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Algebras A_p and B_p and amenability of locally compact groups

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0. Introduction.

Since the appearance of the pioneer work of Eymard [4], the Fourier algebra $A(G)$ of a locally compact group G has been studied by many authors in connection with the theory of unitary representations and the theory of operator algebras. As related algebras, the algebras $A_p(G)$ and the algebras $B_p(G)$ of Herz-Schur multipliers for $1<p<\infty$ have been investigated by Eymard [5] and Herz [8-10] together with the algebras $PF_p(G)$ of pseudofunctions and $PM_p(G)$ of pseudomeasures. Remark here that $A(G)=A_2(G)$. In general the algebra $A_p(G)$ is contractively imbedded in $B_p(G)$. When G is amenable, this imbedding is isometric.

It is shown in [8, 11] that if the group G is amenable, then $A_{p'}(G)$ is contractively included in $A_p(G)$ whenever $1<p<p'\leq 2$ or $2\leq p'<p<\infty$. In particular, $A(G)$ is contractively included in every $A_p(G)$. It is known that the same relation holds also for $B_p(G)$ (see Remark 2.5 (1)). However, according to Pytlik [18], we know that when $\boldsymbol{F}_r$ is a free group with r generators, $2\leq r\leq\infty$, a typical example of non-amenable groups, for any distinct pair p, p' there does not exist any inclusion relation between $A_p(\boldsymbol{F}_r)$ and $A_{p'}(\boldsymbol{F}_r)$ (see Remark 2. 5 (2)). In section 2, we will prove that for every locally compact group G the algebra $B_2(G)$ is contractively included in $B_p(G)$. As a consequence, we show that when $\boldsymbol{F}_r$ is a free group, for any $1<p<\infty$ the algebra $A_p(\boldsymbol{F}_r)$ has an approximate identity $\{u_n\}$ such that $\sup_n\|u_n\|_{B_p}\leq 1$. This result should be compared with the well-known result (e. g. [9], [15]) that $A_p(G)$ has a bounded approximate identity if and only if the group G is amenable.

Nebbia [16] characterized the amenability of G in terms of multipliers of $A(G)$ into the space $M(G)$ of finite complex Radon measures or $L^1(G)$. In section 3, for $1<p<\infty$ and $1\leq p'<\infty$, we define multipliers of $A_p(G)$ into $M(G)$ or $L^{p'}(G)$, and those of $W_p(G)$ (the dual space of $PF_p(G)$) into $M(G)$ or $L^{p'}(G)$. For instance, a multiplier of $A_p(G)$ into $M(G)$ is a bounded linear operator $\Phi: A_p(G)\rightarrow M(G)$ such that $\Phi(uv)=u\Phi(v)$ for all $u, v\in A_p(G)$. Any element of $M(G)$ defines a multiplier in natural way. Extending the results of Nebbia, we present several characterizations of the amenability of G in terms of those newly defined multipliers. We prove, among others, that G is amenable if and only if every multiplier of $A_p(G)$ into $M(G)$ is given by some element of $M(G)$, that is, the space of multipliers of $A_p(G)$ into $M(G)$ is isomorphic with $M(G)$.

1. Definitions and notations

Throughout this paper, let G be a locally compact group with a fixed left Haar measure and $L^p(G)$, $1 \leq p \leq \infty$, the usual Lebesgue spaces on G with the norm $\|\cdot\|_p$. Let $C(G)$ be the Banach space of complex bounded continuous functions on G with the norm $\|\cdot\|_\infty$, $C_0(G)$ the subspace of $C(G)$ consisting of functions vanishing at infinity, and $L(G)$ the subspace of $C(G)$ consisting of functions with compact support. Also let $M(G)$ be the space of finite complex Radon measures on G. In this section, for later convenience, we recall definitions and basic properties of the algebras $A_p(G)$ and several related function algebras on G.

Suppose that $1<p<\infty$ and $1/p+1/q=1$ throughout this section. The algebra $A_p(G)$ introduced by Eymard [5] and Herz [9] is the space of all functions u on G written as $u=\sum_{i=1}^{\infty} f_i * g_i^{\vee}$ for $f_i \in L^p(G)$ and $g_i \in L^q(G)$ with $\sum_{i=1}^{\infty}\|f_i\|_p\|g_i\|_q<+\infty$, where $\overset{\vee}{g}(x)=g(x^{-1})$. The norm on $A_p(G)$ is given by

$$\|u\|_{A_p}=\inf \sum_{i=1}^{\infty}\|f_i\|_p\|g_i\|_q,$$

where the infimum is taken over all such expressions of u. Then $A_p(G)$ is a Banach algebra with pointwise operations. Clearly $A_p(G) \subset C_0(G)$ and $\|u\|_\infty \leq \|u\|_{A_p}$ for $u \in A_p(G)$. we denote by $MA_p(G)$ the space of multipliers of $A_p(G)$ which consists of all functions φ on G such that the pointwise product φu belongs to $A_p(G)$ for every $u \in A_p(G)$. The norm on $MA_p(G)$ is the operator norm on $A_p(G)$. The elements of $MA_p(G)$ are continuous on G and $\|u\|_\infty \leq \|u\|_{MA_p}$ for $u \in MA_p(G)$.

