 0$ as $|z|\to+\infty$ in $\bar{D}$, and $g_j(z)\to 1$ as $j\to\infty$ $(z\in\bar{D})$. Indeed, following [3, Chap.II, Corollary 3.3], it suffices to put $g_j(z)=h_j(a_j\Psi(z))(z\in D)$, where $0<a_j<1$ with $a_j\to 1$, and

$$h_j(w)=\left(\frac{a_j-w}{1-a_j w}\right)^3 \quad (w\in U).$$

For $f \in N_*(D)$ and $s>0$, we have $|f_s(z)| \leqq M$ $(z \in D)$, by the property (C). If we let $f_j = f_s g_j$, then $f_j \in V$, and $d(f_j, f_s) \to 0$ as $j \to \infty$.

LEMMA 5.5. *Let A be a linear isometry of $N_*(D)$ onto $N_*(D)$. Then A transforms V onto V as an $H^3(D)$-isometry.*

PROOF. First note that A transforms $H^1(D)$ onto $H^1(D)$ as an H^1-isometry. Hence Af is written in the form (5.3) for $f \in H^1(D)$, with $p=1$. Let $f \in V$, $|f(z)||z+i|^2 \leqq M$ $(z \in D)$. Then, since $2i(1-(\psi \circ \Psi)(z))^{-1} = (\Psi^{-1} \circ \psi \circ \Psi)(z) + i$, we have

$$\left| \frac{2i}{1-(\psi \circ \Psi)(z)} \right|^2 |f((\Psi^{-1} \circ \psi \circ \Psi)(z))| \leqq M.$$

Moreover, ψ is of the form: $\psi(w) = b(a-w)(1-\bar{a}w)^{-1}$ $(w \in U)$, with $|b|=1$ and $a \in U$, hence $|\psi'(w)| \leqq 2(1-|a|)^{-1}$. From these we see that $|(Af)(z)| \leqq 2M(1-|a|)^{-1}|z+i|^{-2}$ $(z \in D)$, which implies that $Af \in V$. The same argument for A^{-1} shows that A transforms V onto V. Now from

$$|tf^*(x)|,\ |t(Af)^*(x)| \leqq \frac{1}{2(x^2+1)} \quad (x \in \boldsymbol{R})$$

for $0 \leqq t \leqq \delta$ and from $d(tf, 0) = d(A(tf), 0)$, we can conclude that

$$\int_{\boldsymbol{R}} |(Af)^*(x)|^3 dx = \int_{\boldsymbol{R}} |f^*(x)|^3 dx,$$

just as in the proof of Theorem 5.1.

PROOF OF THEOREM 5.2. Since V is dense in $H^3(D)$, there is a linear isometry $\tilde{A}$ of $H^3(D)$ onto $H^3(D)$ such that $\tilde{A} = A$ on V. Thus $\tilde{A}f$ is of the form (5.4), with $p=3$. Now let $f \in V$. Then (5.4) is valid for both $p=1$ and $p=3$; namely,

$$(Af)(z) = c_1 \phi_1'(z) f(\phi_1(z)) = c_3(\phi_3'(z))^{1/3} f(\phi_3(z)) \quad (z \in D).$$

Here ϕ_j $(j=1,3)$ are conformal maps of D onto D, hence

$$\phi_j(z) = \frac{\alpha_j z + \beta_j}{\gamma_j z + \delta_j} \quad (z \in D),$$

where α_j, β_j, γ_j, $\delta_j \in \boldsymbol{R}$ and $D_j := \alpha_j \delta_j - \beta_j \gamma_j > 0$. Thus we have

$$\frac{D_3}{D_1^3} \left| \frac{f(\phi_3(z))}{f(\phi_1(z))} \right|^3 \frac{|\gamma_1 z + \delta_1|^6}{|\gamma_3 z + \delta_3|^2} = 1 \quad (z \in D).$$

Suppose $\gamma_1 \neq 0$, and put $f(z) = (z+i)^{-3}$. Then, by letting $|z| \to +\infty$, we would have a contradiction. Therefore, Af must be of the form $(Af)(z)$

$=cf(\beta z+\alpha)$ $(z\in D)$ for $f\in V$, where $c\in \boldsymbol{C}$, $\beta>0$, and $\alpha\in \boldsymbol{R}$. But $\|Af\|_{H^p}$ $=\|f\|_{H^p}$ $(p=1, 3)$, hence $|c|=\beta=1$. Finally, since V is dense in $N_*(D)$, we conclude that $(Af)(z)=cf(z+\alpha)$ $(z\in D)$ for $f\in N_*(D)$.

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College of General Education
Tôhoku University
Kawauchi, Sendai 980
Japan