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Local solvability of semilinear parabolic equations

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Introduction.

The problem of local in time existence of solutions of semilienar parabolic equations has been studied by a number of authors (e.g. [10, 20, 4, 18, 3]) using several different methods. Friedman [10] made use of the $C^{1+\delta}$ estimates (p. 191 therein), also the Sobolevskii-Tanabe method (cf. [22, 3]) gave a contribution to these studies. However, probably the most important innovation in recent years came from H. Amann (see [3] and references therein), who covered general systems of equations using the idea of an extended "variation of constants" formula. Also in [0] independent studies of this problem are presented. Our approach here is more classical, close in spirit but different in method to that of Friedman [10]. The proofs based on a priori estimates and the iteration technique [2] make it possible to study in compact form the equation (1) with linear boundary conditions of the third type. Our estimates (23), (31) of the life time of solutions seems to be new and to have interesting implications (compare e.g. Lemma 4). The form of equation (1) (with the weak assumption of the local Lipschitz continuity of f) covers most of the recently studied single equations with blowing-up solutions [11, 4, 9] as well as problems with blowing-up derivatives [8], also the formally more complete form of the equation in [10, p. 201] (except that we need the coefficients to be smoother). The technique presented here has been used before by the present author [6, 7] in studies on the long time behaviour of solutions of parabolic problems.

Notation.

The notation of Sobolev spaces is taken from an excellent monograph by R. Adams [1], for the Hölder spaces from Ladyženskaja at al. [13] (except that we use the letter C instead of H in [13] to distinguish the notation of Sobolev space). $|\Omega|$ denotes the Lebesgue measure of Ω, $C := |\Omega|^{1/(2n+2)}$. As throughout this paper the space variable x belongs to a fixed bounded domain $\Omega \subset R^n$, hence all the unspecified integrals are understood to be taken over Ω, also unsepcified sums are taken from 1 to n (the space dimension). By a $C^{1,2}$ solution u of (1)-(3) we mean the classical solution of the problem in $[0, T_{ex})\times\Omega$ ($T_{ex}\leq+\infty$ is the maximal time of existence of u) having, in compact subsets of $[0, T_{ex})\times\bar{\Omega}$, continuous derivatives appearing in (1).

For convenience we collect here certain inequalities of importance for our estimates. For $a, b\geq 0, \varepsilon>0, m>1$ we have the Young inequality:

$$\text{(i)}\qquad ab\leq\frac{1}{m}\varepsilon^m a^m+\frac{m-1}{m}\varepsilon^{-\frac{m}{m-1}}b^{\frac{m}{m-1}},$$

which for $m=2, \varepsilon^2=\tilde{\varepsilon}$ is known as the Cauchy inequality. Let $\Omega\subset R^n$, having the cone property, be bounded ([1, p. 78]), and let ε_0 be a finite positive number, then

$$\text{(ii)}\qquad \underset{K>0}{\exists}\ \underset{0\leq\varepsilon\leq\varepsilon_0}{\forall}\ \underset{0\leq j\leq m-1}{\forall}\ |v|_{j,p}\leq K\varepsilon|v|_{m,p}+K\varepsilon^{-\frac{j}{m-1}}\|v\|_p,$$

for any $v\in W^{m,p}(\Omega), K=K(\varepsilon_0, m, p, \Omega), 1\leq p<\infty$. For $p>n$ we also note [16]:

$$\text{(iii)}\qquad \|v\|_\infty\leq c\|v\|_{1,p}^{n/p}\|v\|_p^{1-n/p}.$$

Preliminaries.

We deal with the parabolic equation:

$$\text{(1)}\qquad u_t=\sum_{i,j}(a_{ij}(t, x)u_{x_j}))_{x_i}+f(t, x, u, u_x)=:L(u),$$

where $t>0$ (which will later be limited from above), $x\in\Omega\subset R^n$ with a bounded domain Ω, $\partial\Omega\in C^{2+\gamma}(\gamma\in(0, 1)$ fixed) and $u_x=(u_{x_1},\dots, u_{x_n})$. Equation (1) will be considered with the conditions:

$$\text{(2)}\qquad u(0, x)=u_0(x)\quad\text{for}\quad x\in\Omega,$$

and, with $\dfrac{\partial}{\partial N}:=\sum_{i,j}a_{ij}(t, x)\dfrac{\partial}{\partial x_j}\cos(n, x_i)$,

$$\text{(3)}\qquad \phi(x)u+\psi(x)\frac{\partial u}{\partial N}=0\text{ on }\partial\Omega,$$

n denotes the inward normal to $\partial\Omega$ in x, where one of the following additional assumptions is required:

(3 a) $\psi(x)\equiv 0$ and $\phi(x)\geq\phi_0>0$ (the Dirichlet problem)

(3 b) $\phi(x)\leq 0$ and $\psi(x)\geq\psi_0>0$ (the third boundary problem).

In most estimates we need only the condition following from (3a) and from (3b), i.e. $u\dfrac{\partial u}{\partial N}\geq 0$ on $\partial\Omega$.

Assumptions.

The following assumptions are valid throughout the paper:

(A1) a_{ij}, $(a_{ij})_{x_i}(i, j=1,\ldots, n)$ are Hölder continuous with respect to x (exponent γ), a_{ij} are Lipschitz continuous in t and $(a_{ij})_{x_i}$ are Hölder continuous in t (exponent $\gamma/2$) all this in sets $[0, \tau]\times\bar{\Omega}(\tau>0$ arbitrary, the Lipschitz, Hölder constants may depend on τ),

(A2) The equation is parabolic:

$$\forall_{\tau>0}\ \exists_{a_0>0}\ \forall_{\substack{x\in\bar{\Omega}\\ t\in[0,\tau]}}\ \forall_{\xi\in R^n}\ \sum_{i,j} a_{ij}(t, x)\xi_i\xi_j\geq a_0|\xi|^2, \tag{4}$$

(A3) f is Lipschitz continuous in t, u and $u_{x_i}(i=1,\ldots, n)$ and Hölder continuous (exponent γ) in x, the Lipschitz, Hölder constants are general in sets $[0, \tau]\times\bar{\Omega}\times[-a, a]\times[-b, b]^n(a, b>0$ arbitrary),

(A4) $\phi, \psi\in C^{1+\gamma}(\partial\Omega)$, $\partial\Omega\in C^{2+\gamma}$,

(A5) $u_0\in C^{2+\gamma}(\bar{\Omega})$ and $u_0=0=L(u_0)$ on $\partial\Omega$ in the case of condition (3a) or $\phi(x)u_0+\psi(x)\dfrac{\partial u_0}{\partial N}=0$ on $\partial\Omega$ in the case of condition (3b).

