Hokkaido Mathematical Journal Vol. 20 (1991) p. 503-518

Lifting Fourier-Stieltjes transforms and transferring cocycles

Wojciech CHOJNACKI
(Received March 26, 1990)

Abstract

We exhibit a class of linear liftings of Fourier-Stieltjes transforms defined on a closed subgroup of a locally compact Abelian group to Fourier-Stieltjes transforms defined on the whole group. Using these liftings, we establish a result about unitary representations associated with cocycles on compact Abelian groups with dense action.

0. Introduction

Let G be a locally compact Abelian group and $\hat{G}$ be the dual group of G. Let $A(G)$ be the space of Fourier transforms of Haar-integrable functions on $\hat{G}$, $B(G)$ be the space of Fourier transforms of complex finite regular Borel measures on $\hat{G}$, $B_{+}^{1}(G)$ be the set of Fourier transforms of regular Borel probability measures on $\hat{G}$, and $B_{s}(G)$ be the space of Fourier transforms of finite regular Borel measures on $\hat{G}$ singular with respect to Haar measure. Let G_{0} be a closed subgroup of G and R be the operator of the restriction to G_{0} of functions defined on G. A well-known elementary result states that $R(A(G))=A(G_{0})$ and $R(B(G))=B(G_{0})$ (cf. [13, Theorems 2.7.2 and 2.7.4]). J. Inoue [10] constructed a linear isometry I from $B(G_{0})$ into $B(G)$, carrying $A(G_{0})$ in $A(G)$, $B_{+}^{1}(G_{0})$ in $B_{+}^{1}(G)$, and $B_{s}(G_{0})$ in $B_{s}(G)$, such that RI is the identity on $B(G_{0})$ and, for each $\psi\in B(G_{0})$, the support of $I\psi$ is contained in the set of all elements of the form $x+y$ with x in the support of ψ and y in any given neighbourhood of 0 in G. Inoue's construction, relying on a subtle reduction to the case in which G_{0} is discrete and in which such an isometry can be expressed by a simple formula (cf. [9, Theorem A. 7. 1]) is fairly complicated and leads to a rather non-transparent formula for I. In this paper, we reveal a class of isometries with properties as above, which have a strikingly simple form. Taking advantage of the special shape of these isometries, we establish a result about transferring cocycles from closed subgroups of compact Abelian groups with dense action to the entire groups. The latter result will provide motivation to the proposed approach.

1. A lifting theorem

With G a locally compact Abelian group, let m_G be the Haar measure on G and $\mathscr{K}(G)$ be the space of all complex continuous functions on G with compact support. With G_0 a closed subgroup of G, let G/G_0 be the corresponding quotient group and π be the canonical epimorphism from G onto G/G_0. Suppose the Haar measure on G/G_0 is normalized so that

$$\int_G f(x)dm_G(x)=\int_{G/G_0}\left[\int_{G_0} f(x+y)\,dm_{G_0}(x)\right]dm_{G/G_0}(\dot{y}) \quad (\dot{y}=\pi(y))$$

for all $f\in\mathscr{K}(G)$; here we adopt the standard notational convention regarding double integrals in which one integration is performed over a subgroup and the other over the corresponding quotient group (cf. [3, p. 44; 5, p. 249]).

Let $C(G)$ be the space of all complex bounded continuous functions on G, $C_0(G)$ be the space of all complex continuous functions on G vanishing at infinity, and, for $1\leq p<+\infty$, let $L^p(G)$ be the pth Lebesgue space based on m_G.

Let $\mathscr{F}$ be the Fourier transformation defined by

$$\mathscr{F}f(\gamma)=\int_G f(x)(x,-\gamma)dm_G(x) \quad (f\in L^1(G),\ \gamma\in\hat{G}).$$

We normalize the Haar measure on $\hat{G}$ so that

$$f(x)=\int_{\hat{G}}\mathscr{F}f(\gamma)(x,\gamma)\ dm_{\hat{G}}(\gamma) \quad (x\in G),$$

whenever $f\in L^1(G)\cap C(G)$ and $\mathscr{F}f\in L^1(\hat{G})$.

Let $G_0^\perp$ be the annihilator of G_0 in $\hat{G}$ defined as

$$\{\gamma\in\hat{G}:(x,\gamma)=1 \text{ for } x\in G_0\}.$$

Let ρ be the canonical epimorphism from $\hat{G}$ onto $\hat{G}/G_0^\perp$. With the normalization of Haar measures on mutually dual groups adopted above, we have

$$\int_{\hat{G}} f(\gamma)dm_{\hat{G}}(\gamma)=\int_{\hat{G}/G_0^\perp}\left[\int_{G_0^\perp} f(\gamma+\xi)\,dm_{G_0^\perp}(\xi)\right]dm_{\hat{G}/G_0^\perp}(\dot{\gamma})(\dot{\gamma}=\rho(\gamma))$$

for all $f\in\mathscr{K}(\hat{G})$.

Let $M(G)$ be the space of all complex finite regular Borel measures on G, $M_s(G)$ be the space of measures in $M(G)$ singular with respect to Haar measure, $M_a(G)$ be the space of atomic measures in $M(G)$, and $M_0(G)$ be the space of measures $\mu\in M(G)$ such that $\mathscr{F}\mu\in C_0(\widehat{G})$, where $\mathscr{F}\mu$, the Fourier transform of μ, is defined by

$$\mathscr{F}\mu(\gamma)=\int_G(x,-\gamma)d\mu(x) \quad (\gamma\in\widehat{G}).$$

We identify the space of measures in $M(G)$ absolutely continuous with respect to Haar measure with the space $L^1(G)$. We also let

$$B_a(\widehat{G})=\{\mathscr{F}\mu:\mu\in M_a(G)\}$$

and

$$B_0(\widehat{G})=\{\mathscr{F}\mu:\mu\in M_0(G)\}.$$

For any space E of functions or measures, we denote by E_+ the set of all non-negative elements of E.

Given a topological space X, let $\mathscr{B}(X)$ be the space of all complex bounded Borel functions on X, and $\mathfrak{B}(X)$ be the σ-algebra of Borel subsets of X.

