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Timelike surfaces in Lorentz 3-space with prescribed mean curvature and Gauss map

Martin A. MAGID

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A timelike surface M^2 in Lorentz 3-space $\boldsymbol{L}^3$ is a surface which inherits a non-degenerate indefinite metric from the standard metric in $\boldsymbol{L}^3$. A Gauss map can be defined on $\boldsymbol{M}^2$ with values in the unit sphere $S_1^2 \subset \boldsymbol{L}^3$.

We will prove in Theorem 3.1 that the Gauss map and mean curvature of a timelike surface satisfies a system of partial differential equations. As a corollary the Gauss map of a timelike minimal surface is shown to satisfy a simple hyperbolic system. This is the precise analogue of the theorem that the Gauss map of a minimal surface in Euclidean space is a holomorphic map into the Riemann sphere. In the latter case the Cauchy-Riemann equations should be thought of as a simple elliptic system of partial differential equations.

In section 4 we find representations for a timelike surface in terms of its Gauss map and mean curvature. The integrability condition for this formula is a pair of partial differential equations (5.2a, b). In Theorem 6.1 we show that given functions defined on a simply connected surface which satisfy the integrability conditions we can find an isometric immersion with these functions as Gauss map and mean curvature.

Let us also note that in Theorem 4.3 we give a Weierstrass representation for timelike minimal surfaces without flat points. Timelike minimal surfaces have recently been the subjects of several papers [Ma2], [Mi2], [Mi3] and [Mi4].

All of these results are timelike versions of the work of K. Kenmotsu [Ke] for a surface in Euclidean 3-space and K. Akutagawa and S. Nishikawa [A-N] for a spacelike surface in $\boldsymbol{L}^3$, and our debt to these authors is clear. They consider M^2 as a Riemann surface, introducing a complex variable via isothermal coordinates. Thus their results are cast in the language of complex analysis. In the timelike case there is no such natural complex structure on M^2. It is somewhat surprising that the same types of results can still be proven, but this really shows that complex analysis is, for the most part, a useful calculational device in [Ke] and

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[A-N] and not involved in the essence of the problems.

1. Notation and preliminaries

$\boldsymbol{L}^3$ denotes the vector space $\boldsymbol{R}^3$ with the metric $((x, y, z), (x, y, z))= -x^2+y^2+z^2$. $X: M^2 \to \boldsymbol{L}^3$ will denote an immersion from a surface M^2 on which the induced metric, g, is *timelike*. A timelike metric is non-degenerate but not definite. We will always use local isothermal coordinates (t,s), so that the metric g has the form

$$(1.1) \qquad g=\lambda^2(-dt^2+ds^2).$$

Such coordinates can always be found ([Mi2], [Ku]).

If X is written in coordinate form $X=(X^1, X^2, X^3)$, we can start to form an orthonormal frame $\{e_1, e_2, e_3\}$ by setting

$$e_1=\frac{1}{\lambda}X_t=\frac{1}{\lambda}\left(X_t^1, X_t^2, X_t^3\right) \text{ and } e_2=\frac{1}{\lambda}X_s=\frac{1}{\lambda}\left(X_s^1, X_s^2, X_s^3\right)$$

Here, for example, $X_t^1=\dfrac{\partial X^1}{\partial t}$. To complete the frame, set $e_3=e_1\times e_2$, the Lorentzian cross product of e_1 and e_2. Thus,

$$(1.2) \qquad e_3=\frac{1}{\lambda^2}(X_s^2X_t^3-X_t^2X_s^3,\ X_s^1X_t^3-X_t^1X_s^3,\ X_t^1X_s^2-X_t^2X_s^1).$$

Occasionally we write this as $e_3=\dfrac{1}{\lambda^2}(C^1, C^2, C^3)=(e_3^1, e_3^2, e_3^3)$ where C is meant to stand for the cross product.

Next set $h_{ij}=(D_{e_i}e_j, e_3)=(Ae_i, e_j)$, $1\le i, j\le 2$, where A is the shape operator of the immersion and D is covariant differentiation in $\boldsymbol{L}^3$. With respect to the basis $\{e_1, e_2\}$ the matrix representing A has the form $\begin{bmatrix} -h_{11} & -h_{12} \\ h_{12} & h_{22} \end{bmatrix}$. The symmetry of A with respect to g is reflected in the fact that the off-diagonal elements are additive inverses.

