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On bicommutators of modules over H-separable extension rings II

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This paper is a continuation of the author's previous paper [13], and is devoted to its application. Therefore we will use the same notation as [13]. In § 1 we will give some supplements to [13], and the results of § 1 and [13] will be applied to Azumaya algebras in § 2. Azumaya algebra is the model of the H-separable extension. In general, when we dealt with modules over Azumaya algebras, they were almost limited to be finitely generated projective. Here we will deal with general modules and show that, if A is an Azumaya R-algebra, then for any left A-module M $A^*=\mathrm{Bic}({}_AM)$ is an Azumaya R^*-algebra, with $R^*=\mathrm{Bic}({}_RM)$ commutative, and we have $R^*\cap\iota(A)=\iota(R)$, $A^*\cong A\otimes_R R^*$, where ι is the canonical map of A to A^* (Theorem 2). Furthermore if M is A-faithful, there exist mutually inverse 1-1-correspondences between the class $\mathscr{S}$ of intermediate rings between R and R^* and the class $\mathscr{T}$ of intermediate rings between A and A^*, given by $T\rightarrow AT$, and $S\rightarrow S\cap R^*$, for $T\varepsilon\mathscr{T}$ and $S\varepsilon\mathscr{S}$. In addition, every ring S belonging to $\mathscr{S}$ is an Azumaya $(S\cap R^*)$-algebra (Proposition 3). In § 3 we deal with the H-separable extension of strongly primitive rings. Strongly primitive ring is the one which has a faithful minimal left, or equivalently right, ideal. Suppose that A and B are strongly primitive rings, and let M and $\mathfrak{m}$ be faithful minimal left ideals of A and B, respectively. If A is left B-finitely generated projective and H-separable over B, then $B^*=\mathrm{Bic}({}_BM)\cong\mathrm{Bic}({}_B\mathfrak{m})$, and $A^*=\mathrm{Bic}({}_AM)$ is an H-separable extension of B^* (See Theorom 3.3 [12]). In this paper we will show under the same condition as above that $A^*=\bar{A}B^*=B^*\bar{A}$, $B^*\cap\bar{A}=\bar{B}$, where $\bar{A}=\iota(A)$ and $\bar{B}=\iota(B)$, and for any strongly primitive subring S of A such that $B\subset S$ and A is left S-projective, A^* is H-separable over S^*, and $S^*=\mathrm{Bic}({}_SM)$ is also a full linear ring (Theorem 4).

1. In this section A will always be a ring with the identity 1 and B a subring of A containing 1. C is the center of A and $D=V_A(B)$, the centralizer of B in A. For a left A-module M we write $A^*=\mathrm{Bic}({}_AM)$, the bicommutator of ${}_AM$, $B^*=\mathrm{Bic}({}_BM)$, $D^*=V_{A*}(B^*)$ and $C^*=V_{A*}(A^*)$, the center of A^*. It is easily seen that $D^*=A^{*B}=V_{A*}(\bar{B})$ and $C^*=A^{*A}=V_{A*}(\bar{A})=$, where $\bar{B}$ and $\bar{A}$ are the images of B and A, respectively, by the canonical map ι of A to A^*. Now suppose that A is an H-separable extension of B. Then we have $D^*\cong D\otimes_C C^*$ and $B^*=V_{A*}(V_{A*}(B^*))$ (See Proposition 1 [13]). If furthermore A is left (resp. right) B-finitely generated projective, then A^* is an H-separable extension of B^*, and A^* is left (resp. right) B^*-finitely generated projective (See Theorem 1 [13] and Remark 2). Therefore it is natural to assume that A is an H-separable extension of B, and M is a left A-module such that A^* is an H-separable extension of B^*. Assume furthermore that M is A-faithful. Under these conditions we have

$$\mathrm{Hom}({}_{C*}D^*, {}_{C*}A^*) \cong \mathrm{Hom}({}_{C*}D\otimes_C C^*, {}_{C*}A^*)$$
$$\cong \mathrm{Hom}({}_C D, {}_C\mathrm{Hom}({}_{C*}C^*, {}_{C*}A^*)) \cong \mathrm{Hom}({}_C D, {}_C A^*)$$