Suppose h is a function in $C_+(\widehat{G})$ such that, for each $\gamma\in\widehat{G}$,

$$\int_{G_0^\perp}h(\gamma+\xi)dm_{G_0^\perp}(\xi)=1. \tag{1. 1}$$

As we shall see shortly, such functions exist in abundance. For each $f\in\mathscr{K}(\widehat{G})$, the function

$$g_f(\gamma)=\int_{G_0^\perp}h(\gamma+\xi)f(\gamma+\xi)dm_{G_0^\perp}(\xi) \quad (\gamma\in\widehat{G})$$

is continuous on $\widehat{G}$ and $g_f(\gamma+\eta)=g_f(\gamma)$ for each $\gamma\in\widehat{G}$ and each $\eta\in G_0^\perp$, so that $g_f=\tilde{g}_f\circ\rho$, where $\tilde{g}_f$ is a uniquely determined continuous function on $\widehat{G}/G_0^\perp$. Moreover, $\tilde{g}_f$ has compact support (cf. [5, Theorem 14. 1. 5. 5]), and if we let $\|\cdot\|_{\infty,X}$ denote the supremum norm over a set X, then $\|\tilde{g}_f\|_{\infty,\widehat{G}/G_0^\perp}=\|g_f\|_{\infty,\widehat{G}}\le\|f\|_{\infty,\widehat{G}}$. Thus, by the Riesz theorem, for each $\mu\in M(\widehat{G}/G_0^\perp)$ the bounded linear functional

$$f\to\int_{\widehat{G}/G_0^\perp}\tilde{g}_f d\mu \quad (f\in\mathscr{K}(G))$$

can be represented as

$$f\to\int_{\widehat{G}}fdJ_\mu$$

for a unique J_μ in $M(\widehat{G})$. We claim that given $f\in\mathscr{B}(\widehat{G})$, g_f is in $\mathscr{B}(\widehat{G}/G_0^\perp)$ and, for each $\mu\in M(\widehat{G}/G_0^\perp)$,

$$\int_{\hat{G}} f dJ_{\mu} = \int_{\hat{G}/G_0^{\perp}} \tilde{g}_f d\mu. \tag{1.2}$$

Let f be a non-negative lower semicontinuous function on $\hat{G}$ and $(f_\alpha)_{\alpha\in A}$ be an increasing net in $\mathscr{K}_+(\hat{G})$ such that $\sup_\alpha f_\alpha = f$. Then, by the generalized monotone convergence theorem (cf. [3, Chapitre 4, §1, n° 1, Théorème 1]), $\tilde{g}_f = \sup_\alpha \tilde{g}_{f_\alpha}$ and, since the $\tilde{g}_{f_\alpha}$ $(\alpha\in A)$ are in $\mathscr{K}_+(\hat{G}/G_0^{\perp})$, $\tilde{g}_f$ is lower semicontinuous. Let $\mu\in M_+(\hat{G}/G_0^{\perp})$. Then, still by the generalized monotone convergence theorem,

$$\int_{\hat{G}} f dJ_{\mu} = \sup_{\alpha} \int_{\hat{G}} f_\alpha dJ_{\mu} = \sup_{\alpha} \int_{\hat{G}/G_0^{\perp}} \tilde{g}_{f_\alpha} d\mu = \int_{\hat{G}/G_0^{\perp}} \tilde{g}_f d\mu.$$

In particular, for each open subset U of $\hat{G}$, if we let 1_U denote the characteristic function of U, then $\tilde{g}_{1_U}$ is lower semicontinuous and (1.2) holds with $f=1_U$. Let

$$\mathscr{D} = \{E\in\mathfrak{B}(\hat{G}) : \tilde{g}_{1_E}\in\mathfrak{B}(\hat{G}/G_0^{\perp}) \text{ and (1.2) holds for } f=1_E\}.$$

Clearly, $\mathscr{D}$ is a Dynkin class: 1° $\hat{G}$ is in $\mathscr{D}$; 2° if E is in $\mathscr{D}$, then $\hat{G}\backslash E$ is in $\mathscr{D}$; 3° if $(E_n)_{n\in N}$ is a sequence of pairwise disjoint sets in $\mathscr{D}$, then $\bigcup_{n\in N} E_n$ is in $\mathscr{D}$. According to the main theorem about Dynkin classes, if Ω is a non-empty set and $\mathscr{E}$ is a family of subsets of Ω closed under finite intersections, then the smallest Dynkin class containing $\mathscr{E}$ coincides with the σ-algebra generated by $\mathscr{E}$ (cf. [2, Theorem 1.2.4]). Applying this result in the situation where $\Omega=\hat{G}$ and where $\mathscr{E}$ is the family of all open subsets of $\hat{G}$, we conclude that $\mathscr{D}=\mathfrak{B}(\hat{G})$, so that the claim is valid for all $f=1_E$ $(E\in\mathfrak{B}(\hat{G}))$, and next, by the usual extension, for all f in $\mathscr{B}(\hat{G})$. The final step consists in an obvious extension of the validity of (1.2) to all measures in $M(\hat{G}/G_0^{\perp})$.

The mapping $J: \mu\to J_\mu$ is clearly a linear operator from $M(\hat{G}/G_0^{\perp})$ into $M(\hat{G})$. Its basic properties are listed in the following

THEOREM 1.1. *The following hold true:*
(i) *J is an isometry;*
(ii) *$J_\mu\in M_+(\hat{G})$ if and only if $\mu\in M_+(\hat{G}/G_0^{\perp})$;*
(iii) *$J_\mu\in L^1(\hat{G})$ if and only if $\mu\in L^1(\hat{G}/G_0^{\perp})$;*
(iv) *if $\mu\in M_a(\hat{G}/G_0^{\perp})$ and G/G_0 is compact, then $J_\mu\in M_a(\hat{G})$.*

PROOF. (i) For each $\mu\in M(\hat{G})$, let $\rho_*\mu$ be the image of μ by ρ given by

$$\rho_*\mu(B) = \mu(\rho^{-1}(B)) \quad (B\in\mathfrak{B}(\hat{G}/G_0^{\perp})).$$

The mapping $\rho_*:\mu\to\rho_*\mu$ is clearly a linear operator from $M(\hat{G})$ into $M(\hat{G}/G_0^\perp)$. One verifies at once that the composition ρ_*J is the identity operator in $M(\hat{G}/G_0^\perp)$. Since $\|J\|\leq 1$ and $\|\rho_*\|\leq 1$, it follows that J is an isometry.