Given the metric in 1.1 we can calculate the Christoffel symbols for $\{\partial/\partial t, \partial/\partial s\}$. Identifying t with the index 1 and s with the index 2, $\Gamma_{11}^1=\Gamma_{12}^2=\Gamma_{22}^1=\dfrac{\lambda_t}{\lambda}$ and $\Gamma_{22}^2=\Gamma_{12}^1=\Gamma_{11}^2=\dfrac{\lambda_s}{\lambda}$. Thus,

$$(1.3) \qquad X_{tt}=\frac{\lambda_t}{\lambda}X_t+\frac{\lambda_s}{\lambda}X_s+\lambda^2h_{11}e_3$$

$$X_{st}=\frac{\lambda_s}{\lambda}X_t+\frac{\lambda_t}{\lambda}X_s+\lambda^2h_{12}e_3$$

$$X_{ss}=\frac{\lambda_t}{\lambda}X_t+\frac{\lambda_s}{\lambda}X_s+\lambda^2 h_{22}e_3$$

and

(1.4 a) $e_{3_t}=h_{11}X_t-h_{12}X_s$

(1.4 b) $e_{3_s}=h_{12}X_t-h_{22}X_s$.

One of the keys to the calculations is defining various functions of the h_{ij}. As usual the mean curvature is defined to be $h=\frac{1}{2}(h_{22}-h_{11})$. We also define the functions $a=-h_{12}$ and $b=\frac{1}{2}(h_{11}+h_{22})$. Thus 1. 4a and 1. 4b become

(1.5 a) $e_{3_t}=-hX_t+bX_t+aX_s$

(1.5 b) $e_{3_s}=-hX_s-aX_t-bX_s$.

The sectional curvature of M^2 is $h_{12}^2-h_{11}h_{22}=h^2-b^2+a^2$. We say that M^2 is *minimal* iff $h=0$. It follows from 1. 3 that $X_{ss}-X_{tt}=0$ iff M^2 is minimal.

2. The Gauss map

The classical Gauss map of a surface in E^3(Euclidean 3-space) is the map which assigns to each point on the surface its unit normal, translated to the origin. This gives a point on S^2. In the case of a timelike surface we take the same definition, where the appropriate sphere $S_1^2=\{\boldsymbol{x}\in\boldsymbol{L}^3:(\boldsymbol{x},\boldsymbol{x})=1\}$ is used. The Gauss map $G:M^2\to S_1^2$ is defined by $G(p)=e_3(p)$. S_1^2, a hyperboloid of one sheet, has constant sectional curvature 1.

Stereographic projection can be defined from S_1^2 to $\boldsymbol{L}^2$, Lorentzian 2-space. Let $U_1=S_1^2-\{z=-1\}$ and $U_2=S_1^2-\{z=1\}$. Define

$$\pi_1(x, y, z)=\left(\frac{x}{1+z}, \frac{-y}{1+z}\right) \text{ for } (x, y, z)\in U_1 \text{ and}$$

$$\pi_2(x, y, z)=\left(\frac{x}{1-z}, \frac{y}{1-z}\right) \text{ for } (x, y, z)\in U_2.$$

This is the intersection of the plane $z=0$ with the line joining (x, y, z) to the appropriate pole.

If $\boldsymbol{L}^2$ has the metric $((u, v), (u, v))=-u^2+v^2$ then each π_i, $i=1, 2$ is conformal. Most of the formulas which follow will be in terms of $\pi_i\circ G$. We write

$$(2.1)\qquad \pi_1\circ G=(\psi_1, \psi_2)=\left(\frac{e_3^1}{1+e_3^3}, \frac{-e_3^2}{1+e_3^3}\right) \text{ and}$$

$$(2.2)\qquad \pi_2\circ G=(\mu_1, \mu_2)=\left(\frac{e_3^1}{1-e_3^3}, \frac{e_3^2}{1-e_3^3}\right)$$

Note that the image of $\pi_i(x, y, z)$ for $(x, y, z)\in U_i$ does not intersect the set $\{(u, v)\in \boldsymbol{L}^2 : 1+v^2-u^2=0\}$. We use the symbol $|(u, v)|=v^2-u^2$ to denote a "norm" in $\boldsymbol{L}^2$. This is simply a notational convenience, since it can obviously take negative values. By a further abuse of notation we set $|\psi|^2=\psi_2{}^2-\psi_1{}^2$ and $|\mu|^2=\mu_2{}^2-\mu_1{}^2$. In this notation $1+|\psi|^2$ and $1+|\mu|^2$ are non-zero. For simplicity all formulas will be given in terms of (ψ_1, ψ_2) in the body of the paper, while the versions with μ_1 and μ_2 will be saved for an appendix. Thus we are always assuming that $z=-1$ is not in the image of the Gauss map.