The composition ϕ of the above isomorphisms is given by $\phi(f)(d)=f(\iota(d))$ for each $f \in \mathrm{Hom}({}_{C*}D^*, {}_{C*}A^*)$ and $d \in D$. Then we have a commutative diagram

$$\begin{tikzcd}
A\otimes_B A \arrow[rr, "\eta"] \arrow[d, "\iota\otimes\iota"'] & & \mathrm{Hom}({}_C D, {}_C A) \arrow[d, "\iota_*"] \\
A^*\otimes_{B*}A^* \arrow[r, "\eta^*"] & \mathrm{Hom}({}_{C*}D^*, {}_{C*}A^*) \arrow[r, "\phi"] & \mathrm{Hom}({}_C D, {}_C A^*)
\end{tikzcd}$$

where $\iota_*=\mathrm{Hom}(D, \iota)$, $\eta(a\otimes b)(d)=adb$ for $a, b \in A$ and $d \in D$, and η^* is defined in the same way as η. Since η and η^* are isomorphisms and ι_* is a monomorphism, $\iota\otimes\iota$ is a monomorphism. Then we have

$$[A\otimes_B A]^A\cong[\mathrm{Hom}({}_C D, {}_C A)]^A\subset[\mathrm{Hom}({}_C D, {}_C A^*)]^A=\mathrm{Hom}({}_C D, {}_C A^{*A})$$
$$=\mathrm{Hom}({}_C D, {}_C A^{*A*})=[\mathrm{Hom}({}_C D, {}_C A^*)]^{A*}\cong[A^*\otimes_{B*}A^*]^{A*}$$

Therefore, we can regard $A\otimes_B A$ and $[A\otimes_B A]^A$ as submodules of $A^*\otimes_{B*}A^*$ and $[A^*\otimes_{B*} A^*]^{A*}$, respectively.

A is an H-separable extension of B if and only if $1\otimes 1 \in [A\otimes_B A]^A D$ by Proposition 1 [9]. When we write $1\otimes 1=\Sigma x_{ij}\otimes y_{ij}d_i$ with $d_i \in D$, $\Sigma x_{ij}\otimes y_{ij} \in [A\otimes_B A]^A$, we call $\{\Sigma x_{ij}\otimes y_{ij}, d_i\}$ an H-system of A over B (See [5]).

Hereafter we will always denote $\iota(a)$ by $\bar{a}$ for each $a \in A$.

LEMMA 1. *Assume that A is an H-separable extension of B, and M a left A-module such that A^* is an H-separable extension of B^*, and let*

$\{\Sigma x_{ij}\otimes y_{ij},\ d_i\}$ *be an* H*-system of* A *over* B. *Then we have*

(1) $\{\Sigma \bar{x}_{ij}\otimes \bar{y}_{ij},\ \bar{d}_i\}$ *is an* H*-system of* A^* *over* B^*.

(2) *The map* $\tilde{\iota}$ *of* $A^*\otimes_B A$ *to* $A^*\otimes_{B*} A^*$ *such that* $\iota\ (a^*\otimes b)=a^*\otimes \bar{b}$ *for* $a^*\ \varepsilon\ A^*$ *and* $b\ \varepsilon\ A$ *is an isomorphism. Similarly we have* $A\otimes_B A^*\cong A^*\otimes_{B*}A^*$.

PROOF. (1). Since $\Sigma \bar{x}_{ij}\otimes \bar{y}_{ij}\ \varepsilon\ [\bar{A}\otimes_{\bar{B}}\bar{A}]^{\bar{A}}\subset[A^*\otimes_{B*}A^*]^{A*}$ by the above discussion and $\bar{d}_i\ \varepsilon\ \bar{D}\subset D^*$, (1) is obvious. (2). For any a^*, $b^*\varepsilon$ A^* we have

$$\tilde{\iota}(\Sigma a^*\bar{d}_i b^*\bar{x}_{ij}\otimes y_{ij})=\Sigma a^*\bar{d}_i b^*\Sigma \bar{x}_{ij}\otimes \bar{y}_{ij}$$
$$=a^*\Sigma \bar{d}_i\bar{x}_{ij}\otimes \bar{y}_{ij}b^*=a^*(1\otimes 1)b^*=a^*\otimes b^*$$