(ii) As J is an isometry and $J_\mu(\hat{G}/G_0^\perp)=\mu(\hat{G})$ for each $\mu\in M(\hat{G}/G_0^\perp)$, it is clear that $J_\mu\geq 0$ if and only if $\mu\geq 0$.

(iii) If $\mu\in M(\hat{G}/G_0^\perp)$ is absolutely continuous with respect to $m_{\hat{G}/G_0^\perp}$ and f is the corresponding density, then, as one directly verifies, J_μ is absolutely continuous with respect to $m_{\hat{G}}$ with density $h(f\circ\rho)$. Conversely, if for $\mu\in M(\hat{G}/G_0^\perp)$, J_μ is absolutely continuous with respect to $m_{\hat{G}}$ and g is the corresponding density, then, as it follows immediately from the identity $\rho_*J_\mu=\mu$, μ is absolutely continuous with respect to $m_{\hat{G}/G_0^\perp}$ with density d defined as

$$d(\dot{\gamma})=\int_{G_0^\perp}g(\gamma+\xi)dm_{G_0^\perp}(\xi)\quad(\dot{\gamma}=\rho(\gamma),\ \gamma\in\hat{G}).$$

(iv) Suppose that G/G_0 is compact. Then $G_0^\perp$ is discrete. Let $\mu=\sum_{s\in S}a_s\delta_s$, where S is a countable subset of $\hat{G}/G_0^\perp$ and the a_s $(s\in S)$ are complex numbers such that $\sum_{s\in S}|a_s|<+\infty$ (for each $s\in S$, δ_s stands, of course, for the Dirac measure at s). For each $s\in S$, choose $\gamma_s\in\hat{G}$ so that $\rho(\gamma_s)=s$. Then, as easily seen,

$$J_\mu=\sum_{s\in S}\sum_{\xi\in G_0^\perp}a_sh(\gamma_s+\xi)\delta_{\gamma_s+\xi}$$

showing that J_μ is atomic along with μ.

The proof is complete.

Now that the role of functions satisfying (1. 1) is clear, we turn to the question of the existence of such functions.

Let $\mathscr{U}$ be a locally finite covering of $\hat{G}/G_0^\perp$ consisting of open relatively compact sets. For each $U\in\mathscr{U}$, let f_U be a function in $\mathscr{K}_+(\hat{G})$ such that $\rho(\{\gamma:f_U(\gamma)>0\})\supset U$. Let g_U be the function on $\hat{G}$ defined by

$$g_U(\gamma)=\begin{cases}f_U(\gamma)\left[\int_{G_0^\perp}f_U(\gamma+\xi)dm_{G_0^\perp}(\xi)\right]^{-1}, & \text{if } \gamma\in\rho^{-1}(U)\,;\\ 0, & \text{if } \gamma\in\hat{G}\backslash\rho^{-1}(U).\end{cases}$$

The latter definition makes sense for if $\gamma\in\rho^{-1}(U)$, then $f_U(\gamma+\chi)\neq 0$ for some $\chi\in G_0^\perp$, whence

$$\int_{G_0^\perp}f_U(\gamma+\xi)dm_{G_0^\perp}(\xi)>0.$$

Notice that g_U is lower semicontinuous and, for each $\gamma\in\rho^{-1}(U)$,

$$\int_{G_0^\perp} g_U(\gamma+\xi)dm_{G_0^\perp}(\xi)=1. \tag{1. 3}$$

Let $(\varphi_U)_{U\in\mathscr{U}}$ be a partition of unity subordinate to $\mathscr{U}$. Set $f=\sum_{U\in\mathscr{U}}(\varphi_U\circ\rho)g_U$. Clearly, f is a non-negative lower semicontinuous function on $\hat{G}$. If we arbitrarily fix an element γ of $\hat{G}$, then, for each $\xi\in G_0^\perp$,

$$f(\gamma+\xi)=\sum_{U\in\mathscr{U}}\varphi_U(\rho(\gamma))g_U(\gamma+\xi).$$

The right-hand sum has only a finite number of non-zero summands corresponding to those U's for which $\varphi_U(\rho(\gamma))>0$ or, equivalently, for which $\gamma\in\rho^{-1}(U)$. Hence

$$\int_{G_0^\perp} f(\gamma+\xi)dm_{G_0^\perp}(\xi)=\sum_{U\in\mathscr{U}}\varphi_U(\rho(\gamma))\int_{G_0^\perp} g_U(\gamma+\xi)dm_{G_0^\perp}(\xi).$$

Taking into account (1. 3), we see that f satisfies (1. 1).

Let p be a function in $\mathscr{K}_+(G_0^\perp)$ such that

$$\int_{G_0^\perp} p(\eta)dm_{G_0^\perp}(\eta)=1. \tag{1. 4}$$

For each $\gamma\in\hat{G}$, set

$$r(\gamma)=\int_{G_0^\perp} f(\gamma+\eta)p(\eta)dm_{G_0^\perp}(\eta).$$

Repeating *mutatis mutandis* the argument used in the proof of (1. 2), we see that r is a non-negative lower semicontinuous function on $\hat{G}$. By (1. 1) applied to f and by (1. 4),

$$\begin{aligned}\int_{G_0^\perp} r(\gamma+\xi)dm_{G_0^\perp}(\xi)&=\int_{G_0^\perp}\left[\int_{G_0^\perp} f(\gamma+\xi+\eta)p(\eta)dm_{G_0^\perp}(\eta)\right]dm_{G_0^\perp}(\xi)\\&=\int_{G_0^\perp}\left[\int_{G_0^\perp} f(\gamma+\xi+\eta)dm_{G_0^\perp}(\xi)\right]g(\eta)dm_{G_0^\perp}(\eta)\\&=1,\end{aligned}$$

so r satisfies (1. 1) and $\|r\|_{\infty,\hat{G}}\leq\|p\|_{\infty,G_0^\perp}$. Now, letting $*$ denote convolution, if s is a function in $\mathscr{K}_+(\hat{G})$ such that $\int_{\hat{G}} s dm_{\hat{G}}=1$, then $h=r*s$ is a function in $C_+(\hat{G})$ satisfying (1. 1).