Let $V_p(G)$ be the space of pointwise multipliers of the projective tensor product space $L^p(G) \otimes_\gamma L^q(G)$, that is, the space of all complex functions ψ on $G \times G$ such that for every $F \in L^p(G) \otimes_\gamma L^q(G)$ the pointwise product ψF belongs to $L^p(G) \otimes_\gamma L^q(G)$ (the elements of $L^p(G) \otimes_\gamma L^q(G)$ can be regarded as locally integrable functions on $G \times G$). The norm on $V_p(G)$ is the operator norm on $L^p(G) \otimes_\gamma L^q(G)$. Then the space $B_p(G)$ of Herz-Schur multipliers is the space of all functions φ on G such that the function K_φ on $G \times G$ defined by $K_\varphi(x, y)=\varphi(xy^{-1})$ belongs to $V_p(G)$. The norm $\|\varphi\|_{B_p}$ is given by $\|\varphi\|_{B_p}=\|K_\varphi\|_{V_p}$.

Let $\mathbf{B}(L^p(G))$ be the Banach space of all bounded linear operators on $L^p(G)$. An operator T in $\mathbf{B}(L^p(G))$ is called a convolution operator if $(Tf)*g=T(f*g)$ for all $f, g\in L(G)$ where the symbol $*$ denotes the convolution. We denote by $CV_p(G)$ the space of all convolution operators in $\mathbf{B}(L^p(G))$, which becomes a Banach algebra with the operator norm on $L^p(G)$. For each $\mu\in M(G)$, the operator $\lambda(\mu): f\mapsto \mu * f$ belongs to $CV_p(G)$ with $\|\lambda(\mu)\|_{CV_p}\leq\|\mu\|$. We consider that $(L^1(G)\subset)$ $M(G)\subset CV_p(G)$ in this sense. As is well-known, $\mathbf{B}(L^p(G))$ is identified with the dual Banach space of $L^p(G)\otimes_\gamma L^q(G)$, so that the w^*-topology from $L^p(G)\otimes_\gamma L^q(G)$ can be considered on $\mathbf{B}(L^p(G))$. The space $CV_p(G)$ is closed in this topology. We denote by $PF_p(G)$ and $PM_p(G)$ the norm closure and the w^*-closure of $L^1(G)$ in $CV_p(G)$, respectively, which are Banach algebras with the operator norm. Then $PF_p(G)\subset PM_p(G)\subset CV_p(G)$, and moreover $M(G)\subset PM_p(G)$ in the sense stated above. Herz [9] called the elements of $PF_p(G)$ pseudofunctions and those of $PM_p(G)$ pseudomeasures. In particular when $p=2$, $PF_2(G)$ is the reduced C^*-algebra of G and $PM_2(G)$ is the group von Neumann algebra of G.

Finally let $W_p(G)$ be the dual Banach space of $PF_q(G)$ with the dual norm. The elements of $W_p(G)$ can be regarded as functions in $L^\infty(G)$. The spaces $MA_p(G)$, $B_p(G)$ and $W_p(G)$ are Banach algebras with respect to respective norms and pointwise operations.

We always have $A_p(G)\subset W_p(G)\subset B_p(G)\subset MA_p(G)$, where each imbedding is contractive. Moreover $MA_p(G)$ is isometrically isomorphic with $W_p(G)$ when G is amenable. In this case we have $W_p(G)=B_p(G)=MA_p(G)$ and the three corresponding norms coincide. The dual Banach space of $A_p(G)$ is isometrically isomorphic with $PM_q(G)$, where the duality is given by

$$\langle T, u\rangle=\sum_{i=1}^{\infty}\langle Tg_i, f_i\rangle,\quad T\in PM_q(G),\ u=\sum_{i=1}^{\infty}f_i*g_i^\vee\in A_p(G).$$

In particular, $A_2(G)$ is the so-called Fourier algebra $A(G)$ of G which becomes the predual of the group von Neumann algebra of G.

For details on these algebras see [17].

2. Inclusion relation of $B_p(G)$

The main aim of this section is to show that $B_2(G)$ is included in $B_p(G)$ for any $1<p<\infty$. For the convenience of reference, we first mention two known results.

The following was given by Herz [10, Lemmes 1, 2].

PROPOSITION 2.1. *Let G_d denote the group G with the discrete topol-*

ogy. Suppose $1<p<\infty$. *If* ψ *is a continuous function on* $G\times G$, *then* $\psi\in V_p(G)$ *if and only if* $\psi\in V_p(G_d)$. *Moreover* $\|\psi\|_{V_p(G)}=\|\psi\|_{V_p(G_d)}$ *in this case.*

The following fact is found in Cowling and Haagerup [2, §0] (without proof).

PROPOSITION 2.2. *Let* φ *be a complex-valued function on* G. *Then* φ *belongs to* $B_2(G)$ *if and only if there exist a Hilbert space* $\mathscr{H}$ *and* $\mathscr{H}$*-valued bounded continuous functions* ξ, η *on* G *such that*

$$\varphi(xy^{-1})=\langle\xi_x, \eta_y\rangle, \quad x, y\in G.$$

Moreover the norm $\|\varphi\|_{B_2}$ *is the minimum of* $\sup_{x,y\in G}\|\xi_x\|\,\|\eta_y\|$ *for all such expressions.*

For an arbitrary set X, the space $V_p(X)$ of multipliers of $\ell^p(X)\otimes_\gamma\ell^q(X)$ $(1/p+1/q=1)$ is defined in the same way as $V_p(G)$. Then we have:

PROPOSITION 2.3. *Let* X *be a set and* $\mathscr{H}$ *a Hilbert space. Let* ξ *and* η *be* $\mathscr{H}$*-valued bounded functions on* X, *and define a function* K *on* $X\times X$ *by*

$$K(x, y)=\langle\xi_x, \eta_y\rangle, \quad x, y\in X.$$

Then for any $1<p<\infty$, K *belongs to* $V_p(X)$ *and* $\|K\|_{V_p(X)}\leq\sup_{x,y\in X}\|\xi_x\|\,\|\eta_y\|$.