3. Lemmas comparing the partial derivatives of X and ψ

The basis for all the theorems are the relationships between the partial derivatives of X and ψ. Each equation found in [Ke] or [A-N] usually will have two counterparts. Roughly speaking this is because their complex equations have a real and an imaginary part while we have only real equations.

LEMMA 3.1. *If $X : M^2 \to \boldsymbol{L}^3$ is a timelike isometric immersion then*

(3.1 a) $X_t^1-\psi_1 X_t^3=X_s^2+\psi_2 X_s^3$

(3.1 b) $X_t^2+\psi_2 X_t^3=X_s^1-\psi_1 X_s^3$.

PROOF: This is essentially a verification. We will give the calculation only for 3.1a. Examine

$$X_t^1-\psi_1 X_t^3-X_s^2+\psi_2 X_s^3=X_t^1-X_s^2-\frac{e_3^1}{1+e_3^3}X_t^3+\frac{e_3^2}{1+e_3^3}X_s^3.$$

Using 1.2 this equals $(\lambda^2+C^3)^{-1}((\lambda^2+C^3)(X_t^1-X_s^2)-C^1X_t^3+C^2X_s^3)$. Writing λ^2 as either $-(X_s^1)^2+(X_s^2)^2+(X_s^3)^2$ or $(X_t^1)^2-(X_t^2)^2-(X_t^3)^2$ this becomes

$$\begin{aligned}(\lambda^2+C^3)^{-1}((-(X_s^1)^2+(X_s^2)^2+(X_s^3)^2)X_t^1-((X_t^1)^2-(X_t^2)^2-(X_t^3)^2)X_s^2+\\ (X_t^1X_s^2-X_t^2X_s^1)(X_t^1-X_s^2)-(X_s^2X_t^3-X_t^2X_s^3)X_t^3+(X_s^1X_t^3-X_t^1X_s^3)X_s^3)\\ =(\lambda^2+C^3)^{-1}((X_s^1+X_t^2)(-X_s^1X_t^1+X_s^2X_t^2+X_s^3X_t^3))=0.\end{aligned}$$

Note that $(X_s, X_t)=0$ is the last factor above. QED

LEMMA 3.2. *If $X : M^2 \to \boldsymbol{L}^3$ is a timelike isometric immersion then*

(3.2 a) $X_t^3=\psi_1(X_t^1+X_s^2)+\psi_2(X_s^1+X_t^2)$

(3.2 b) $X_s^3=+\psi_2(X_t^1+X_s^2)+\psi_1(X_s^1+X_t^2)$.

PROOF: We will only verify 3.2a.

$$\psi_1(X_t^1+X_s^2)+\psi_2(X_s^1+X_t^2)-X_t^3=$$
$$(\lambda^2+C^3)^{-1}(C^1(X_t^1+X_s^2)-C^2(X_s^1+X_t^2)-(\lambda^2+C^3)X_t^3).$$

After substituting for C^1, C^2, C^3 and using $\lambda^2=-(X_s^1)^2+(X_s^2)^2+(X_s^3)^2$ this equals

$$(\lambda^2+C^3)^{-1}(X_s^3(X_t^1X_s^1-X_t^2X_s^2-X_s^3X_t^3))=0.\quad \text{QED}$$

LEMMA 3.3. *If $X: M^2 \to \boldsymbol{L}^3$ is a timelike isometric immersion then*

$$(3.3\text{a})\quad \frac{4\lambda^2\psi_1}{(1+|\psi|^2)^2}=X_t^3(X_t^1+X_s^2)-X_s^3(X_s^1+X_t^2)$$

$$(3.3\text{b})\quad \frac{4\lambda^2\psi_2}{(1+|\psi|^2)^2}=-X_t^3(X_s^1+X_t^2)+X_s^3(X_t^1+X_s^2).$$

Before this lemma can be proved note that we have an identity which is easily established using $-(e_3^1)^2+(e_3^2)^2+(e_3^3)^2=1$.