Let $\mathscr{F}h$ be the Fourier transform of h in the sense of pseudomeasures, that is, $\mathscr{F}h$ is the element of the dual space $A(G)'$ of $A(G)$ given by

$$<\mathscr{F}h,\ \varphi>=\int_{\hat{G}} h(\xi)w(-\xi)dm_{\hat{G}}(\xi)\quad (\varphi\in A(G)\,;\ \varphi=\mathscr{F}w,\ w\in L^1(\hat{G})).$$

It turns out that the support of $\mathscr{F}h$ may always be assumed to be arbirarily small.

Indeed, let U be a neighbourhood of 0 in G and V be a compact symmetric neighbourhood of 0 in G such that $V+V\subset U$, where $V+V=\{x+y : x, y\in V\}$. Set $\psi=(m_G(V))^{-1}(\mathscr{F}1_V)^2$. A standard argument shows that ψ is a non-negative element of $C_0(\hat{G})\cap L^1(\hat{G})$ with $\int_{\hat{G}}\psi dm_{\hat{G}}=1$ whose Fourier transform, $(m_G(V))^{-1}(1_V * 1_V)$, is suppored by U. Now, if we set $\tilde{h}=\psi * h$, then $\tilde{h}$ is an element of $C_+(\hat{G})$ satisfying (1. 1) whose Fourier transform, $\mathscr{F}\psi\mathscr{F}h$, has support in U.

Given a function f on G and an element x of G, let T_xf be the translate of f by x, that is,

$$T_xf(y)=f(x+y) \quad (y\in G).$$

We recall that if $x\in G$ and $S\in A(G)'$, then the translate T_xS of S by x is the pseudomeasure

$$<T_xS, \varphi>=<S, T_{-x}\varphi> \quad (\varphi\in A(G)).$$

Let $\mathscr{A}(G)$ be the space of all compactly supported Fourier transforms of elements of $L^1(\hat{G})\cap C(\hat{G})$. With S in $A(G)'$, we shall say that the function $G_0\ni x\rightarrow T_xS\in A(G)'$ is weakly integrable whenever $G_0\ni x\rightarrow <T_xS, \varphi>$ is in $L^1(G_0)$ for every $\varphi\in\mathscr{A}(G)$. Notice that if f is any $m_{\hat{G}}$-essentially bounded $m_{\hat{G}}$-measurable function on $\hat{G}$ whose Fourier transform has compact support, then $G_0\ni x\rightarrow T_x\mathscr{F}f$ is weakly integrable. In the light of the previous paragraph it is clear that the assumptions about the function h appearing in the theorem to follow are consistent.

THEOREM 1. 2. *Let h be a function in $C_+(\hat{G})$ satisfying* (1. 1) *such that $G_0\ni x\rightarrow T_x \mathscr{F}h\in A(G)'$ is weakly integrable. Then, for each $\mu\in M(\hat{G}/G_0^\perp)$ and each $\varphi\in\mathscr{A}(G)$,*

$$\int_G\varphi\mathscr{F}J_\mu dm_G=\int_{G_0}<T_{-x}\mathscr{F}h, \varphi>\mathscr{F}\mu(x)dm_{G_0}(x).$$

PROOF. For each $\varphi\in\mathscr{A}(G)$ with $\varphi=\mathscr{F}w$ $(w\in L^1(\hat{G})\cap C(\hat{G}))$, we have

$$\int_{\hat{G}/G_0^\perp}\left[\int_{G_0^\perp}h(\gamma+\xi)|w(-\gamma-\xi)|dm_{G_0^\perp}(\xi)\right]dm_{\hat{G}/G_0^\perp}(\dot{\gamma})$$
$$\leq\|h\|_{\infty,\hat{G}}\|w\|_{1,\hat{G}} \quad (\dot{\gamma}=\rho(\gamma)),$$

where $\|\cdot\|_{1,\hat{G}}$ denotes the $L^1(\hat{G})$ norm, so that $\tilde{g}_{w^*}$, where $w^*(\gamma)=w(-\gamma)$ for all $\gamma\in\hat{G}$, is in $L^1(\hat{G}/G_0^\perp)\cap C(\hat{G}/G_0^\perp)$ (cf. [3, Chapitre 7, § 2, n° 3, Proposition 3; 5, Theorem 14.4.5]). Remembering that $(\hat{G}/G_0^{\perp})^{\widehat{}}$ can canonically be identified with G_0, for each $x \in G_0$, we have

$$\begin{aligned}\mathscr{F}\tilde{g}_{w^*}(x) &= \int_{\hat{G}/G_0^{\perp}} \tilde{g}_{w^*}(\dot{\gamma})(x, -\dot{\gamma})\, dm_{\hat{G}/G_0^{\perp}}(\dot{\gamma}) \\ &= \int_{\hat{G}/G_0^{\perp}} \left[\int_{G_0^{\perp}} h(\gamma+\xi) w(-\gamma-\xi)(x, -\gamma-\xi)\, dm_{G_0^{\perp}}(\xi)\right] dm_{\hat{G}/G_0^{\perp}}(\dot{\gamma}).\end{aligned}$$

Given $\gamma \in \hat{G}$, set $v(\gamma) = w(\gamma)(x, \gamma)$. Then, of course, $\mathscr{F}v = T_{-x}\varphi$ and

$$\begin{aligned}\mathscr{F}\tilde{g}_{w^*}(x) &= \int_{\hat{G}} h(\gamma) v(-\gamma)\, dm_{\hat{G}}(\gamma) = <\mathscr{F}h, T_{-x}\varphi> \\ &= <T_x\mathscr{F}h, \varphi>.\end{aligned}$$

Since $G_0 \ni x \to <T_x\mathscr{F}h, \varphi> \in \boldsymbol{C}$ is in $L^1(G_0)$ and $\tilde{g}_{w^*}$ is continuous, it follows that

$$\tilde{g}_{w^*}(\dot{\gamma}) = \int_{G_0} <T_x\mathscr{F}h, \varphi>(x, \dot{\gamma})\, dm_{G_0}(x) \quad (\dot{\gamma} = \rho(\gamma)).$$

Hence, in view of (1.2), for each $\mu \in M(\hat{G}/G_0^{\perp})$,

$$\begin{aligned}\int_G \varphi \mathscr{F} J_\mu\, dm_G &= \int_{\hat{G}} w^*\, dJ_\mu = \int_{\hat{G}/G_0^{\perp}} \tilde{g}_{w^*}\, d\mu \\ &= \int_{G_0} <T_x\mathscr{F}h, \varphi> \left[\int_{\hat{G}/G_0^{\perp}} (x, \dot{\gamma})\, d\mu(\dot{\gamma})\right] dm_{G_0}(x) \\ &= \int_{G_0} <T_x\mathscr{F}h, \varphi> \mathscr{F}\mu(-x)\, dm_{G_0}(x) \\ &= \int_{G_0} <T_{-x}\mathscr{F}h, \varphi> \mathscr{F}\mu(x)\, dm_{G_0}(x).\end{aligned}$$

The proof is complete.