PROOF: Let $1<p<\infty$. Suppose first that $\mathscr{H}$ is separable. Then it is known ([8], [6, Lemma 8.4.4]) that $\mathscr{H}$ is isometrically isomorphic with a closed subspace $\mathfrak{M}$ of the Lebesgue space $L^p(0,1)$ (with respect to the Lebesgue measure). Let T denote the isometry from $\mathscr{H}$ onto $\mathfrak{M}$. For each $v\in\mathscr{H}$, define $\Phi_v(Tu)=\langle u, v\rangle$, $Tu\in\mathfrak{M}$ $(u\in\mathscr{H})$. Then Φ_v is a bounded linear functional on $\mathfrak{M}$ with $\|\Phi_v\|=\|v\|$. By the Hahn-Banach theorem, there exists $\tilde{v}\in L^q(0,1)$ $(1/p+1/q=1)$ such that $\|\tilde{v}\|_q=\|v\|$ and $\langle Tu, \tilde{v}\rangle=\langle u, v\rangle$, $u\in\mathscr{H}$. Hence there exists an $L^q(0,1)$-valued bounded function $\tilde{\eta}$ on X with $\|\tilde{\eta}_y\|_q=\|\eta_y\|$, $y\in X$, such that

$$\langle T\xi_x, \tilde{\eta}_y\rangle=\langle\xi_x, \eta_y\rangle, \quad x, y\in X.$$

Therefore by replacing ξ and η with $T\xi$ and $\tilde{\eta}$ respectively, we can suppose that ξ and η are $L^p(0,1)$-valued and $L^q(0,1)$-valued functions on X respectively.

Now let us prove that if f and g are finitely supported functions on X, then

$$\|K\cdot f\otimes g\|_{\ell^p\otimes_\gamma\ell^q}\leq\Big(\sup_{x,y\in X}\|\xi_x\|\,\|\eta_y\|\Big)\|f\|_p\|g\|_q.$$

The assertion then follows from linearity and density. For $x, y\in X$, we have

$$K(x, y)f(x)g(y)=\langle\xi_x, \eta_y\rangle f(x)g(y)=\int_0^1 F_t(x)G_t(y)dt,$$

where $F_t(x)=f(x)\xi_x(t)$ and $G_t(y)=g(y)\eta_y(t)$. Hence we have

$$K\cdot f\otimes g=\int_0^1 F_t\otimes G_t\ dt.$$

The integral should be considered as a Bochner integral in $\ell^p(X)\otimes_\gamma\ell^q(X)$. Since f and g are finitely supported, we can easily see that $\{F_t\otimes G_t : t\in(0, 1)\}$ is separable in $\ell^p(X)\otimes_\gamma\ell^q(X)$ and the function $t\mapsto F_t\otimes G_t$ is weakly measurable. Hence it follows ([12, Theorem 3. 5. 3]) that the function $t\mapsto F_t\otimes G_t$ is strongly measurable. Moreover by Hölder's inequality,

$$\begin{aligned}\int_0^1\|F_t\otimes G_t\|_{\ell^p\otimes_\gamma\ell^q}dt&=\int_0^1\|F_t\|_p\|G_t\|_q dt\\&\leq\Big(\int_0^1\|F_t\|_p^p dt\Big)^{\frac{1}{p}}\Big(\int_0^1\|G_t\|_q^q dt\Big)^{\frac{1}{q}}\\&=\Big(\int_0^1\sum_{x\in X}|f(x)\xi_x(t)|^p dt\Big)^{\frac{1}{p}}\Big(\int_0^1\sum_{y\in X}|g(y)\eta_y(t)|^q dt\Big)^{\frac{1}{q}}\\&\leq\Big(\sup_{x,y\in X}\|\xi_x\|\,\|\eta_y\|\Big)\|f\|_p\|g\|_q.\end{aligned}$$

Therefore we have

$$\|K\cdot f\otimes g\|_{\ell^p\otimes_\gamma\ell^q}\leq\Big(\sup_{x,y\in X}\|\xi_x\|\,\|\eta_y\|\Big)\|f\|_p\|g\|_q,$$

as claimed.