which means that $\tilde{\iota}$ is surjective. Next suppose that $\Sigma\ a_k^*\otimes \bar{b}_k=0$ in $A^*\otimes_{B*}A^*$ with $a_k^*\ \varepsilon\ A^*$ and $b_k\ \varepsilon\ A$. Then $\eta^*(\Sigma\ a_k^*\otimes \bar{b}_k)(\bar{d}_i)=\Sigma a_k^*\bar{d}_i\bar{b}_k=0$ for each i, and we have in $A^*\otimes_B A$ that

$$\Sigma a_k^*\otimes b_k=\Sigma a_k^*\bar{d}_i\bar{x}_{ij}\otimes y_{ij}b_k=\Sigma a_k^*\bar{d}_i\bar{b}_k\bar{x}_{ij}\otimes y_{ij}=0$$

which means that $\tilde{\iota}$ is a monorphism. Since H-system is left and right symmetry, we have also that $A\otimes_B A^*\cong A^*\otimes_{B*}A^*$.

PROPOSITION 1. *With the same notation as Lemma* 1 *assume furthermore that* B^* *is a right (resp. left)* B^**-direct summand of* A^*. *Then we have* $A^*=B^*\bar{A}\cong B^*\otimes_B A$ *(resp,* $A^*=\bar{A}B^*\cong A\otimes_B B^*$*).*

PROOF. By the assumption there exists a right B^*-projection p of A^* to B^*. Then, since $\{\Sigma \bar{x}_{ij}\otimes \bar{y}_j,\ \bar{d}_i\}$ is an H-system of A^* over B^* by Lemma 1, we have $x^*=\Sigma p(\bar{d}_i x^*\bar{x}_{ij})\bar{y}_{ij}\ \varepsilon\ B^*\bar{A}$ for each $x^*\ \varepsilon\ A^*$ (See page 53 [10]), which implies that $A^*=B^*\bar{A}$. On the other hand we have the isomorphism $\tilde{\iota}: A^*\otimes_B A\rightarrow A^*\otimes_{B*}A^*$ by Lemma 1. Since B^* is a right B^*-direct summand of A^*, we have $B^*\otimes_B A\subset A^*\otimes_B A$. Then $\tilde{\iota}(B^*\otimes_B A)=\{1\otimes\Sigma\ b_i^*a_i : b_i^*\ \varepsilon\ B^*$ and $a_i\ \varepsilon\ A\}$. But the map μ of A^* to $A^*\otimes_{B*}A^*$ such that $\mu(a^*)=1\otimes a^*$ for any $a^*\ \varepsilon\ A^*$ is a B^*-split monomorphism. Hence we have $\tilde{\iota}(B^*\otimes_B A)=\mu(B^*\bar{A})$, which concludes $B^*\otimes_B A\cong B^*\bar{A}$.

We are now ready to have our main theorem

THEOREM 1. *Let* A *be an* H*-separable extension of* B *such that* A *is left or right* B*-finitely generated projective. Then if* B *is a left (resp. right)* B*-direct summand of* A, *we have* $A^*=\bar{A}B^*\cong A\otimes_B B^*$ *(resp.* $A^*=B^*\bar{A}\cong B^*\otimes_B A$*) and* $B^*\cap\bar{A}=\bar{B}$.

PROOF. By Theorem 1 [13] A^* is an H-separable extension of B^*, and B^* is a left (resp. right) B^*-direct summand of A^*. Hence we have the first assertion of the theorem. By Proposition 1. 5 [7] and Proposition 3. 4 [10] we have that $\bar{A}$ is an H-separable extension of $\bar{B}$ such that $\bar{B}$ is a left (resp. right) $\bar{B}$-direct summand of $\bar{A}$. Then by Proposition 1. 2 [7] and Proposition 1 [13] we have $\bar{B}=V_{\bar{A}}(V_{\bar{A}}(\bar{B}))=B^*\cap\bar{A}$.