Now we are in a position to state the main result of this section. It will be a minor generalisation of Inoue's result mentioned in the introduction.

THEOREM 1.3. *Let h be a function in $C_+(\hat{G})$ satisfying* (1.1), *and let I be the linear operator from $B(G_0)$ into $B(G)$ defined by*

$$I\psi = \mathscr{F}J_\mu \ (\psi \in B(G_0);\ \psi = \mathscr{F}\mu,\ \mu \in M(\hat{G}/G_0^{\perp})).$$

Then

(i) *I is an isometry such that RI is the identity on $B(G)$;*

(ii) $I(A(G_0)) \subset I(A(G))$;

(iii) $I(B_+^1(G_0)) \subset B_+^1(G)$;

(iv) $I(B_s(G_0)) \subset B_s(G)$;

(v) *if G/G_0 is compact, then $I(B_a(G_0)) \subset B_a(G)$.*

If, for a given neighbourhood U of 0 in G, h is such that supp $\mathscr{F}h \subset U$, then

(vi) *supp $I\psi \subset$ supp $\psi + U$ for each $\psi \in B(G_0)$.*

PROOF. (i) That I is an isometry follows immediately from Theorem 1.1(i). By (1.1) and (1.2), for each $\psi \in B(G_0)$ with $\psi = \mathscr{F}\mu$ ($\mu \in M(\hat{G}/G_0^{\perp})$) and each $x \in G_0$, we have

$$\begin{aligned} I\psi(x) &= \int_{\hat{G}/G_0^{\perp}} \left[\int_{G_0^{\perp}} h(\gamma+\xi)(x, -\gamma-\xi)\, dm_{G_0^{\perp}}(\xi) \right] d\mu(\dot{\gamma}) \\ &= \int_{\hat{G}/G_0^{\perp}} \left[\int_{G_0^{\perp}} h(\gamma+\xi)\, dm_{G_0^{\perp}}(\xi) \right] (x, -\gamma)\, d\mu(\dot{\gamma}) \\ &= \psi(x), \end{aligned}$$

showing that RI is the identity in $B(G)$.

(ii), (iii), (iv) and (v) are consequences of suitable statements of Theorem 1.1.

(vi) results from Theorem 1.2.

2. A refinement

In this section, we single out a class of functions satisfying (1.1) and examine the corresponding lifting operators. The results of this section will be of direct use in the next section.

Let G be a locally compact Abelian group satisfying the second axiom of countability and G_0 be a closed subgroup of G such that G/G_0 is compact. Let η be a section of the canonical epimorphism π over G/G_0, that is, η is a Borel right inverse of π (for the existence of at least one such section see [14, Theorem 8.11]).

For each $\gamma \in \hat{G}$, set

$$h(\gamma) = \left| \int_{G/G_0} (\eta(\dot{x}), \gamma)\, dm_{G/G_0}(\dot{x}) \right|^2; \tag{2.1}$$

here we assume that m_{G/G_0} has mass equal to 1. Clearly, h is a function in $C_+(\hat{G})$ with values no greater than 1. It turns out that h satisfies (1.1).

To see this, notice first that for each $x \in G$, $\pi(x - \eta\pi(x)) = 0$, so $x - \eta\pi(x)$ lies in G_0. Hence, for each $\gamma \in \hat{G}$ and each $\xi \in G_0^{\perp}$,

$$(\eta\pi(x), \gamma+\xi) = (\eta\pi(x), \gamma)(x, \xi). \tag{2.2}$$

Since $G_0^{\perp}$ is the discrete dual of G/G_0, it follows from Parseval's identity and the above identity that

$$\sum_{\xi\in G_0^\perp} h(\gamma+\xi)=\sum_{\xi\in G_0^\perp}\left|\int_{G/G_0}(\eta(\dot{x}),\gamma)(\dot{x},\xi)dm_{G/G_0}(\dot{x})\right|^2$$
$$=\int_{G/G_0}|(\eta(\dot{x}),\gamma)|^2 dm_{G/G_0}(\dot{x})=1,$$

as was to be shown.

As we saw earlier, for each $x\in G$, $x-\eta\pi(x)$ is an element of G_0. We shall denote it by $[x]$ and refer to it as the integral part of x. Such a terminology fits in with the one employed in the special case in which $G=\boldsymbol{R}$, $G_0=\boldsymbol{Z}$, and, for each $\dot{x}\in\boldsymbol{R}/\boldsymbol{Z}$, $\eta(\dot{x})$ is the unique element of $[0,1)$ such that $\pi\eta(\dot{x})=\dot{x}$.

Now we can state our major result.

THEOREM 2.1. *If I is the lifting operator associated with h given by* (2.1), *then, for each $\psi\in B(G_0)$ and each $x\in G$,*

$$I\psi(x)=\int_{G/G_0}\psi([x+\eta(\dot{y})])dm_{G/G_0}(\dot{y}).$$

PROOF. In view of (2.2) and Parseval's identity, for each $x\in G$ and each $\gamma\in\hat{G}$, we have

$$\sum_{\xi\in G_0^\perp} h(\gamma+\xi)(x,-\gamma-\xi)$$
$$=(x,-\gamma)\sum_{\xi\in G_0^\perp}(x,-\xi)\left|\int_{G/G_0}(\eta(\dot{y}),\gamma+\xi)dm_{G/G_0}(\dot{y})\right|^2$$
$$=(x,-\gamma)\sum_{\xi\in G_0^\perp}\int_{G/G_0}(\eta(\dot{x}+\dot{y}),\gamma)(\dot{y},\xi)dm_{G/G_0}(\dot{y})$$
$$\times\overline{\int_{G/G_0}(\eta(\dot{y}),\gamma)(\dot{y},\xi)dm_{G/G_0}(\dot{y})}$$
$$=(x,-\gamma)\int_{G/G_0}(\eta(\dot{x}+\dot{y})-\eta(\dot{y}),\gamma)dm_{G/G_0}(\dot{y}).$$