Suppose that $\mathscr{H}$ is arbitrary. Let f and g be functions of finite support. Put

$$\xi'_x=\begin{cases}\xi_x, & x\in\mathrm{supp}f,\\ 0, & \text{otherwise},\end{cases}\qquad \eta'_y=\begin{cases}\eta_y, & y\in\mathrm{supp}\ g,\\ 0, & \text{otherwise},\end{cases}$$

and

$$K'(x, y)=\langle\xi'_x, \eta'_y\rangle,\qquad x, y\in X.$$

Then $K'\cdot f\otimes g=K\cdot f\otimes g$. Since f and g are finitely supported, the Hilbert space generated by $\{\xi'_x, \eta'_y : x, y\in X\}$ is separable. Hence the above argument yields

$$\|K\cdot f\otimes g\|_{\ell^p\otimes_\gamma\ell^q}=\|K'\cdot f\otimes g\|_{\ell^p\otimes_\gamma\ell^q}$$
$$\leq\Big(\sup_{x,y\in X}\|\xi'_x\|\,\|\eta'_y\|\Big)\|f\|_p\|g\|_q$$
$$\leq\Big(\sup_{x,y\in X}\|\xi_x\|\,\|\eta_y\|\Big)\|f\|_p\|g\|_q,$$

from which the assertion follows. ■

THEOREM 2.4. *For any* $1<p<\infty$, $B_2(G)$ *is included in* $B_p(G)$ *and* $\|\varphi\|_{B_p}\leq\|\varphi\|_{B_2}$ *for all* $\varphi\in B_2(G)$.

PROOF: Let $\varphi\in B_2(G)$, $K_\varphi(x,y)=\varphi(xy^{-1})$ and $1<p<\infty$. By Propositions 2.2 and 2.3, we see that $K_\varphi\in V_p(G_d)$ and $\|K_\varphi\|_{V_P(G_d)}\leq\|\varphi\|_{B_2}$. Since K_φ is continuous on $G\times G$, Proposition 2.1 implies that $K_\varphi\in V_p(G)$ and $\|K_\varphi\|_{V_P(G)}=\|K_\varphi\|_{V_P(G_d)}$. Hence $\varphi\in B_p(G)$ and $\|\varphi\|_{B_p}\leq\|\varphi\|_{B_2}$. ■

REMARKS 2.5. (1) When G is amenable, it holds $B_p(G)=W_p(G)$ and there exists the inclusion relation among $CV_p(G)$ spaces ([11], [17, Proposition 18.18]). Since $PF_p(G)$ is the norm closure of $L^1(G)$ in $CV_p(G)$, the same inclusion relation holds for $PF_p(G)$. Since the dual space of $PF_p(G)$ is $W_q(G)$ $(1/p+1/q=1)$, $B_{p'}(G)$ is contractively included in $B_p(G)$ whenever $1<p<p'\leq 2$ or $2\leq p'<p<\infty$.

(2) Pytlik [18] proved that if $\boldsymbol{F}_r$ is a free group with r generators $(r\geq 2)$ and if $1<p,p'<\infty$, $p\neq p'$ then there exists an element of $CV_p(\boldsymbol{F}_r)$ which does not belong to $CV_{p'}(\boldsymbol{F}_r)$. This implies by duality that under the same assumption there exists an element of $A_p(\boldsymbol{F}_r)$ (resp. $W_p(\boldsymbol{F}_r)$) which does not belong to $A_{p'}(\boldsymbol{F}_r)$ (resp. $W_{p'}(\boldsymbol{F}_r)$).

The following theorem is a partial extension of [3, Corollary 3.9] (see also [19, Remark 3.3 (2)]).

THEOREM 2.6. *Let* $\boldsymbol{F}_r$ *be a free group with* r *generators* $(2\leq r\leq\infty)$, *and* $1<p<\infty$. *Then there exists a sequence* $\{\varphi_n\}$ *in* $A_p(\boldsymbol{F}_r)$ *such that*

$$\sup_n\|\varphi_n\|_{B_p}\leq 1,$$
$$\lim_{n\to\infty}\|\varphi_n u-u\|_{A_p}=0,\qquad u\in A_p(\boldsymbol{F}_r).$$

PROOF: For each element x in $\boldsymbol{F}_r$, $|x|$ denotes the length of x. First suppose that $\boldsymbol{F}_r$ is finitely generated, i.e. $2\leq r<\infty$. For $m\in\boldsymbol{N}$, let χ_m be the characteristic function of the set $\{x\in\boldsymbol{F}_r:|x|=m\}$. By [19, Corollary 1] we know that $\chi_m\in B_2(\boldsymbol{F}_r)$ and $\|\chi_m\|_{B_2}\leq e(m+1)$. Therefore we have $\chi_m\in B_p(\boldsymbol{F}_r)$ and $\|\chi_m\|_{B_p}\leq e(m+1)$ by Theorem 2.4. For each $\sigma>0$ and $m\in\boldsymbol{N}$, define

$$\varphi_{\sigma,m}(x)=\begin{cases} e^{-\sigma|x|}, & |x|\leq m, \\ 0, & \text{otherwise,}\end{cases}$$

and $\varphi_\sigma(x)=e^{-\sigma|x|}$. For fixed $\sigma>0$, since

$$\|\varphi_{\sigma,m}-\varphi_\sigma\|_{B_p}\leq\sum_{n=m+1}^{\infty}e^{-\sigma n}\|\chi_n\|_{B_p}\leq\sum_{n=m+1}^{\infty}e(n+1)e^{-\sigma n},$$

we have

$$(2.1)\qquad \lim_{m\to\infty}\|\varphi_{\sigma,m}-\varphi_\sigma\|_{B_p}=0.$$

Since φ_σ is positive definite by [7, Lemma 1.2], it follows from Proposition 2.2 that $\|\varphi_\sigma\|_{B_2}\leq\varphi_\sigma(e)=1$. Also $1=\|\varphi_\sigma\|_\infty\leq\|\varphi_\sigma\|_{B_p}$, so that $\|\varphi_\sigma\|_{B_p}=1$ by Theorem 2.4. Let $\psi_{\sigma,m}=\varphi_{\sigma,m}/\|\varphi_{\sigma,m}\|_{B_p}$. Since $\psi_{\sigma,m}$ has finite support, it belongs to $A_p(\boldsymbol{F}_r)$. Moreover by (2.1)

$$\|\psi_{\sigma,m}\|_{B_p}=1,\qquad \lim_{m\to\infty}\|\psi_{\sigma,m}-\varphi_\sigma\|_{B_p}=0.$$