${}_AM$ is called to be balanced if the canonical map ι is surjective. As an immediate consequence of Theorem 1 we have

COROLLARY 1. *Under the same condition as Theorem* 1, *we have that* ${}_AM$ *is balanced if and only if* ${}_BM$ *is balanced*

PROOF. If ${}_BM$ is balanced, we have $B^*=\bar{B}$ and $A^*=\bar{A}B^*=\bar{A}$. If conversely $\bar{A}=A^*(\supset B^*)$, we have $\bar{B}=V_{\bar{A}}(V_{\bar{A}}(\bar{B}))=B^*\cap\bar{A}=B^*$ for the same reason as the second part of Theorem 1.

Another application of Theorem 1 [13] is

PROPOSITION 2. *Let* A *be an* H*-separable extension of* B *and* M *a left* A*-module such that* A^* *is an* H*-separable extension of* B^*. *Then for any subring* P *of* A *such that* P *is a separable extension of* B *and a* P-P*-direct summand of* A, $P^*=Bic({}_PM)$ *is a separable extension of* B^* *and a* P^*-P^**-direct summand of* A^*.

PROOF. By Theorem 1 [6] $E=V_A(P)$ is a separable C-algebra, while A is an H-separable extension of P by Proposition 2. 2 [8]. Therefore by Theorem 1 [13] we have that $P^*=V_{A^*}(V_{A^*}(P^*))$ and $V_{A*}(P^*))\cong E\otimes_C C^*$. But $E\otimes_C C^*$ is a separable C^*-subalgebra of $D\otimes_c C^*(\cong D^*)$. Hence again by Theorom 1 [6] P^* is a separable extension of B^* and a P^*-P^*-direct summand of A^*.

REMARK 1. In the disccuion above Lemma 1 let us drop the condition that A^* is an H-separable extension of B^*. But η^* always exists, and ϕ is an isomorphism. Therefore $\iota\otimes\iota: A\otimes_B A\rightarrow A^*\otimes_{B*}A^*$ is a monomorphism. If η^* is a monomorphism, then $[A\otimes_B A]^A\subset[A^*\otimes_{B*}A^*]^{A*}$, and A^* is an H-separable extension of B^*.

REMARK 2. Let S be a ring and T a subring of S, and assume that there exists a ring homomorphism $\varkappa$ of A into S such that $\varkappa\ (B)\subset T$. If furthermore $C'=V_S(S)=V_S(\varkappa(A))$ and $V_s(T)=V_S(\varkappa(B))$, then all assertions in this section exept Theorem 1 and Corollary 1 are valid when we replace A^* and B^* with S and T, respectively. Because, all of them depend only on the facts that $V_{A*}(\bar{A})=C^*$ and $D^*=V_{A*}(\bar{B})$. If we assume furthermore that $T=V_S(V_S(T))$, then also Theorem 1 [13] and Theorem 1 hold for S and T.

2. In this section we will apply the results in [13] or § 1 of this paper to the theory on Azumaya algebra. Let R be a commutative ring. An Azumaya R-algebra A is always an H-separable extension of R and is R-finitely generated projective, and R is an R-direct summand of A. Conversely if an R-algebra A is an H-separable extension of R, and R is an R-direct summand of A, then A is an Azumaya R-algebra (See Corollary 1. 1 [7]). Therefore applying Theorem 1 [13] and Theorem 1 we have

THEOREM 2. *Let A be an Azumaya R-algebra and M a left A-module, and write $A^*=Bic({}_AM)$ and $R^*=Bic({}_RM)$. Then we have*

(1) *A^* is an Azumaya R^*-algebra with $A^*\cong A\otimes_R R^*$.*

(2) *${}_AM$ has the double centralizer property if and only if ${}_RM$ does.*

(3) *For any separable R-subalgebra B of A, $B^*=Bic({}_BM)$ is a separable R^*-subalgebra of A^*.*

PROOF. If B is a separable R-subalgebra of an Azumaya R-algebra A, then B is a B-B-direct summand of A by Proposition 1. 5 [8]. Therefore we need only to prove the following

LEMMA 2. *Let R be a commutative ring M an R-module and $\Lambda=End({}_RM)$. Then $R^*=Bic({}_RM)$ coinsides with $C(\Lambda)$, the center of Λ.*

PROOF. Since R is commutative, it is obvious that $R^*\subset\Lambda$. Furthermore we have $V_\Lambda(R^*)=\mathrm{End}({}_{R*}M)=\mathrm{End}({}_RM)=\Lambda$, which means $R^*\subset C(\Lambda)$. That $C(\Lambda)\subset R^*$ is clear, since $R^*=\mathrm{End}({}_\Lambda M)$.