A preliminary lemma.

We start by formulating a lemma making it possible to estimate the space derivatives of u through the time derivative u_t and u alone. To do so let us first define a set:

$$X:=\left\{(t, x, u, p):t\in[0, T_0], x\in\bar{\Omega}, |u|\leq M_1, \sum_i|p_i|\leq M_2\right\},$$

where $p=(p_1\ldots, p_n)$, $T_0>0$ is fixed. For $(t, x, u, u_x)\in X$ let us denote by A the Lipschitz constant of a_{ij} with respect to $t(i, j=1,\ldots, n)$, and by L_t, L_u and L_x the Lipschitz constants of f with respect to, respectively, t, u and $u_{x_i}(i=1,\ldots, n)$. Further, let $|f(t, x, 0, 0)|\leq N$. We have:

LEMMA 1: *As long as the $C^{1,2}$ solution u of* (1)-(3) *remains in X, it fulfills, with sufficiently small $\delta>0$ satisfying* (15), *the condition*:

$$\forall_{\delta}\ \exists_{C_\delta>0}\ \sum_i\|u_{x_i}(t, \cdot)\|_\infty\leq\delta(\|u_t(t,\cdot)\|_{2n+2}+NC) + C_\delta\|u(t,\cdot)\|_\infty, \tag{5}$$

where $C_\delta = const.\ \delta^{-\frac{3n+2}{n+2}}$. *Also, with the same* δ *and* C_δ:

$$(6)\qquad \sum_i \|(u_0)_{x_i}\|_\infty \le \delta\Big(\|\sum_{i,j}(a_{ij}(0,\cdot)(u_0)_{x_j})_{x_i} + f(0,\cdot,\ u_0,(u_0)_x)\|_{2n+2} + NC\Big) + C_\delta \|u_0\|_\infty .$$

PROOF: As shown in [21, p. 138], for weak solutions of elliptic problems

$$(7)\qquad Sv := \sum_{ij}(A_{ji}(x)v_{x_j})_{x_i} = F(x),\ x\in\Omega,$$

$$(8)\qquad \sum_{i,j} A_{ij}(x)v_{x_j}\cos(n, x_i) = \Phi(x) \text{ on } \partial\Omega,$$

with $A_{ij}\in W^{1,s}(\Omega)$, $\Phi\in W^{1-1/s,s}(\partial\Omega)$, $s>n$, the Calderon-Zygmund estimates are valid:

$$(9)\qquad \|v\|_{2,s} \le c_1(\|Sv\|_s + \|\Phi\|_{1-1/s,s,\partial\Omega}) + c_2\|v\|_s .$$

In the case of our more general than (8) boundary condition (3 b), using (9) with $\Phi = \frac{\phi}{\psi}v$ and the trace estimate [1, p. 217] we get:

$$(10)\qquad \|v\|_{2,s} \le c_1\Big(\|Sv\|_s + \|\frac{\phi}{\psi}v\|_{1-1/s,s,\partial\Omega}\Big) + c_2\|v\|_s$$
$$\le c_1\Big(\|Sv\|_s + K_1\|\frac{\phi}{\psi}v\|_{1,s}\Big) + c_2\|v\|_s .$$

Since from [1, p. 115], for $mp>n$, the $W^{m,p}(\Omega)$ is a Banach Algebra, then for $s>n$:

$$\|v\|_{2,s} \le c_1\Big(\|Sv\|_s + K_1K^*\|\frac{\phi}{\psi}\|_{1,s}\|v\|_{1,s}\Big) + c_2\|v\|_s ,$$

and finally, as a consequence of (ii) applied to $|v|_{1,s}$:

$$(11)\qquad \|v\|_{2,s} \le \text{const.}\ (\|Sv\|_s + \|v\|_s) \le c_{s,\infty}(\|Sv\|_s + \|v\|_\infty).$$

For the case of bundary condition (3a) the estimate (11) is stated explicitly in [14, Chap. III].

Then from (1) (for fixed $t>0$), when denoting:

$$Pu := \sum_{i,j}(a_{ij}(t, x)u_{x_j})_{x_i},$$

we have

$$(12)\qquad \begin{aligned}\|Pu\|_p &= \|u_t - f(t,\cdot,u,u_x)\|_p \\ &\leq \|u_t(t,\cdot)\|_p + \|f(t,\cdot,u,u_x) - f(t,\cdot,u,0)\|_p \\ &\quad + \|f(t,\cdot,u,0) - f(t,\cdot,0,0)\|_p + \|f(t,\cdot,0,0)\|_p \\ &\leq \|u_t(t,\cdot)\|_p + L_x \sum_i \|u_{x_i}(t,\cdot)\|_p + L_u \|u(t,\cdot)\|_p + N\tilde{C},\end{aligned}$$

as long as the solution remains in X (here $\tilde{C} := C^{\frac{2n+2}{p}}$).