Since

$$\begin{aligned}\|\psi_{\sigma,m}u-\varphi_\sigma u\|_{A_p}&\leq\|\psi_{\sigma,m}-\varphi_\sigma\|_{MA_p}\|u\|_{A_p}\\&\leq\|\psi_{\sigma,m}-\varphi_\sigma\|_{B_p}\|u\|_{A_p},\quad u\in A_p(\boldsymbol{F}_r),\end{aligned}$$

we have

$$(2.2)\qquad \lim_{m\to\infty}\|\psi_{\sigma,m}u-\varphi_\sigma u\|_{A_p}=0,\qquad u\in A_p(\boldsymbol{F}_r).$$

On the other hand,

$$\lim_{\sigma\to 0}\|\varphi_\sigma\delta_x-\delta_x\|_{A_p}=\lim_{\sigma\to 0}|\varphi_\sigma(x)-1|=0,\qquad x\in\boldsymbol{F}_r,$$

where $\delta_x(t)=1$ if $t=x$ and $\delta_x(t)=0$ otherwise. Therefore, since $A_p(\boldsymbol{F}_r)\cap L(\boldsymbol{F}_r)$ is dense in $A_p(\boldsymbol{F}_r)$ and $\|\varphi_\sigma\|_{B_p}=1$, we have

$$\lim_{\sigma\to 0}\|\varphi_\sigma u-u\|_{A_p}=0,\qquad u\in A_p(\boldsymbol{F}_r).$$

From this combined with (2.2) it follows that for all $u\in A_p(\boldsymbol{F}_r)$,

$$\lim_{\substack{m\to\infty\\ \sigma\to 0}}\|\psi_{\sigma,m}u-u\|_{A_p}=0.$$

Now the existence of the sequence with the required property is shown by the separability of $A_p(\boldsymbol{F}_r)$ as in the last part of the proof of [3, Theorem 4.6].

Now let $\boldsymbol{F}_\infty$ be a free group with infinitely many generators. Let a, b

be the free generators of $\boldsymbol{F}_2$. Then the subgroup $\boldsymbol{F}$ of $\boldsymbol{F}_2$ generated by $\{b^n ab^{-n} : n\in\boldsymbol{N}\}$ can be identified with $\boldsymbol{F}_\infty$. Let $\{\varphi_n\}$ be a sequence of the theorem obtained for $\boldsymbol{F}_2$. Put $\psi_n=\varphi_n|_{\boldsymbol{F}}$. Then ψ_n belongs to $A_p(\boldsymbol{F})$ ([9, Theorem 1]) and $\|\psi_n\|_{B_p(\boldsymbol{F})}\leq\|\psi_n\|_{B_p(\boldsymbol{F}_2)}\leq 1$ ([10, p. 146]). For $u\in A_p(\boldsymbol{F})$ define the function $\tilde{u}$ by $\tilde{u}(x)=u(x)$ if $x\in\boldsymbol{F}$ and $\tilde{u}(x)=0$ if $x\in\boldsymbol{F}_2\backslash\boldsymbol{F}$. Then by [9, Proposition 5] we have $\tilde{u}\in A_p(\boldsymbol{F}_2)$ and

$$\lim_{n\to\infty}\|\psi_n u-u\|_{A_p(\boldsymbol{F})}\leq\lim_{n\to\infty}\|\varphi_n\tilde{u}-\tilde{u}\|_{A_p(\boldsymbol{F}_2)}=0.$$

Hence the sequence $\{\psi_n\}$ has a required property. ■

REMARKS 2.7. (1) It can be shown that if G is weakly-amenable (in the sense of [2]), then $A_p(G)$ has an approximate identity $\{u_\alpha\}_{\alpha\in I}$ such that $\sup_{\alpha\in I}\|u_\alpha\|_{B_p}<+\infty$. In fact let $\{v_\alpha\}_{\alpha\in I}$ be an approximate identity of $A(G)=A_2(G)$ such that $\sup_{\alpha\in I}\|v_\alpha\|_{B_2}<+\infty$ and let $f\in L(G)$ be a non-negative function such that $\|f\|_1=1$. Then the net $\{f*v_\alpha\}_{\alpha\in I}$ has a required property.

(2) Since $\boldsymbol{F}_r$ is not amenable, $A_p(\boldsymbol{F}_r)$ does not possess a bounded approximate identity, so that the function φ_σ in the proof of Theorem 2.6 does not belong to $A_p(\boldsymbol{F}_r)$ for small σ. In fact, for $\sigma<\min\{1/p, 1/q\}\log(2r-1)$ where $1/p+1/q=1$, φ_σ does not belong to $A_p(\boldsymbol{F}_r)$, which is seen from the following fact: If $1<p\leq 2$ (resp. $2\leq p<\infty$), and if a function φ on $\boldsymbol{F}_r$ belongs to $A_p(\boldsymbol{F}_r)$, then the function $x\mapsto\varphi(x)e^{-\sigma|x|}$ belongs to $\ell^q(\boldsymbol{F}_r)$ (resp. $\ell^p(\boldsymbol{F}_r)$) for every $\sigma>0$. This can be deduced from [6, Lemma 8.4.7] and [7, Theorem 3.1].