$$(3.4)\qquad (1+|\psi|^2)(1+e_3^3)=2.$$

PROOF of 3.3b: Using 3.4 the left-hand side of 3.3b is

$$\lambda^2e_3^2(1+e_3^3)=C^2\left[1+\frac{C^3}{\lambda^2}\right]=(X_s^1X_t^3-X_t^1X_s^3)\left[1+\frac{X_t^1X_s^2-X_t^2X_s^1}{\lambda^2}\right].$$

Then

$$\frac{4\lambda^2\psi_2}{(1+|\psi|^2)^2}+X_t^3(X_s^1+X_t^2)-X_s^3(X_t^1+X_s^2)$$ is easily seen to be zero.

QED

The next lemma gives expressions for $\psi_{1t}-\psi_{2s}$ and $\psi_{1s}-\psi_{2t}$ in terms of X_t^i and X_s^j, for i, $j=1, 2, 3$.

LEMMA 3.4. *If $X: M^2\to \boldsymbol{L}^3$ is a timelike isometric immersion then*

$$(3.5\text{a})\quad \psi_{1t}-\psi_{2s}=-(h/2)(1+|\psi|^2)^2(X_t^1+X_s^2)$$
$$(3.5\text{b})\quad \psi_{1s}-\psi_{2t}=-(h/2)(1+|\psi|^2)^2(X_s^1+X_t^2).$$

PROOF of 3.5a:

$$\psi_{1t}-\psi_{2s}=\left[\frac{e_3^1}{1+e_3^3}\right]_t+\left[\frac{e_3^2}{1+e_3^3}\right]_s$$
$$=(1+e_3^3)^{-1}(e_{3t}^1-\psi_1e_{3t}^3+e_{3s}^2+\psi_2e_{3s}^3).$$

Using 1.5 and 3.4, we have

$$\psi_{1t}-\psi_{2s}=\tfrac{1}{2}(1+|\psi|^2)(-hX^1_t+bX^1_t+aX^1_s-\psi_1(-hX^3_t+bX^3_t+aX^3_s)$$
$$-hX^2_s-aX^2_t-bX^2_s+\psi_2(-hX^3_s-aX^3_t-bX^3_s))$$
$$=\tfrac{1}{2}(1+|\psi|^2)(h(-X^1_t+\psi_1X^3_t-X^2_s-\psi_2X^3_s)) \text{ by } 3.1\text{a, b.}$$

Using 3.2, this becomes

$$=-\tfrac{1}{2}h(1+|\psi|^2)^2(X^1_t+X^2_s). \quad \text{QED}$$

The next lemma in this sequence gives expressions for $\psi_{1t}+\psi_{2s}$ and $\psi_{2t}+\psi_{1s}$.

LEMMA 3.5. *If $X: M^2\to \boldsymbol{L}^3$ is a timelike isometric immersion then*

$$(3.6\text{a})\quad \psi_{1t}+\psi_{2s}=\frac{\left[1+|\psi|^2\right]^2}{2}[b(X^1_t+X^2_s)+a(X^1_s+X^2_t)]$$

$$(3.6\text{b})\quad \psi_{1s}+\psi_{2t}=\frac{-\left[1+|\psi|^2\right]^2}{2}\left[a(X^1_t+X^2_s)+b(X^1_s+X^2_t)\right]$$

PROOF: First we prove 3.6a. As above

$$\psi_{1t}+\psi_{2s}=\tfrac{1}{2}(1+|\psi|^2)(e^1_{3t}-\psi_1e^3_{3t}-e^2_{3s}-\psi_2e^3_{3s}).$$

Using 1.5, 3.1 and then 3.2, this becomes

$$\tfrac{1}{2}(1+|\psi|^2)^2(a(X^1_s+X^2_t)+b(X^1_t+X^2_s)). \quad \text{QED}$$

For later use we record

COROLLARY 3.1. *If $X: M^2\to \boldsymbol{L}^3$ is a timelike isometric immersion then*

$$(3.6\text{c})\quad aX^3_t+bX^3_s=\frac{2}{\left[1+|\psi|^2\right]^2}(-\psi_1(\psi_{1s}+\psi_{2t})+\psi_2(\psi_{1t}+\psi_{2s}))$$

$$(3.6\text{d})\quad bX^3_t+aX^3_s=\frac{2}{\left[1+|\psi|^2\right]^2}(\psi_1(\psi_{1t}+\psi_{2s})-\psi_2(\psi_{1s}+\psi_{2t})).$$