Finally, using (iii) for $v := w_{x_i}$ and summing over i, we get:

$$\begin{aligned}\sum_i \|w_{x_i}\|_\infty &\leq c \sum_i \|w_{x_i}\|_{1,p}^{n/p} \|w_{x_i}\|_p^{1-n/p} \\ &\leq c \|w\|_{2,p}^{n/p} \sum_i \|w_{x_i}\|_p^{1-n/p},\end{aligned}$$

and, with the use of (i) with $m = \frac{p}{n}$ and (ii) we obtain:

$$\sum_i \|w_{x_i}\|_\infty \leq cn \left\{ \frac{n}{p} \varepsilon_1^{\frac{p}{n}} \|w\|_{2,p} + \frac{p-n}{p} \varepsilon_1^{-\frac{p}{p-n}} (K\varepsilon |w|_{2,p} + K\varepsilon^{-1} \|w\|_p) \right\}$$

Choosing $\varepsilon = \varepsilon_1^{\frac{p^2}{n(p-n)}}$, $\tilde{\delta} := \text{const}, \ \varepsilon_1^{\frac{p}{n}}$ we thus verify that

$$(13)\qquad \sum_i \|w_{x_i}\|_\infty \leq \tilde{\delta} \|w\|_{2,p} + C_{\tilde{\delta}} \|w\|_p \ ).$$

with $C_{\tilde{\delta}} = \text{const.} \ \ \tilde{\delta}^{-\frac{p+n}{p-n}} (p > n$ as usual).

The three estimates (11), (12) and (13) together (with $p = s = 2n+2$, $v = w = u(t,\cdot)$, $P = S$), give:

$$(14)\qquad \begin{aligned}\sum_i \|u_{x_i}(t,\cdot)\|_\infty \leq \tilde{\delta} \{ c_{2n+2,\infty} [\|u_t(t,\cdot)\|_{2n+2} + L_x \sum_i \|u_{x_i}(t,\cdot)\|_{2n+2} \\ + L_u \|u(t,\cdot)\|_{2n+2} + NC] + \|u(t,\cdot)\|_\infty \} + C_{\tilde{\delta}} \|u(t,\cdot)\|_{2n+2}.\end{aligned}$$

Since $\|v\|_{2n+2} \leq C \|v\|_\infty$, then for $\tilde{\delta}$ so small that:

$$(15)\qquad \tilde{\delta} c_{2n+2,\infty} \max\{L_x C\,;1\} \leq \frac{1}{2},$$

denoting $\delta := 2\tilde{\delta} c_{2n+2,\infty} (\delta \leq 1$ as a consequence of (15)), it follows from (14) that

$$\begin{aligned}\frac{1}{2} \sum_i \|u_{x_i}(t,\cdot)\|_\infty \leq \frac{1}{2} \bar{\delta} (\|u_t(t,\cdot)\|_{2n+2} + NC) + \\ + \left[\frac{1}{2} \delta (L_u C + 1) + C_{\tilde{\delta}} C \right] \|u(t,\cdot)\|_\infty.\end{aligned}$$

The square bracket above will be dominated by the largest component inside it multiplied by a suitable constant, hence (5) follows. To get (6) we may use the equality:

$$\|Pu_0|_{t=0}\|_p=\|[Pu_0|_{t=0}+f(0,\cdot,u_0,(u_0)_x)]-f(0,\cdot,u_0,(u_0)_x)\|_p$$

and then repeat our arguments starting from (12), with the only evident changes being:

$$u(t,\cdot)\longrightarrow u_0,\ u_t(t,\cdot)\longrightarrow Pu_0|_{t=0}+f(0,\cdot,u_0,(u_0)_x).$$

The estimate (6) is not optimal but will be used in this form in the sequel. The proof of Lemma 1 is completed.

It is worth noticing that Lemma 1 is not necessary in the proof of the following Theorem 1 (c. f. [5]), if the coefficients a_{ij} in (1) are time independent for all $i, j=1,\dots, n$.

The main result.

We are now in a position to formulate the main result of the paper. This theorem gives a priori estimates of the solution u.

THEOREM 1. *Under the conditions stated in the Assumptions, for arbitrary positive numbers m_1 and m_2 and two further positive numbers M_1 and M_2, such that $m_1<M_1$ and $m_2<M_2$, there exists a time $T>0$ such that a $C^{1,2}$ solution u of* (1)-(3) *with*

$$\|u_0\|_\infty\le m_1 \text{ and } \delta(\|L(u_0)|_{t=0}\|_{2n+2}+NC)+C_\delta\|u_0\|_\infty\le m_2{}^* \tag{16}$$

for some δ as in Lemma 1, *satisfies*

$$\|u(t,\cdot)\|_\infty\le M_1 \text{ and } \sum_i\|u_{x_i}(t,\cdot)\|_\infty\le M_2 \tag{17}$$

at least until the time T.

The proof of Theorem 1 is given in two lemmas which are designed to obtain a priori estimates of $\|u(t,\cdot)\|_\infty$ and $\|u_t(t,\cdot)\|_{2n+2}$, respectively. An iteration technique given by N. D. Alikakos is used in the first lemma (c. f. [3, 5, 6]). Our estimate (18) below generalizes the Maximum Principle for solutions of (1)-(3).

REMARK. The second half oa the condition (16) imposed on the initial function u_0 is evidently satisfied by u_0 fulfilling (A5) and belonging to a neighbourhood of zero in $L^\infty(\Omega)$. To cover arbitrary u_0 fulfilling (A5) we shall replaced (1)-(3) by the equivalent problem for the new unknown $v:=u-u_0$:

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*This quantity dominates $\sum_i\|(u_0)_{x_i}\|_\infty$ as shown in (6). From now on δ and C_δ are fixed.

(1′) $\quad v_t=\sum_{i,j}(a_{ij}(t,x)v_{x_j})_{x_i}+\bar{f}(t,x,v,v_x),$

(2′) $\quad v(0,x)=0 \quad \text{for } x\in\Omega,$

(3′) $\quad \phi(x)v+\psi(x)\frac{\partial v}{\partial N}=0 \quad \text{on } \partial\Omega,$

where $\bar{f}(t,x,v,v_x):=\sum_{i,j}(a_{ij}(t,x)u_{0xj})_{x_i}+f(t,x,v+u_0,(v+u_0)_x)$. The properities of f are preserved by $\bar{f}$ and Theorems 1,2 applied to (1′)-(3′) show local existence of the solution v.