3. Characterizations of amenability

In this section, we characterize the amenability of G in several ways using the notions of multipliers of $A_p(G)$ and $W_p(G)$.

Suppose that $1<p<\infty$, $1/p+1/q=1$ and $1\leq p'\leq\infty$. A bounded linear operator $\Phi: A_p(G)\to M(G)$ is called a multiplier of $A_p(G)$ into $M(G)$ if $\Phi(uv)=u\Phi(v)$ for every $u, v\in A_p(G)$, where $d(u\mu)=ud\mu$ for $u\in C_0(G)$ and $\mu\in M(G)$. We denote by $\mathscr{M}(A_p, M)$ the space of multipliers of $A_p(G)$ into $M(G)$. Similarly we define the space $\mathscr{M}(W_p, M)$ of multipliers of $W_p(G)$ into $M(G)$, the space $\mathscr{M}(A_p, L^{p'})$ of multipliers of $A_p(G)$ into $L^{p'}(G)$ and the space $\mathscr{M}(W_p, L^{p'})$ of multipliers of $W_p(G)$ into $L^{p'}(G)$.

For each $\mu\in M(G)$ the operator $\Phi_\mu: u\mapsto u\mu$ of $A_p(G)$ (resp. $W_p(G)$) into $M(G)$ is clearly an element of $\mathscr{M}(A_p, M)$ (resp. $\mathscr{M}(W_p, M)$) such that $\|\Phi_\mu\|\leq\|\mu\|$. Hence $M(G)$ is contractively imbedded in $\mathscr{M}(A_p, M)$ or $\mathscr{M}(W_p, M)$ by the natural imbedding $\mu\mapsto\Phi_\mu$. Analogously $L^{p'}(G)$ is contractively imbedded in $\mathscr{M}(A_p, L^{p'})$ or $\mathscr{M}(W_p, L^{p'})$.

Let $Q_p(G)$ denote the Banach space consisting of all functions $h=\sum_{i=1}^{\infty} u_i g_i$ with $u_i \in A_p(G)$ and $g_i \in C_0(G)$ satisfying $\sum_{i=1}^{\infty}\|u_i\|_{A_p}\|g_i\|_\infty < +\infty$, where the norm $\|h\|_{Q_p}$ is the infimum of $\sum_{i=1}^{\infty}\|u_i\|_{A_p}\|g_i\|_\infty$ for all such expressions. Also let $R_p(G)$ denote the Banach space consisting of all functions $h=\sum_{i=1}^{\infty} u_i g_i$ with $u_i \in W_p(G)$ and $g_i \in C_0(G)$ satisfying $\sum_{i=1}^{\infty}\|u_i\|_{W_p}\|g_i\|_\infty < +\infty$, where the norm $\|h\|_{R_p}$ is the infimum of $\sum_{i=1}^{\infty}\|u_i\|_{W_p}\|g_i\|_\infty$ for all such expressions. Note that $Q_p(G)$ and $R_p(G)$ are the subspaces of $C_0(G)$. Moreover $\|h\|_\infty \leq \|h\|_{Q_p}$ for $h \in Q_p(G)$, and $\|h\|_\infty \leq \|h\|_{R_p}$ for $h \in R_p(G)$.

LEMMA 3.1. *Let* $1<p<\infty$.

(1) *$\mathscr{M}(A_p, M)$ is the dual Banach space of $Q_p(G)$. The duality is given by*

$$\langle \Phi, h\rangle = \sum_{i=1}^{\infty}\langle \Phi(u_i), g_i\rangle, \quad \Phi \in \mathscr{M}(A_p, M), \quad h=\sum_{i=1}^{\infty} u_i g_i \in Q_p(G).$$

(2) *$\mathscr{M}(W_p, M)$ is the dual Banach space of $R_p(G)$. The duality is given by*

$$\langle \Phi, h\rangle = \sum_{i=1}^{\infty}\langle \Phi(u_i), g_i\rangle, \quad \Phi \in \mathscr{M}(W_p, M), \quad h=\sum_{i=1}^{\infty} u_i g_i \in R_p(G).$$

PROOF: We only prove (1) (the proof of (2) is analogous). Let $F \in Q_p(G)^*$ and $u \in A_p(G)$. Define $\mathscr{F}_u(g)=F(ug)$ for $g \in C_0(G)$. Then $\mathscr{F}_u$ is a bounded linear functional on $C_0(G)$ and $|\mathscr{F}_u(g)| \leq \|F\|\,\|u\|_{A_p}\|g\|_\infty$. Therefore there exists $\mu_u \in M(G)$ such that $\mathscr{F}_u(g)=\langle \mu_u, g\rangle$, $g \in C_0(G)$ and $\|\mu_u\| \leq \|F\|\,\|u\|_{A_p}$. The mapping $\Phi : u \mapsto \mu_u$ defines a bounded linear operator of $A_p(G)$ into $M(G)$ with $\|\Phi\| \leq \|F\|$. It is easily verified that $\Phi \in \mathscr{M}(A_p, M)$.