Since, for any $x,y\in G$, $[x+\eta(\dot{y})]=x+\eta(\dot{y})-\eta(\dot{x}+\dot{y})$, we see that

$$\sum_{\xi\in G_0^\perp} h(\gamma+\xi)(x,-\gamma-\xi)=\int_{G/G_0}([x+\eta(\dot{y})],-\gamma)dm_{G/G_0}(\dot{y}).$$

Now, if $\mu\in M(\hat{G}/G_0^\perp)$ is such that $\psi=\mathscr{F}\mu$, then, in view of (1.2), for each $x\in G$,

$$I\psi(x)=\mathscr{F}J_\mu(x)=\int_{\hat{G}/G_0^\perp}\left[\sum_{\xi\in G_0^\perp} h(\gamma+\xi)(x,-\gamma-\xi)\right]d\mu(\dot{\gamma})$$
$$=\int_{\hat{G}/G_0^\perp}\left[\int_{G/G_0}([x+\eta(\dot{y})],-\gamma)dm_{G/G_0}(\dot{y})\right]d\mu(\dot{\gamma})$$
$$=\int_{G/G_0}\mathscr{F}\mu([x+\eta(\dot{y})])dm_{G/G_0}(\dot{y})$$

$$=\int_{G/G_0}\psi([x+\eta(\dot{y})])dm_{G/G_0}(\dot{y}).$$

The proof is complete.

It is worth noticing that if one takes $\boldsymbol{R}$ for G, $\boldsymbol{Z}$ for G_0, and the natural mapping from $\boldsymbol{R}/\boldsymbol{Z}$ onto $[0,1)$ for η, then Theorem 2. 1 in conjunction with Theorem 1. 3 implies the following theorem due to R. Goldberg [7].

THEOREM 2. 2. *If $(a_n)_{n\in Z}$ is the sequence of Fourier coefficients of a finite Borel measure on $[0, 2\pi)$, then the function whose graph consists of the line segments successively joining the points (n, a_n) is the Fourier transform of a finite Borel measure on $\boldsymbol{R}$.*

More generally, the extension to $\boldsymbol{R}^n$ of Goldberg's result due to C. C. Graham and A. Maclean [8] can immediately be deduced from Theorems 1. 3 and 2. 1.

We close this section with the following

THEOREM 2. 3. *If I is the lifting operator associated with h given by (2. 1), then*

$$I(B_0(G_0))\subset B_0(G).$$

PROOF. Let ψ be a non-zero element of $B_0(G_0)$. Since every Borel measure on locally compact space satisfying the second axiom of countability is regular, given $\varepsilon>0$ there exists a compact subset C of G such that $\eta_* m_{G/G_0}(C) >1-\varepsilon/4\|\psi\|_{\infty,G_0}$. Since G/G_0 is compact, the set $\pi^{-1}(\pi(C))$ is also compact. Passing if necessary to $\pi^{-1}(\pi(C))$, we may assume with no loss of generality that $\pi^{-1}(\pi(C))=C$. For each $\dot{x}\in G/G_0$, we have

$$\begin{aligned}
&m_{G/G_0}(G/G_0\backslash\pi(C)\cap(\pi(C)-\dot{x}))\\
&\quad\leq m_{G/G_0}(G/G_0\backslash(\pi(C))+m_{G/G_0}(G/G_0\backslash(\pi(C)-\dot{x}))\\
&\quad=2(1-\eta_* m_{G/G_0}(C)\\
&\quad<\frac{\varepsilon}{2\|\psi\|_{\infty,G_0}}
\end{aligned}$$

whence

$$\int_{G/G_0\backslash\pi(C)\cap(\pi(C)-\dot{x})}\psi([x+\eta(\dot{y})])dm_{G/G_0}(\dot{y})<\frac{\varepsilon}{2}; \tag{2. 3}$$

here, of course,

$$\pi(C)-\dot{x}=\{\dot{z}\in G/G_0:\dot{z}=\dot{y}-\dot{x}\text{ with }\dot{y}\in\pi(C)\}.$$

Let K be a compact subset of G_0 such that $|\psi(z)|<\varepsilon/2$ for $z\in G_0\backslash K$. The set $K+C-C$ is compact, so to end the proof, it suffices to show that $|I\psi(x)|<\varepsilon$ for $x\in G\backslash(K+C-C)$.

Let $x\in G\backslash(K+C-C)$. Note that if $\dot{y}$ is in $\pi(C)\cap(\pi(C)-\dot{x})$, then $\eta(\dot{y})$ and $\eta(\dot{x}+\dot{y})$ are in $\eta(\pi(C))$. But $\pi(\eta(\pi(C)))=\pi(C)$ and so $\eta(\pi(C))\subset\pi^{-1}(\pi(C))=C$. Thus $\eta(\dot{y})$ and $\eta(\dot{x}+\dot{y})$ are in C, which implies that $x+\eta(\dot{y})-\eta(\dot{x}+\dot{y})$ is in $G_0\backslash K$ and next that $\psi([x+\eta(\dot{y})])<\varepsilon/2$. Consequently

$$\int_{\pi(C)\cap(\pi(C)-\dot{x})}\psi([x+\eta(\dot{y})])dm_{G/G_0}(\dot{y})<\frac{\varepsilon}{2}.$$

The latter inequality together with (2. 3) implies that $|I\psi(x)|<\varepsilon$.

The proof is complete.

3. An application

Let G be a locally compact non-compact Abelian group satisfying the second axiom of countability and Σ be a compact Abelian group satisfying the second axiom of countability. Suppose that there is a one-to-one continuous homomorphism α from G onto a dense subgroup of Σ.