LEMMA 2: *(First a priori estimate). Under the assumptions of Theorem* 1 *the following estimate of the uniform norm of $u(t,\cdot)$ holds for sufficiently small t ($t\leq T_1$, T_1 given in* (23)):

(18) $$\|u(t,\cdot)\|_\infty^2 \leq\left[\|u_0\|_\infty^2+\frac{N}{L_u+N}(1-\exp(-2t(L_u+N)))\right]\exp(2t(L_u+N)).$$

PROOF: Multiplying (1) by u^{2k-1}; $k=1,2,\ldots$, and integrating over Ω we obtain (to simplify notation the arguments (t,x) or $(t,\cdot)$ will be neglected):

$$\int u_t u^{2k-1}dx=\int\sum_{i,j}(a_{ij}u_{xj})_{xi}u^{2k-1}dx+\int f\,u^{2k-1}dx.$$

The components are then transformed and estimated in the following way:

$$\int u_t u^{2k-1}dx=\frac{1}{2k}\frac{d}{dt}\int u^{2k}dx,$$

$$\int\sum_{i,j}(a_{ij}u_{xj})_{x_i}u^{2k-1}dx=-\int_{\partial\Omega}u^{2k-1}\frac{\partial u}{\partial N}ds$$

$$-\frac{2k-1}{k^2}\int\sum_{i,j}a_{ij}(u^k)_{x_i}(u^k)_{xj}dx\leq-\frac{2k-1}{k^2}a_0\int\sum_i[(u^k)_{xi}]^2dx,$$

because of (4) and the constraints of the boundary conditions. Adding and subtracting, the last component will be estimated, with the use of the inequality $|a|^{2k-1}\leq a^{2k}+a^{2k-2}$ and the Cauchy inequality, as follows:

$$\int f(t,x,u,u_x)u^{2k-1}dx\leq L_x\int\sum_i|u_{xi}u^{2k-1}|dx$$

$$+L_u\int u^{2k}dx+N\int|u^{2k-1}|dx$$

$$\leq\frac{L_x}{k}\int\sum_i|(u^k)_{xi}u^k|dx+L_u\int u^{2k}dx$$

$$+N\int(u^{2k}+u^{2k-2})dx\leq\frac{L_x\varepsilon_2}{2k}\int\sum_i[(u^k)_{xi}]^2dx$$

$$+\left[\frac{L_x n}{2\varepsilon_2 k}+L_u+N\right]\int u^{2k}dx+N\int u^{2k-2}dx.$$

Hence the resulting estimate takes the form ($\bar{C}:=|\Omega|^{1/k}$):

$$\frac{1}{2k}\frac{d}{dt}\int u^{2k}dx\leq\frac{1}{k}\left[-\frac{2k-1}{k}a_0+\frac{L_x\varepsilon_2}{2}\right]\int\sum_i[(u^k)_{xi}]^2dx \tag{19}$$
$$+\left[\frac{L_x n}{2\varepsilon_2 k}+L_u+N\right]\int u^{2k}dx+N\bar{C}^2\left(\int u^{2k}dx\right)^{\frac{k-1}{k}},$$

where the relation between the L^{2k-2} and L^{2k} norms of u has been used. Noting that for $k=1, 2,\dots$ we have $1\leq\frac{2k-1}{k}\leq 2$, we fix $\varepsilon_2=\varepsilon_2^0:=\frac{2a_0}{L_x}$, such that

$$-\frac{2k-1}{k}a_0+\frac{L_x\varepsilon_2^0}{2}\leq 0,\ k=1, 2,\dots,$$

to obtain the differential inequality for functions $y_k(t):=\int u^{2k}dx$:

$$\frac{d}{dt}y_k(t)\leq 2k\left[\frac{L_x^2 n}{4a_0 k}+L_u+N\right]y_k(t) \tag{20}$$
$$+2kN\bar{C}^2(y_k(t))^{\frac{k-1}{k}}.$$

Solving (20) we obtain:

$$y_k^{1/k}(t)\leq\left[y_k^{1/k}(0)+\frac{\beta_k}{\alpha_k}\left(1-\exp(-\frac{\alpha_k}{k}t)\right)\right]\exp\left(\frac{\alpha_k}{k}t\right) \tag{21}$$

with

$$\alpha_k=2k\left[\frac{L_x^2 n}{4a_0 k}+L_u+N\right],\ \beta_k=2kN\bar{C}^2. \tag{22}$$

The limit passage with k to $+\infty$ in (21) and (22) leads to the final estimate (18) of Lemma 2.

We define τ_1 to be given by the condition:

$$\left[m_1^2+\frac{N}{L_u+N}(1-\exp(-2\tau_1(L_u+N)))\right]\exp(2\tau_1(L_u+N))=M_1^2. \tag{23}$$

From now on we will restrict our considerations to the time interval $[0, T_1]$ with $T_1:=\min\{T_0;\tau_1\}$. The proof of Lemma 2 is finished.

We are now able to give the a priori estimate of the time derivative $u_t(t,\cdot)$ in $L^{2n+2}(\Omega)$. As shown in (5), both the first and second a priori estimate are sufficient to estimate also the space gradient u_x of u.

LEMMA 3: (*Second a priori estimate*). *With the assumptions of Theorem* 1 *we have an estimate*:

$$\left(\int u_t^{2n+2}dx\right)^2 \leq \left[r^2+\frac{\beta}{\alpha}\left(1-\exp\left(-\frac{2\alpha t}{n+1}\right)\right)\right]\exp\left(\frac{2\alpha t}{n+1}\right), \tag{24}$$

where the constants α *and* β *are defined in* (28) *and* $r:=\|L(u_0)|_{t=0}\|_{2n+2}$.