PROOF: This follows from substituting 3.6a, b in 3.2a, b. QED

COROLLARY 3.2. *If $X: M^2\to \boldsymbol{L}^3$ is a timelike isometric immersion then*

$$(3.7)\quad \frac{4\lambda^2}{(1+|\psi|^2)^2}=(X^1_t+X^2_s)^2-(X^1_s+X^2_t)^2$$

$$(3.8\text{a})\quad (\psi_{1t}-\psi_{2s})^2-(\psi_{1s}-\psi_{2t})^2=\lambda^2h^2(1+|\psi|^2)^2$$

$$(3.8\text{b})\quad (\psi_{1t}+\psi_{2s})^2-(\psi_{1s}+\psi_{2t})^2=\lambda^2(b^2-a^2)(1+|\psi|^2)^2.$$

PROOF: To prove 3.7 we note that from 3.3a and 3.2

$$\frac{4\lambda^2\psi_1}{(1+|\psi|^2)^2}=\psi_1((X_t^1+X_s^2)^2-(X_s^1+X_t^2)^2).$$

In the same way we also have $\dfrac{4\lambda^2\psi_2}{\left[1+|\psi|^2\right]^2}=\psi_2((X_t^1+X_s^2)^2-(X_s^1+X_t^2)^2)$, if we begin with 3.3b. If $\psi_1(p)$ or $\psi_2(p)\neq 0$ then 3.7 holds at p. If $\psi_1(p)=0$ and $\psi_2(p)=0$ then 3.1 implies that $X_t^1(p)=X_s^2(p)$ and $X_t^2(p)=X_s^1(p)$. Because $\{t, s\}$ is an isothermal coordinate system, $X_t^3(p)=0=X_s^3(p)$ and $\lambda^2(p)=(X_t^1)^2-(X_t^2)^2$. At p, then, 3.7 reduces to showing that $4((X_t^1)^2-(X_t^2)^2)=(2X_t^1)^2-(2X_t^2)^2$, which is true.

To prove 3.8a we first use 3.5 a, b to obtain

$$(\psi_{1t}-\psi_{2s})^2-(\psi_{1s}-\psi_{2t})^2=(h/2)^2(1+|\psi|^2)^4((X_t^1+X_s^2)^2-(X_s^1+X_t^2)^2).$$

This equals $h^2(1+|\psi|^2)^2\lambda^2$ by equation 3.7.

3.8b follows in the same way if we begin with 3.6a, b. QED

At this point we can see that the Gauss map satisfies a system of partial differential equations.

THEOREM 3.1. *The Gauss map of a timelike isometric immersion* $X: M^2\to \boldsymbol{L}^3$ *satisfies*

(3.9 a) $-h(\psi_{1t}+\psi_{2s})=b(\psi_{1t}-\psi_{2s})+a(\psi_{1s}-\psi_{2t})$

(3.9 b) $h(\psi_{1s}+\psi_{2t})=a(\psi_{1t}-\psi_{2s})+b(\psi_{1s}-\psi_{2t})$.

The proof is immediate from 3.5 and 3.6.

COROLLARY 3.3. *If* $X: M^2\to \boldsymbol{L}^3$ *is a timelike isometric immersion then*

1. $h(p)=0$ *iff* $\psi_{1t}(p)-\psi_{2s}(p)=0=\psi_{1s}(p)-\psi_{2t}(p)$
2. $a(p)=0=b(p)$ *iff* $\psi_{1t}(p)+\psi_{2s}(p)=0$ *and* $\psi_{1s}(p)+\psi_{2t}(p)=0$.

PROOF: By 3.8a, if $\psi_{1t}(p)-\psi_{2s}(p)=0$ and $\psi_{1s}(p)-\psi_{2t}(p)=0$ then $h=0$. The converse of 1. follows from 3.5.