A (G,Σ)-cocycle is a Borel function A from $\Sigma\times G$ into the circle group $\boldsymbol{T}$ such that

$$A(\sigma, x+y)=A(\sigma, x)A(\sigma+\alpha(x), y)$$

for all $\sigma\in\Sigma$ and all $x, y\in G$. Given a (G,Σ)-cocycle A, one defines a unitary strongly continuous representation U of G in $L^2(\Sigma)$ by setting

$$(U(x)f)(\sigma)=A(\sigma, x)f(\sigma+\alpha(x)) \quad (x\in G,\ \sigma\in\Sigma,\ f\in L^2(\Sigma)).$$

In virtue of the Stone-Naïmark-Ambrose-Godement theorem (cf. [1, Theorem 6. 2. 1]), there is a unique regular projection-valued measure E on $\mathfrak{B}(\hat{G})$, taking values in a Boolean algebra of projections in $L^2(\Sigma)$, such that, for each $x\in G$,

$$U(x)=\int_{\hat{G}}(x, -\gamma)dE(\gamma),$$

where the integral is to be interpreted in the sense of strong convergence. If, for each $f, g\in L^2(\Sigma)$, $E_{f,g}$ is the Borel measure on $\hat{G}$ given by

$$E_{f,g}(A)=(E(A)f, g) \qquad (A\in\mathfrak{B}(\hat{G})),$$

where $(\cdot\,,\cdot)$ denotes the scalar product in $L^2(\Sigma)$, and if

$$\mathscr{I}=\{E_{f,g}: f, g\in L^2(\Sigma)\},$$

then, as one can show, either $\mathscr{I}\subset M_a(\hat{G})$, or $\mathscr{I}\subset(M_s(\hat{G})\backslash M_a(\hat{G}))\cup\{0\}$, or $\mathscr{I}\subset L^1(\hat{G})$. One expresses this property by saying that A is either trivial, or of singular type, or of Haar type, respectively. Moreover, one has either $\mathscr{I}\subset M_0(\hat{G})$ or $\mathscr{I}\cap M_0(\hat{G})=\{0\}$ and, correspondingly, one says that A is either of type (C_0) or of oscillatory type. Cocycles of different types exist and play a vital role in harmonic analysis, ergodic theory, and differential equations (cf. [4]).

Let $\hat{\alpha}$ be the homomorphism from $\hat{\Sigma}$ into $\hat{G}$ given by

$$(x, \hat{\alpha}(\chi))=(\alpha(x), \chi) \qquad (x\in G, \chi\in\hat{\Sigma}).$$

Let Γ be a subgroup of $\hat{\alpha}(\hat{\Sigma})$ that is discrete under the topology inherited from $\hat{G}$. Let G_0 be the annihilator of Γ in G and K be the closure of $\alpha(G_0)$ in Σ. Of course, G_0 is a closed subgroup of G and G/G_0 is compact. Let η be a section over G/G_0 of the canonical epimorphism π from G onto G/G_0. It turns out that by means of η each (G_0, K)-cocycle can be transferred into a (G, Σ)-cocycle of the same type. The description of this transfer and its properties is the main objective of the present section.

For each $\dot{x}\in G/G_0$ and each $k\in K$, set

$$\theta(\dot{x}, k)=\alpha(\eta(\dot{x}))+k.$$

We first show that θ is a bijection from $G/G_0\times K$ onto Σ inducing an isomorphism of the Borel structures of the two groups.

Suppose that $\theta(\dot{x}_1, k_1)=\theta(\dot{x}_2, k_2)$ for $x_1, x_2\in G$ and $k_1, k_2\in K$. Since Γ is contained in $\hat{\alpha}(\hat{\Sigma})$, it follows that $\Gamma=\hat{\alpha}(K^\perp)$. Now $\alpha(\eta(\dot{x}_1)-\eta(\dot{x}_2))=k_2-k_1$ is annihilated by $K^\perp$, so $\eta(\dot{x}_1)-\eta(\dot{x}_2)$ is annihilated by $\hat{\alpha}(K^\perp)$ and hence $\eta(\dot{x}_1)-\eta(\dot{x}_2)$ is in G_0. Consequently, $\pi\eta(\dot{x}_1)=\pi\eta(\dot{x}_2)$ which amounts to $\dot{x}_1=\dot{x}_2$ and next implies that $k_1=k_2$. Thus θ is injective.

Given $\sigma\in\Sigma$, let $(x_n)_{n\in N}$ be a sequence in G such that $\sigma=\lim_{n\to\infty}\alpha(x_n)$. (Note that, in view of second countability, both G and Σ are metrizable.) By the compactness of G/G_0, there exists x in G, a subsequence $(x_{n_k})_{k\in N}$ of $(x_n)_{n\in N}$, and a sequence $(y_k)_{k\in N}$ in G_0 such that $x=\lim_{k\to\infty}(x_{n_k}-y_k)$. In view of the compactness of K, we may assume that the sequence $(\alpha(y_k))_{k\in N}$ is convergent. Let j be the limit of this suquence. Then $\sigma=\alpha(x)+j$. With this representation, it is now easy to see that

$$\sigma=\theta(\dot{x}, j+\alpha(x-\eta(\dot{x}))) \quad (\dot{x}=\pi(x)).$$

This establishes the surjectiveness of θ.

It is clear that θ is a Borel map. Since the Borel structures of $G/G_0\times K$ and Σ are standard, it follows that θ^{-1} is also Borel (cf. [11]).

Now we shall show that

$$\theta_*(m_{G/G_0}\otimes m_K)=m_\Sigma. \tag{3.1}$$

Notice first that

$$\int_{G/G_0}(\eta(\dot{x}), \gamma)dm_{G/G_0}(\dot{x})=\begin{cases}1, & \text{if } \gamma=0;\\ 0, & \text{if } \gamma\in\Gamma\backslash\{0\}.\end{cases} \tag{3.2}$$

In fact, for any $x, y\in G$, $\eta(\dot{x})+\eta(\dot{y})-\eta(\dot{x}+\dot{y})$ is in G_0, so $(\eta(\dot{x})+\eta(\dot{y}), \gamma)=(\eta(\dot{x}+\dot{y}), \gamma)$ whatever $\gamma\in\Gamma$. Hence

$$\begin{aligned}((y, \gamma)-1)&\int_{G/G_0}(\eta(\dot{x}), \gamma)dm_{G/G_0}(\dot{x})\\ &=((\eta(\dot{y}), \gamma)-1)\int_{G/G_0}(\eta(\dot{x}), \gamma)dm_{G/G_0}(\dot{x})\\ &=\int_{G/G_0}(\eta(\dot{x})+\eta(\dot{y}), \gamma)-(\eta(\dot{x}), \gamma))dm_{G/G_0}(\dot{x})\\ &=\int_{G/G_0}((\eta(\dot{x}+\dot{y}), \gamma)-(\eta(\dot{x}), \gamma))dm_{G/G_0}(\dot{x})\\ &=0,\end{aligned}$$

from which (3.2) follows immediately.