PROOF: The proof is exactly similar to that of Lemma 2. Since we must, in practice, differentiate (1) with respect to t, and the second derivative u_{tt} does not necessarily exist, we will study instead the difference quotients. Let us put, for a fixed at the moment $h>0$ and $t+h\leq T_1$:

$$u_h(t,x):=h^{-1}(u(t+h,x)-u(t,x)).$$

Subtracting (1) written for t from (1) written for $t+h$ and multiplying by h^{-1}, we get

$$u_{ht}=\sum_{i,j}(a_{ij}u_{x_j})_{xih}+h^{-1}(f|_{t+h}-f|_t) \tag{25}$$

where $f|_s:=f(s,x,u(s,x),\ u_x(s,x))$. Multiplying (25) by u_h^{2n+1} and integrating over Ω:

$$\frac{1}{2n+2}\ \frac{d}{dt}\int u_h^{2n+2}dx=\int\sum_{i,j}(a_{ij}u_{x_j})_{hx_i}u_h^{2n+1}dx$$
$$+\int h^{-1}(f|_{t+h}-f|_t)u_h^{2n+1}dx. \tag{26}$$

The right hand side components are transformed as follows:

$$\int\sum_{i,j}(a_{ij}u_{x_j})_{hx_i}u_h^{2n+1}dx=-\int_{\partial\Omega}\left[\frac{\partial u}{\partial N}\right]_h u_h^{2n+1}ds$$
$$-\int\sum_{i,j}(a_{ij}u_{x_j})_h(u_h^{2n+1})_{x_i}dx$$
$$\leq-\int\sum_{i,j}[a_{ijh}(t,x)u_{x_j}(t+h,x)+a_{ij}(t,x)u_{hx_j}(t,x)](u_h^{2n+1})_{x_i}dx$$
$$=-\frac{2n+1}{(n+1)^2}\int\sum_{i,j}a_{ij}(u_h^{n+1})_{x_i}(u_h^{n+1})_{x_j}dx$$
$$-\frac{2n+1}{n+1}\int\sum_{i,j}a_{ijh}u_{x_j}(t+h,x)u_h^n(u_h^{n+1})_{x_i}dx$$
$$\leq-\frac{2n+1}{(n+1)^2}a_0\int\sum_i[(u_h^{n+1})_{x_i}]^2dx+J,$$

where

$$J:=\frac{2n+1}{n+1}A\sum_i\|(u_h^{n+1})_{x_i}\|_2\,\|u_h^n\|_{\frac{2n+2}{n}}\left(\sum_j\|u_{x_j}(t+h,\cdot)\|_{2n+2}\right).$$

In the last estiamte the Hölder inequality $(\frac{1}{2}+\frac{n}{2n+2}+\frac{1}{2n+2}=1)$, condition (4) and the non-negativity of the boundary integral were used. Then from (5) it follows that:

$$\sum_j \|u_{x_j}(t,\cdot)\|_\infty \le C \text{ (right hand side of (5))},$$

hence J will be estimated using the inequality $(a+b+c)^2 \le 3(a^2+b^2+c^2)$ and the Cauchy inequality as stated:

$$\begin{aligned} J &\le \frac{2n+1}{n+1} A \sum_i \|(u_h^{n+1})_{x_i}\|_2 \, \|u_h^n\|_{\frac{2n+2}{n}} \times \\ &\times C[\delta(\|u_t(t+h,\cdot)\|_{2n+2}+NC)+C_\delta \|u(t+h,\cdot)\|_\infty] \\ &\le \frac{2n+1}{n+1} A \Big\{ \frac{\varepsilon_3}{2} \sum_i \|(u_h^{n+1})_{x_i}\|_2^2 + \frac{3}{2\varepsilon_3} \|u_h^n\|_{\frac{2n+2}{n}}^2 \times \\ &\times C^2[\delta^2(\|u_t(t+h,\cdot)\|_{2n+2}^2 + N^2C^2)+C_\delta^2 M_1^2] \Big\}, \end{aligned}$$

provided that $t+h \le T_1$. Adding and subtracting, the last component in (26) is estimated using the Cauchy and $|a|^{2n+1} \le a^{2n+2}+a^{2n}$ inequalities, as follows:

$$\begin{aligned} &\int h^{-1}(f|_{t+h}-f|_t) u_h^{2n+1} dx \le L_t \int |u_h|^{2n+1} dx \\ &+ L_u \int u_h^{2n+2} dx + L_x \int \sum_i |u_{hx_i} u_h^{2n+1}| dx \\ &\le \frac{L_x \varepsilon_4}{2(n+1)} \int \sum_i [(u_h^{n+1})x_i]^2 dx + \Big[L_t + L_u + \frac{L_x n}{2(n+1)\varepsilon_4}\Big] \int u_h^{2n+2} dx \\ &+ L_t \int u_h^{2n} dx. \end{aligned}$$

The resulting estimate integrated over $(0,s)$ with $s \in (0, T_1-h)$ (s arbitrary), takes the form:

(27)
$$\begin{aligned} &\frac{1}{2}\Big[\int u_h^{2n+2}(s,x)dx - \int u_h^{2n+2}(0,x)\ dx\Big] \\ &\le \int_0^s \Big\{\Big[-\frac{2n+1}{n+1} a_0 + (2n+1)A\frac{\varepsilon_3}{2} + \frac{L_x \varepsilon_4}{2}\Big] \int \sum_i [(u_h^{n+1})_{x_i}]^2 dx \\ &+ (2n+1)\frac{2AC^2}{2\varepsilon_3}[\delta^2 \|u_t(t+h,\cdot)\|_{2n+2}^2 + N^2C^2\delta^2 + C_\delta^2 M_1^2] \|u_h^n\|_{\frac{2n+2}{n}}^2 \\ &+ \Big[L_t + L_u + \frac{L_x n}{2(n+1)\varepsilon_4}\Big](n+1) \int u_h^{2n+2} dx + L_t(n+1) \int u_h^{2n} dx \Big\} dt. \end{aligned}$$