Conversely let $\Phi \in \mathscr{M}(A_p, M)$. Define $F(h)=\sum_{i=1}^{\infty}\langle \Phi(u_i), g_i\rangle$ for $h \in Q_p(G)$, $h=\sum_{i=1}^{\infty} u_i g_i$. Let us show that F is well defined, that is, if $h\equiv 0$ then $F(h)=0$. Let I be the set of all compact subsets of G, and $\{u_K\}_{K\in I}$ a net of elements of $A_p(G)$ such that $v_K \equiv 1$ on K and $0 \leq v_K \leq 1$ ([4, Lemme 3.2]). For given $\varepsilon>0$, choose $N \geq 1$ such that $\sum_{i>N}\|u_i\|_{A_p}\|g_i\|_\infty < \varepsilon$. Let $K \in I$ be such that $\sum_{i=1}^{N}\|u_i\|_{A_p}\|g_i v_K - g_i\|_\infty < \varepsilon$. Since $v_K \in A_p(G)$, we have

$$\sum_{i=1}^{\infty}\langle \Phi(u_i), g_i v_K\rangle = \sum_{i=1}^{\infty}\langle \Phi(u_i v_K), g_i\rangle = \sum_{i=1}^{\infty}\langle \Phi(v_K), u_i g_i\rangle = \langle \Phi(v_K), h\rangle = 0.$$

Hence we have

$$\begin{aligned} |F(h)| &= \left|\sum_{i=1}^{\infty}\langle \Phi(u_i), g_i - g_i v_K\rangle\right| \\ &\leq \sum_{i=1}^{\infty}\|\Phi\|\,\|u_i\|_{A_p}\|g_i - g_i v_K\|_\infty \end{aligned}$$

$$\leq \|\Phi\| \sum_{i=1}^{N} \|u_i\|_{A_p} \|g_i - g_i v_K\|_\infty + \|\Phi\| \sum_{i>N} \|u_i\|_{A_p} \|g_i\|_\infty$$
$$\leq 2\|\Phi\|\varepsilon.$$

Since $\varepsilon>0$ is arbitrary, we have $F(h)=0$. Now it is clear that $F\in Q_p(G)^*$ and $\|F\|\leq\|\Phi\|$. ■

THEOREM 3.2. *Let $1<p<\infty$ and $1\leq p'<\infty$. Then the following conditions are equivalent:*

(1) *G is amenable;*
(2) $C_0(G)=A_p(G)\cdot C_0(G)$;
(3) $C_0(G)=W_p(G)\cdot C_0(G)$;
(4) $\mathscr{M}(A_p, M)\simeq M(G)$;
(5) $\mathscr{M}(W_p, M)\simeq M(G)$;
$(6)_{p'}$ $\mathscr{M}(A_p, L^{p'})\simeq L^{p'}(G)$;
$(7)_{p'}$ $\mathscr{M}(W_p, L^{p'})\simeq L^{p'}(G)$.

Here each of (4)-$(7)_{p'}$ means that the natural imbedding is surjective isomorphism. Moreover each isomorphism in (4)-$(7)_{p'}$ is isometric if G is amenable.

PROOF: (1) ⇒ (2). Since G is amenable, $A_p(G)$ has a bounded approximate identity ([9, 15]). Therefore (2) follows from [13, (32. 22)]. (2) ⇒ (3) is clear because $A_p(G)\subset W_p(G)$ in general.

(2) ⇒ (4). By (2), $Q_p(G)$ is isomorphic with $C_0(G)$. Therefore it follows from Lemma 3.1(1) that $\mathscr{M}(A_p, M)\simeq C_0(G)^*\simeq M(G)$. (3) ⇒ (5) is similarly shown by Lemma 3.1(2).

(4) ⇒ (1). By (4) there exists a positive constant C such that

$$(3.2)\qquad \|f\|_1\leq C\|\Phi_f\|_{\mathscr{M}(A_p,M)},\qquad f\in L^1(G),$$

where $\Phi_f : u\mapsto uf$ for $u\in A_p(G)$. Also we have

$$(3.2)\qquad \begin{aligned}\|\Phi_f\|_{\mathscr{M}(A_p,M)}&=\sup\{|\langle uf, g\rangle| : u\in A_p(G), \|u\|_{A_p}\leq 1, g\in C_0(G), \|g\|_\infty\leq 1\}\\ &=\sup\{|\langle u, fg\rangle| : u\in A_p(G), \|u\|_{A_p}\leq 1, g\in C_0(G), \|g\|_\infty\leq 1\}\\ &\leq\sup\{\|\lambda(fg)\|_{PM_q} : g\in C_0(G), \|g\|_\infty\leq 1\}.\end{aligned}$$

Let $f\in L^1(G)$ and $f\geq 0$. Then for every $g\in C_0(G)$ and $h\in L^q(G)$, we have

$$\|gf*h\|_q\leq\| \, \|g\|_\infty f*|h| \, \|_q\leq\|g\|_\infty\|\lambda(f)\|_{PM_q}\|h\|_q,$$

so that $\|\lambda(gf)\|_{PM_q}\leq\|g\|_\infty\|\lambda(f)\|_{PM_q}$. From this and (3.1) and (3.2) it follows that

$$(3.3)\qquad \|f\|_1\leq C\|\lambda(f)\|_{PM_q},\qquad f\in L^1(G),\ f\geq 0.$$

For $0\leq f\in L^1(G)$, we have

$$\|f\|_1^n=\|f^{(*n)}\|_1\leq C\|\lambda(f^{(*n)})\|_{PM_q}\leq C\|\lambda(f)\|_{PM_q}^n$$

by (3.3), so that $\|f\|_1\leq C^{1/n}\|\lambda(f)\|_{PM_q}$ for every $n\in \boldsymbol{N}$. Hence we have

$$\|f\|_1=\|\lambda(f)\|_{PM_q}, \qquad f\in L^1(G),\ f\geq 0.$$

It follows from [14] that G is amenable. (5) $\Rightarrow$ (1) is analogously shown.