For any $\chi\in\hat{\Sigma}$, we have

$$\begin{aligned}(\mathscr{F}\theta_*(&m_{G/G_0}\otimes m_K))(\chi)\\ &=\int_{G/G_0}(\alpha(\eta(\dot{x})), \chi)dm_{G/G_0}(\dot{x})\int_K(k, \chi)dm_K(k).\end{aligned}$$

If $\chi\notin K^\perp$, then

$$\int_K(k, \chi)dm_K(k)=0;$$

if $\chi\in K^\perp$, then

$$\int_K(k, \chi)dm_K(k)=1$$

and, moreover, since $\hat{\alpha}(\chi)$ is in Γ, it follows from (3.1) that

$$\int_{G/G_0}(\alpha(\eta(\dot{x})), \chi)dm_{G/G_0}(\dot{x})=\begin{cases}1, & \text{if } \chi=0;\\ 0, & \text{if } \chi\neq0.\end{cases}$$

Thus

$$(\mathscr{F}\theta_*(m_{G/G_0}\otimes m_K))(\chi)=\begin{cases}1, \text{ if } \chi=0\,;\\ 0, \text{ if } \chi\neq 0,\end{cases}$$

which establishes (3. 1).

Now we are ready to discuss transference of cocycles.

Let A be a (G_0, K)-cocycle. For each $x\in G$ and each $\sigma\in\Sigma$, put

$$\tilde{A}(\sigma, x)=A(k,\ [x+\eta(\dot{y})]),$$

where $(\dot{y}, k)=\theta^{-1}(\sigma)$ $(\dot{y}\in G/G_0,\ k\in K)$. One verifies by a direct computation that $\tilde{A}$ is a (G, Σ)-cocycle. The definition of $\tilde{A}$ in the case where $G=\boldsymbol{R}$ and $G_0=\boldsymbol{Z}$ is due to T. W. Gamelin [6]. The above general definition of $\tilde{A}$ parallels the one employed by J. Mathew and M. G. Nadkarni in [12].

The following is the main result of the present section:

THEOREM 3. 1. If a (G_0, K)-cocycle A is trivial (resp. of singular type, of Haar type, of type (C_0), of oscillatory type), then the corresponding (G, Σ)-cocycle $\tilde{A}$ is also trivial (resp. of singular type, of Haar type, of type (C_0), of oscillatory type).

PROOF. Let U be the unitary representation of G_0 in $L^2(K)$ associated with A and V be the unitary representation of G in $L^2(\Sigma)$ associated with $\tilde{A}$. Let E and F be the corresponding projection-valued measures. Then, for each $x\in G$,

$$\begin{aligned}\mathscr{F}F_{1,1}(x)&=(V(x)1, 1)=\int_\Sigma\tilde{A}(x, \sigma)dm_\Sigma(\sigma)\\ &=\int_{G/G_0}\left[\int_K A(k, [x+\eta(\dot{y})])dm_K(k)\right]dm_{G/G_0}(\dot{y})\\ &=\int_{G/G_0}(U([x+\eta(\dot{y})])1, 1)dm_{G/G_0}(\dot{y})\\ &=\int_{G/G_0}\mathscr{F}E_{1,1}([x+\eta(\dot{y})])dm_{G/G_0}(\dot{y}).\end{aligned}$$

Hence, by virtue of Theorem 2. 1, $\mathscr{F}F_{1,1}$ is the image of $\mathscr{F}E_{1,1}$ by the lifting operator corresponding to the function h given by (2. 1). That $\tilde{A}$ is trivial (resp. of singular type, of Haar type, of oscillatory type) now follows upon applying Theorem 1. 3. To conclude that $\tilde{A}$ is of type (C_0) whenever A is so, it suffices to invoke Theorem 2. 3.

The proof is complete.

References

[1] A. O. BARUT and R. RĄCZKA, Theory of Group Representations and Applications, Państwowe Wydawnictwo Naukowe, Warszawa, 1980.

[2] H. BAUER, Probability Theory and Elements of Measure Theory, Academic Press, London/New York, 1981.
[3] N. BOURBAKI, Éléments de mathématique, Livre 6, Intégration, Chapitres 1-8, Hermann, Paris, 1963.
[4] W. CHOJNACKI, Some non-trivial cocycles, J. Funct. Anal. 77 (1988), 9-31.
[5] J. DIEUDONNÉ, Treatise on Analysis, vol. II, Academic Press, New York/London, 1970.
[6] T. W. GAMELIN, Remarks on compact groups with ordered duals, Rev. Un. Math. Argentina 23 (1967), 97-108.
[7] R. GOLDBERG, Restrictions of Fourier transforms and extension of Fourier sequences, J. Approx. Theory 3 (1970), 149-155.
[8] C. C. GRAHAM and A. MACLEAN, A multiplier theorem for continuous measures, Studia Math. 66 (1980), 213-225.
[9] C. C. GRAHAM and O. C. MCGEHEE, Essays in Commutative Harmonic Analysis, Springer-Verlag, New York, 1979.
[10] J. INOUE, On a lifting problem of Fourier-Stieltjes tranforms of measures, Hokkaido Math. J. 12 (1983), 113-118.
[11] G. W. MACKEY, Borel structure in groups and their duals, Trans. Amer. Math. Soc. 85 (1957), 134-165.
[12] J. MATHEW and M. G. NADKARNI, On systems of imprimitivity on locally compact abelian groups with dense action, Ann. Inst. Fourier (Grenoble) 28 (1978), 1-23.
[13] W. RUDIN, Fourier Analysis on Groups, Interscience Publishers, New York, 1962.
[14] V. S. VARADARAJAN, Geometry of Quantum Theory, vol. II, Van Nostrand, New York, 1970.

Instytut Matematyki Stosowanej i Mechaniki
Uniwersytet Warszawski
ul. Banacha 2
00-913 Warszawa, Poland
and
School of Information Science and Technology
The Flinders University of South Australia
GPO Box 2100, Adelaide, SA 5001, Australia