Let us take $\varepsilon_3 = \varepsilon_3^0 := \frac{a_0}{(2n+1)A}$, $\varepsilon_4 = \varepsilon_4^0 := \frac{a_0}{L_x}$. Since $\frac{3}{2} \le \frac{2n+1}{n+1} \le 2$ for n

$=1, 2,\ldots,$ then the first square bracket on the right side of (27) becomes non-positive. Passing with h to 0^+ in (27) (with $\varepsilon_i=\varepsilon_i^0$, $i=3, 4$), using the relation between the L^{2n} and L^{2n+2} norms and denoting $z(t):=\int u_t^{2n+2}(t,x)dx$,

$$\alpha:=\frac{3}{2a_0}(2n+1)^2A^2C^2\delta^2+(L_t+L_u)\ (n+1)+\frac{L_x^2 n}{2a_0}, \tag{28 a}$$

$$\beta:=C^2\Big[\frac{3}{2a_0}(2n+1)^2A^2C^2(N^2C^2\delta^2+C_\delta^2M_1^2)+L_t(n+1)\Big], \tag{28 b}$$

we obtain from (27) an integral inequality of the Bihari type:

$$z(s)\le z(0)+2\int_0^s\{\alpha z(t)+\beta z^{\frac{n}{n+1}}(t)\}dt. \tag{29}$$

The solution of (29) for the explicit form of z is given by:

$$\|u_t(t,\cdot)\|_{2n+2}^2\le\|u_t(0,\cdot)\|_{2n+2}^2\exp\Big(\frac{2\alpha t}{n+1}\Big)+\frac{\beta}{\alpha}[\exp\Big(\frac{2\alpha t}{n+1}\Big)-1]. \tag{30}$$

The quantity $u_t(0,x)$ in (30) will be found from (1) with $t=0$ (u is smooth enough to do so), hence:

$$r=\|u_t(0,\cdot)\|_{2n+2}=\|Pu_0|_{t=0}+f(0,\cdot,u_0,(u_0)_x)\|_{2n+2}.$$

The proof of Lemma 3 is completed.

We are now able to determine the time T introduced in the formulation of Theorem 1. Noting (5) let τ_2 be given by the condition:

$$\delta\Big\{\Big[r^2\exp\Big(\frac{2\alpha\tau_2}{n+1}\Big)+\frac{\beta}{\alpha}\Big(\exp\Big(\frac{2\alpha\tau_2}{n+1}\Big)-1\Big)\Big]^{\frac{1}{2}}+NC\Big\}$$
$$+C_\delta(\text{right hand side of (18) with } t=\tau_2)=M_2, \tag{31}$$

and let $T:=\min\{T_1;\tau_2\}$. We are now sure that for $t\in[0, T]$ the solution u remains in X. Hence all the considerations of Lemmas 1, 2, 3 remain valid for t in $[0, T]$. The proof of Theorem 1 is completed.

Local existence.

The *a* priori estimates of Theorem 1 are sufficient to establish the existence of the uniformly Hölder continuous solution to (1)-(3). Since the proof is relatively standard (c. f. [6, 7]) only an outline will be given.

THEOREM 2. *Under the Assumptions taken, there exists a unique*

$C^{1+\frac{\nu}{2},2+\nu}([0,\ T]\times\bar{\Omega})$ *solution* u *of* (1)-(3) (*here* $\nu:=\min\left\{\gamma;\frac{1}{2}\ \frac{\mu}{\mu+1}\right\}$, $\mu\geq\frac{1}{2}$ as in (35)).

PROOF: We start with the proof of uniqueness which is the result of the local Lipschitz continuity of f with respect to u and u_x. Let us denote $w:=u-v$, for two different solutions u, v of (1)-(3) (satisfying the same initial-boundary condition) remaining in X for $t\leq T$. Then from (1):

$$w_t=\sum_{i,j}(a_{ij}(t,x)w_{x_j})_{x_i}+f(t,x,u,u_x)-f(t,x,v,v_x). \tag{32}$$

Multiplying (32) by w, integrating over Ω and by parts, adding and subtracting, we have:

$$\begin{aligned}\frac{1}{2}\ \frac{d}{dt}\int w^2dx&=-\int_{\partial\Omega}w\frac{\partial w}{\partial N}ds-\int\sum_{i,j}a_{ij}(t,x)w_{x_i}w_{x_j}dx\\&+\int[f(t,x,u,u_x)-f(t,x,u,v_x)+f(t,x,u,v_x)-f(t,x,v,v_x)]\ w\ dx\\&\leq-a_0\int\sum_i(w_{x_i})^2dx+L_x\int\sum_i|w_{x_i}w|\ dx+L_u\int w^2\ dx\\&\leq\left[-a_0+\frac{\varepsilon_5L_x}{2}\right]\int\sum_i(w_{x_i})^2dx+\left[\frac{L_x}{2\varepsilon_5}+L_u\right]\int w^2\ dx.\end{aligned}$$

With the use of the Gronwall inequality, the last estimate with $\varepsilon_5=\varepsilon_5^0:=\frac{2a_0}{L_x}$ finishes the proof of uniqueness of u.

We proceed to an outline of the proof of existence. Below we show how the *a* priori extimates of Theorem 1 will be strengthened to get *a* priori estimates of u and u_x in the Hölder space of the argument (t,x). The proper proof of existence is then standard ([13, 6]).