COROLLARY 2. *With the same notation as Lemma 2, for any Azumaya R^*-subalgebra A of Λ we have that $A=V_\Lambda(V_\Lambda(A))$, i. e., $A=A^*$.*

PROOF. Note that $R^*=\mathrm{Bic}({}_RM)$, and apply Theorem 2 (2).

PROPOSITION 3. *With the same notation as Theorem 2 assume furthermore that ${}_AM$ is faithful. Denote the class of subrings of R^* containing R by $\mathscr{T}$, and the class of subrings of A^* containing A by $\mathscr{S}$. Each T in $\mathscr{S}$ is an Azumaya $(S\cap R^*)$-algabra, and there exist mutually inverse 1-1-correspondences between $\mathscr{T}$ and $\mathscr{S}$ given by $T\to AT$ and $S\to S\cap R^*$, for $T\ \varepsilon\ \mathscr{T}$ and $S\ \varepsilon\ \mathscr{S}$.*

PROOF. Let $S\ \varepsilon\ \mathscr{S}$. S is also an R-algebra. Hence we have $S=AV_S(A)\cong A\otimes_R V_S(A)$. But $V_S(A)=S\cap V_{A*}(A)=S\cap R^*\varepsilon\ \mathscr{T}$. Then S is an Azumaya $(S\cap R^*)-$algebra, and we have $S=A(S\cap R^*)$. Next let $T\ \varepsilon\ \mathscr{T}$. We have $AT\cong A\otimes_R T$, since $A\otimes_R T\subset A\otimes_R R^*\cong AR^*$. Thus AS is an Azumaya S-algebra. Then by the same urgument as above we see that $AS\cap R^*$ is the center of AS, manely, $AS\cap R^*=S$.

For any ring A and left A-modules M and N we write $M\sim N$, in case M and N are isomorphic to direct summands of some finite direct sums of copies of N and M, respectively, as A-module. The next lemma is an immediate consequence of Proposition 1. 5 [4] and Morita Theorem. But we will give here a proof by the direct computations.

LEMMA 3. *Let A be a ring and M and N left A-modules such that $M\sim N$. Let $\Lambda=End({}_AM)$ and $\Omega=End({}_AN)$. Then we have $C(\Lambda)\cong C(\Omega)$.*

PROOF. By the assumption there exist f_i, k_j ε Hom$({}_AM, {}_AN)$ and g_i, h_j ε Hom$({}_AN, {}_AM)$ such that $\Sigma g_if_i=1_M$ and $\Sigma k_jh_j=1_N$. Then for any f ε $C(\Lambda)$ and g ε Ω we have Σk_jfh_j ε Ω and

$$\Sigma k_jfh_jg=\Sigma k_jfh_jg\Sigma k_mh_m=\Sigma k_jh_jgk_mfh_m=g\Sigma k_mfh_m$$

since h_jgk_m ε Λ and $\Sigma k_jh_j=1_N$. Hence we have Σk_jfh_j ε $C(\Omega)$, and we can define the map Φ of $C(\Lambda)$ to $C(\Omega)$ by $\Phi(f)=\Sigma k_jfh_j$ for f ε $C(\Lambda)$. It is easily seen that Φ is a ring homorphism. Similarly we can define Ψ: $C(\Omega)\rightarrow C(\Lambda)$ by $\Psi(g)=\Sigma g_igf_i$ for g $\varepsilon C(\Omega)$. We can easily see that $\Psi\Phi$ and $\Phi\Psi$ are identity maps on $C(\Lambda)$ and $C(\Omega)$, respectively. Thus Φ and Ψ are isomorphisms.

Now we have our main theorem of this section.