(1) $\Rightarrow$ $(6)_{p'}$. Let $\Phi\in\mathscr{M}(A_p, L^{p'})$ and let $\{u_\alpha\}_{\alpha\in I}$ be a bounded approximate identity of $A_p(G)$ with $\|u_\alpha\|_{A_p}\leq 1$. First consider the case $p'>1$. Since $\{\Phi(u_\alpha)\}_{\alpha\in I}$ has a w^*-accumulation point f in $L^{p'}(G)$ by boundedness, it may be assumed that $w^*\text{-}\lim_\alpha\Phi(u_\alpha)=f$. Then for every $u\in A_p(G)$, we have

$$\begin{aligned}\Phi(u)&=w^*\text{-}\lim_\alpha \Phi(uu_\alpha)\\ &=w^*\text{-}\lim_\alpha u\Phi(u_\alpha)=uf.\end{aligned}$$

Moreover since

$$\|f\|_{p'}\leq\liminf_\alpha \|\Phi(u_\alpha)\|_{p'}\leq\|\Phi\|,$$

we obtain $\Phi=\Phi_f$ and $\|\Phi\|=\|f\|_{p'}$. Next when $p'=1$, since $\Phi(u_\alpha)\in L^1(G)\subset M(G)$, we may assume that the net $\{\Phi(u_\alpha)\}_{\alpha\in I}$ converges to some element $\mu\in M(G)$ in the w^*-topology. The same argument as above shows that $\Phi(u)=u\mu\in L^1(G)$ for every $u\in A_p(G)$. It follows that $\mu\in L^1(G)$. Thus we have $(6)_{p'}$ for any $p'\geq 1$. (1) $\Rightarrow$ $(7)_{p'}$ is clear because (1) implies $1\in MA_p(G)=W_p(G)$ ([1]).

$(6)_{p'}\Rightarrow(6)_1$. We may suppose that $p'>1$. Let I be the set of all compact subsets of G ordered by inclusion. For each $K\in I$, take $u_K\in A_p(G)\cap L(G)$ such that $u_K\equiv 1$ on K and $0\leq u_K\leq 1$. Let $\Phi\in\mathscr{M}(A_p, L^1)$. For any $K_1, K_2\in I$, since $\Phi(u_{K_1})u_{K_2}=\Phi(u_{K_1}u_{K_2})=u_{K_1}\Phi(u_{K_2})$, we have $\Phi(u_{K_1})|_{K_1\cap K_2}=\Phi(u_{K_2})|_{K_1\cap K_2}$. Hence there is a measurable function h on G such that $h|_K=\Phi(u_K)|_K$ for all $K\in I$. If $u\in A_p(G)\cap L(G)$, then we have

$$\Phi(u)=\lim_K \Phi(uu_K)=\lim_K u\Phi(u_K)=uh,$$

hence

$$\begin{aligned}\|u|h|^{1/p'}\|_{p'}^{p'}&\leq\|u\|_\infty^{p'-1}\|uh\|_1\\ &\leq\|u\|_\infty^{p'-1}\|u\|_{A_p}\|\Phi\|\leq\|\Phi\|\ \|u\|_{A_p}^{p'}.\end{aligned}$$

This shows that $|h|^{1/p'}$ defines a multiplier of $A_p(G)$ into $L^{p'}(G)$. Hence by $(6)_{p'}$ we have $|h|^{1/p'}\in L^{p'}(G)$, so that $h\in L^1(G)$. Thus we obtain $(6)_1$. The proof of $(7)_{p'}\Rightarrow(7)_1$ is similar.

If we replace $\mathscr{M}(A_p, M)$ with $\mathscr{M}(A_p, L^1)$ in the proof of (4) ⇒ (1), we can also obtain the proof of $(6)_1$ ⇒ (1), and $(7)_1$ ⇒ (1) is analogously shown.

Finally if G is amenable, the same argument as in the proof of (1) ⇒ $(6)_{p'}$ shows that each isomorphism in (4)-$(7)_{p'}$ is isometric. ■

REMARKS 3.3. (1) When $p'=\infty$, the implication $(6)_{p'}$ ⇒ (1) of Theorem 3.2 does not hold. In fact, if G is discrete and $\Phi \in \mathscr{M}(A_p, \ell^\infty)$, then the function $\varphi(x)=\Phi(\delta_x)(x)$ belongs to $\ell^\infty(G)$ and $\Phi(u)=u\varphi$ for $u \in A_p(G)$. Hence for any discrete group G, $\mathscr{M}(A_p, \ell^\infty)$ is isomorphic to $\ell^\infty(G)$.

(2) From [6, Lemma 8.4.7] it can be shown that for a free group $G = \boldsymbol{F}_r$, the function $\varphi(x)=(2r-1)^{-|x|}$ belongs to $\mathscr{M}(A_p, \ell^1)$, but not to $\ell^1(G)$.

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