Consider (1), with fixed $t\in(0,\ T]$, as a linear elliptic equation:

$$\begin{aligned}\sum_{i,j}a_{ij}(t,x)u_{x_jx_i}+\sum_j\Big(\sum_i a_{ij}(t,x)_{x_i}\Big)u_{x_j}\\=u_t-f(t,x,u,u_x)=:\tilde{f}(t,x).\end{aligned} \tag{33}$$

For $t\in(0,\ T]$, as a consequence of (18), (24) and (5), the function $\tilde{f}(t,\cdot)$ belongs to $L^{2n+2}(\Omega)$ (with the norm estimated uniformly with respect to t). Also, the coefficients $a_{ij}(t,\cdot)$. $\sum_i a_{ij}(t,\cdot)_{x_i}$ are smooth enough for the Calderon-Zygmund estimates ([14, 17, 21, 6]) to be satisfied:

$$\|u(t,\cdot)\|_{2,2n+1}\leq c_3(\|\tilde{f}(t,\cdot)\|_{2n+1}+\|u(t,\cdot)\|_1)\leq\text{const.} \tag{34}$$

for $t\leq T$. Next, as a consequence of the Sobolev imbeddings we also have:

(35) $\|u_{x_i}(t,\cdot)\|_{C^{\mu}(\bar{\Omega})}\leq c_4\|u_{x_i}(t,\cdot)\|_{2,2n+1},\ i=1,\ldots,n,$

for $\mu:=1-\frac{n}{2n+1}(\geq\frac{1}{2})$. In particular, $u_{x_i}(t,\cdot)\in C^{\frac{1}{2}}(\bar{\Omega})$ with the norm bouded independently of $t\in[0, T]$. As a consequence of Lemma 3 $u_t(t,\cdot)\in L^{2n+2}(\Omega)$, with the norm bounded uniformly for $t\in[0, T]$, hence as a consequence of Theorem 1 we have the *a* priori estimates:

(36) $\|u_{x_i}\|_Y\leq$const., $\|u_t\|_Y\leq$const.,

$Y:=L^{\infty}([0, T]; L^{2n+2}(\Omega))$. Using the Sobolev theorems once more (now in R^{n+1}), as a result of (36) and (18) we verify that ($Y\subset L^{2n+2}([0, T]\times\Omega)$)

(37) $u\in C^{\frac{1}{2},\frac{1}{2}}([0, T]\times\bar{\Omega})$,

then from Lemma 3.1, Chapt. II of [13], (35) and (37) it follows that:

(38) $u_{x_i}\in C^{\frac{\omega}{2},\omega}([0, T]\times\bar{\Omega})$ with $\omega:=\frac{1}{2}\frac{\mu}{\mu+1}$,

in particular $u_{x_i}\in C^{\frac{1}{12},\frac{1}{6}}([0, T]\times\bar{\Omega})$. Due to (37) and (38) the nonlinear term f in (1) will be considered as the uniformly Hölder continuous "right hand side". The standard use of Schauder type estimates for linear parabolic equations (c. f. [10, p. 65] for the Dirichlet condition, [12] for the third boundary condition) and the Leray-Schauder Principle [14, 13, 6] shows the existence of a $C^{1+\frac{\nu}{2},2+\nu}([0, T]\times\bar{\Omega})$ solution of (1)-(3) for $\nu:=\min\{\gamma, \omega\}$. The proof of Theorem 2 is completed.

To show further possible consequences of the *a* priori estimates of Theorem 1 we formulate, for the case of the autonomous equation, the following observation concerning the life time of solutions in a neighborhood of the trivial solution.

LEMMA 4: *If the coefficients a_{ij}, $f(i, j=1,\ldots, n)$ are time independent (then the limitation $t\leq T_0$ in the definition of X is not necessary) and the trivial solution for* (1)-(3) *is admitted, there are solutions to our problem different from $\bar{u}\equiv 0$ (and close to $\bar{u}$ for $t=0$) having arbitrarily long life time T_{ex}.*

PROOF: Let us note that the time τ_1 in (23) is a decreasing function of m_1 and provided that $N=0$ (since $\bar{u}$ is a solution to (1)-(3)) we could find solutions corresponding to sufficiently small m_1, with arbitrarily long τ_1. Also it follows from (31) that the time τ_2 is a decreasing function of

r. Again, if $\beta=0$ (which is true (28) if $A=L_t=0$, that is if a_{ij}, f are time independent) we could, for fixed M_2, find solutions with arbitrarily long τ_2 (and r correspondingly small). Since $T=\min\{\tau_1; \tau_2\}$, the proof is finished.

Comments.

As was shown in [6] our technique is applicable to the problems with a_{ij} depending also on u, and [7] to systems of parabolic equations. Our estimates of the time $T(<T_{ex})$ (23), (31) are fully effective at least for equations with a_{ij} time independent ($i, j=1,\dots, n$) and f independent on u_x, as shown by the examples which follow. When the nonlinearity f depends on the space gradient of u or a_{ij} depend on t, there are some constants (K in (ii), c in (iii), c_1, c_2 in (9), $c_{s,\infty}$ in (11)) characterizing the domain Ω and also boundary conditions (3a) or (3b), which in the general case had to be found (compare [15] and related) before giving our etimates. This is the reason why we restrict our examples here to simpler problems. Taking the cited constants (or rather the synthetic constant C_δ in (6)) for the actual domain Ω and boundary condition, the life time T_{ex} will be estimated using (23) and (31) with (28) for general equation (1).

There is an interesting observation made in [19] that the Hölder continuity in x of the right hand side of the linear parabolic equation is not necessary for the existence of the classical solution. Nowhere in our paper, except at the end of the proof of Theorem 2, has the Hölder continuity of a_{ij}, a_{ijx_i}, f with respect to x been used. Also the Hölder constants for a_{ij}, a_{ijx_i}, f were not defined anywhere. Thus it would appear to be possible that at least the assumption of the Hölder continuity of f with respect to x is not necessary for local existence of the classical solution of (1)-(3).

Examples.

We close our paper giving examples of estimates of the life time T_{ex} for solutions to simple parabolic equations. Here Lemma 1 is effectively used.