THEOREM 4. *Let A be an Azumaya R-algebra and M and N left A-modules such that $M\sim N$ as R-module. Then we have $Bic({}_AM)\cong Bic({}_AN)$.*

PROOF. Since $M\sim N$ as R-module, we have $C(\Lambda)\cong C(\Omega)$, where $\Lambda=$End$({}_RM)$ and $\Omega=$End$({}_RN)$. Then by Theorem 2 and Lemma 2 we have

$$\begin{aligned}\mathrm{Bic}({}_AM)&\cong A\otimes_R\mathrm{Bic}({}_RM)\cong A\otimes_R C(\Lambda)\\ &\cong A\otimes_R C(\Omega)\cong A\otimes_R\mathrm{Bic}({}_RN)\cong\mathrm{Bic}({}_AN).\end{aligned}$$

3. In this section we will apply the results of § 1 or [13] to the theory on strongly primitive rings. Again we will use the same notation as § 1, and consider the same situation as Theorem 3. 3 [12].

Let A and B be strongly primitive rings and M and $\mathfrak{m}$ faithful minimal left ideals of A and B, respectively. Suppose that A is an H-separable extension of B such that A is left B-finitely generated projective. Then M is isomorphic to a finite direct sum of copis of $\mathfrak{m}$ as left B-module, and $B^*\cong \mathrm{Bic}({}_B\mathfrak{m})$ by Theorem 3. 3 [12]. In addition Theorem 1 [13] shows that A^* is an H-separable extension of B^*. Furthermore we have

THEOREM 4. *Let A, B, M and $\mathfrak{m}$ be as above. Suppose that T is a strongly primitive subring of A such that $B\subset T$ and A is left T-finitely generated projective. Then we have*

(1) *$A^*=AB^*\cong A\otimes_B B^*$, and $A^*=B^*A\cong B^*\otimes_B A$. In addition $B'=V_A(V_A(B))$ is strogly primitive.*

(2) *M is isomorphic to a finite direct sum of copies of $\mathfrak{n}$ as left T-module, and $T^*=Bic({}_TM)$ is isomorphic to $Bic({}_T\mathfrak{n})$, where $\mathfrak{n}$ is a faithful minimal left ideal of T.*

(3) *A^* is an H-separable extension of T^* such that $V_{A*}(V_{A*}(T^*))=T^*$, and $V_{A*}(T^*)$ is finite dimensional simple C^*-algebra.*

PROOF. Since $B\subset V_A(V_A(B))=B^*\cap A\subset B^*$, B$'$ is strongly primitive. As is remarked above, we have $B^*\cong \mathrm{End}({}_\Delta\mathfrak{m})$, where $\Delta=\mathrm{End}({}_B\mathfrak{m})$ is a division ring. Therefore A^* is an H-separable extension of a left full linear ring B^*, and A^* is left, as well as right, B^*-finitely generated free by Theorem 4 [11]. Then A^* is a left, as well as right, B^*-generator, which implies that B^* is a left, as well as right, B^*-direct summand of A^* by B. Müller's Lemma. Then we can apply Proposition 1 to have (1). Next, let T be a subring of A which satisfies the condition of the theorem, and denote the socles of A, T and B by S, $\mathfrak{Z}$ and $\mathfrak{z}$, respectively. By Lemma 3. 1 and Theorem 3. 2 [12] we have $\mathfrak{z}=B\cap S\subset T\cap S=\mathfrak{Z}$, and $S=\mathfrak{z}A\subset\mathfrak{Z}A\subset S$. Thus we have $M\subset S=\mathfrak{Z}A=\Sigma\oplus\mathfrak{n}$, while M is T-finitely generated. Then we have (2) (See Remark § 3 [12]). Let furthermore S^*, $\mathfrak{Z}^*$ and $\mathfrak{z}^*$ be the socles of A^*, T^* and B^*, respectively. Then we have $S^*=SAC^*$, $\tilde{\mathfrak{z}}^*=\mathfrak{Z}T^*$ and $\mathfrak{z}^*=\mathfrak{z}B^*$ by Corollary 2. 1 [12]. Then $S^*=\mathfrak{z}AA^*=\mathfrak{z}A^*\subset\tilde{\mathfrak{z}}T^*A^*=\mathfrak{Z}^*A^*$, which implies that $V_{A*}(T^*)$ is a finite dimensional simple C^*-algebra and $T^*=V_{A*}(V_{A*}(T^*))$ by Theorem 36.4 [2]. This implies that A^* is an H-separable extesion of T^* by Thetrem 4 [11].

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