EXAMPLE 1. Consider the problem:

(39) $$u_t=\Delta u+u|u|^{p-1},\ p>1$$

(40) $$u(0, x)=u_0(x) \text{ in } \Omega,\ u=0 \text{ on } \partial\Omega.$$

The second order term has constant coefficients and the nonlinearity is gradient independent. Thus only L_u is non-zero and hence taking $m_1=10$,

$M_1=10^4$ we have $L_u=p\ 10^{4(p-1)}$ and the time T is given by formula (23):

$$10^2 \exp(2Tp\ 10^{4(p-1)})=10^8,$$

from which

$$T=\frac{3\ \ln 10}{p\ 10^{4(p-1)}}<T_{ex}. \tag{41}$$

Here the restriction $t\leq T_0$ is not necessary as the coefficients are time independent, the same being true for their Lipschitz constants.

EXAMPLE 2. Consider the problem similar to that studied by J. W. Bebernes:

$$u_t=\Delta u+(1+t^3)\ \exp(u),\ x\in\Omega, \tag{42}$$

with the condition (40). We must now limit t in advance in order to have uniform (inside X) Lipschitz constants; set $t\leq T_0=1$. Put $m_1=1$, $M_1=10^3$, so that $L_u=2\exp(1000)$, $N=1$. The formula (23) then reads:

$$\left\{1+[2\exp(1000)+1]^{-1}\ [1-\exp(-2T(2\exp(1000)+1))]\right\}\times \\ \times\exp[2T(2\exp(1000)+1)]=10^6,$$

hence

$$T=[2(2\exp(1000)+1)]^{-1}\ \ln\left(\frac{10^6+(2\exp(1000)+1)^{-1}}{1+(2\exp(1000)+1)^{-1}}\right)<T_{ex}. \tag{43}$$

References

[0] P. ACQUISTAPACE and B. TERRENI, *On quasilinear parabolic systems,* Math. Ann. **282** (1988), 315-335.

[1] R. A. ADAMS, "Sobolev Spaces," Academic Press, New York, 1975.

[2] N. D. ALIKAKOS, *An application of the invariance principle in reaction-diffusion equations,* J. Differential Equations **33, 2** (1979), 201-225.

[3] H. AMANN, *Parabolic evolution equations and nonlinear boundary conditions,* J. Differential Equations **72, 2** (1988), 201-269.

[4] J. W. BEBERNES and K. SMITT, *On the existence of maximal and minimal solutions for parabolic partial differential equations,* Proc. Amer. Math. Soc. **73, 2** (1979), 211-218.

[5] T. DLOTKO, *Global solutions of reaction-diffusion equations,* Funkcial. Ekvac. **30, 1** (1987), 34-43.

[6] T. DLOTKO, "Parabolic Equations in Divergence Form. Theory of Global Solutions," Wydawnictwo Uniwersytetu Slaskiego, Katowice, 1987.

[7] T. DLOTKO, *A priori estimates for a Navier-Stokes like system,* Riv. Mat. Univ. Parma. **(4) 13** (1987), 187-199.

[8] T. DLOTKO, *Examples of parabolic problems with blowing-up derivatives,* J. Math. Anal.

Appl. 154 (1991), 226-237.

[9] P. C. FIFE, *Mathematical Aspects of Reacting and Diffusing Systems,* "Lecture Notes in Biomathematics **28**," Springer-Verlag, New York, 1979.

[10] A. FRIEDMAN, "Partial Differential Equations of Parabolic Type," Prentice-Hall, Englewood Cliffs, 1964.

[11] A. FRIEDMAN, *Blow-up of solutions of nonlinear parabolic equations,* in "Nonlinear Diffusions Equations and Their Equilibrium States I, W.-M. Ni, L. A. Peletier and J. Serrin (Eds.)," Springer-Verlag, Berlin, Heidelberg, New York, 1988, pp. 301-318.

[12] L. I. KAMYNIN and V. N. MASLENNIKOVA, *Boundary estimates of the solution to the problem involving a skew derivative for a parabolic equation in non-cylindrical region,* Dokl. Akad. Nauk SSSR **160, 3** (1965), 523-526.

[13] O. A. LADYŽENSKAJA, V. A. SLONNIKOV and N. N. URAL'CEVA, "Linear and Quasiliniar Equations of Parabolic Type," Amer. Math Soc., Providence, R. I., 1968.

[14] O. A. LADYŽHENSKAYA and N. N. URAL'TSEVA, "Linear and Quasilinear Elliptic Equations," Academic Press, New York, 1968.

[15] E. LIEB, *Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities,* Ann. of Math. **118, 2** (1983), 349-374.

[16] G. MAZJA, "Sobolev Spaces," Springer-Verlag, Berlin, 1985.

[17] C. B. MORREY, "Multiple Integrals in the Calculus of Variations," Springer-Verlag, New York, 1966.

[18] F. ROTHE, *A priori estimates for reaction-diffusion systems,* in "Nonlinear Diffusion Equations and Their Equilibrium States II, W.-M. Ni, L. A. Peletier and J. Serrin (Eds.)," Springer-Verlag, Berlin, Heidelberg, New York, 1988, pp. 235-244.

[19] E. SINESTRARI and W. von WAHL, *On the solutions of the first boundary value problem for the linear parabolic equations,* Proc. Roy. Soc, Edinburgh Sect **A, 108** (1988), 339-355.

[20] P. E. SOBOLEVSKII, *Local and non-local theorems of existence for non-linear second -order parabolic equations,* Dokl. Akad. Nauk SSSR **160, 3** (1965), 523-526.

[21] V. A. SOLONNIKOV, On Lp estimates of solutions of elliptic and parabolic systems, Trudy Mat. Inst. Steklov **102** (1967), 137-160.

[22] H. TANABE, "Equations of Evolution," Pitman Publishing Ltd., London, 1979.

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