Now suppose that $a=b=0$. By 3.6 we find that $\psi_{1t}(p)+\psi_{2s}(p)=0$ and $\psi_{2t}(p)+\psi_{1s}(p)=0$. On the other hand, if $\psi_{1t}(p)+\psi_{2s}(p)=0$ and $\psi_{1s}(p)+\psi_{2t}(p)=0$ then 3.8b implies that $a^2=b^2$. If a and b were nonzero, 3.6 would imply that $X_t^1+X_s^2=\pm(X_s^1+X_t^2)$. But from 3.7 it would follow that $\lambda^2=0$, a contradiction. QED

Theorem 3.1 and Corollary 3.3 are the analogues of the positive definite results which state that the Gauss map is holomorphic iff the surface is minimal and the Gauss map is anti-holomorphic iff the surface is umbilical. In the timelike setting M^2 is minimal or umbilical is equivalent to the requirement that the Gauss map satisfy a simple hyperbolic system, rather than, for example, the elliptic Cauchy-Riemann equations.

L. M. Berard's thesis [B] contains many examples of timelike minimal surfaces obtained by rotation of plane curves.

4. Representation theorems

In this section we will find representations for X_t^i and X_s^j in terms of the Gauss map and h or a and b. The representation in terms of $h \neq 0$ will eventually allow us to construct surfaces of prescribed mean curvature, while those in terms of a and b will enable us to find a Weierstrass-type representation formula for minimal surfaces.

THEOREM 4.1. *If $X: M^2 \to \boldsymbol{L}^3$ is a timelike isometric immersion with coordinates (X^1, X^2, X^3), mean curvature h and Gauss map ψ_1, ψ_2 then*

$$\text{(4.1)} \qquad hX_t^1 = \frac{1}{\left[1+|\psi|^2\right]^2}\left(-(1+\psi_1^2+\psi_2^2)(\psi_{1_t}-\psi_{2s})-2\psi_1\psi_2(\psi_{1_s}-\psi_{2t})\right)$$

$$hX_s^1 = \frac{1}{\left[1+|\psi|^2\right]^2}\left(-2\psi_1\psi_2(\psi_{1_t}-\psi_{2s})-(1+\psi_1^2+\psi_2^2)(\psi_{1_s}-\psi_{2t})\right)$$

$$hX_t^2 = \frac{1}{\left[1+|\psi|^2\right]^2}\left(2\psi_1\psi_2(\psi_{1_t}-\psi_{2s})+(\psi_1^2+\psi_2^2-1)(\psi_{1_s}-\psi_{2t})\right)$$

$$hX_s^2 = \frac{1}{\left[1+|\psi|^2\right]^2}\left((\psi_1^2+\psi_2^2-1)(\psi_{1_t}-\psi_{2s})+2\psi_1\psi_2(\psi_{1_s}-\psi_{2t})\right)$$

$$hX_t^3 = \frac{2}{\left[1+|\psi|^2\right]^2}\left(-\psi_1(\psi_{1_t}-\psi_{2s})-\psi_2(\psi_{1_s}-\psi_{2t})\right)$$

$$hX_s^3 = \frac{2}{\left[1+|\psi|^2\right]^2}\left(-\psi_2(\psi_{1_t}-\psi_{2s})-\psi_1(\psi_{1_s}-\psi_{2t})\right).$$

PROOF: In 3.1a, written as $X_s^2 - X_t^1 = -\psi_2 X_s^3 - \psi_1 X_t^3$, substitute using 3.2 to arrive at:

$$X_s^2 - X_t^1 = -(\psi_1{}^2+\psi_2{}^2)(X_t^1+X_s^2)-2\psi_1\psi_2(X_s^1+X_t^2).$$

With 3.5 this can be transformed to

$$-\tfrac{1}{2}h(1+|\psi|^2)^2(X_s^2-X_t^1) = -(\psi_1{}^2+\psi_2{}^2)(\psi_{1_t}-\psi_{2s})-2\psi_1\psi_2(\psi_{1_s}-\psi_{2t}).$$

Combining this with 3.5a we can find the expressions for X_t^1 and X_s^2.

If we had begun with 3.1b and used 3.5b we would have gotten the representations for X_s^1 and X_t^2.

To find the expressions for X^3_s and X^3_t we use 3.2a, b with 3.5a, b. QED

In order to state the next theorem efficiently, set

$$(4.2)\qquad F_1=\frac{b(\psi_{1t}+\psi_{2s})+a(\psi_{1s}+\psi_{2t})}{(1+|\psi|^2)^2}$$
$$F_2=\frac{-a(\psi_{1t}+\psi_{2s})-b(\psi_{1s}+\psi_{2t})}{(1+|\psi|^2)^2}.$$

THEOREM 4.2. *If $X: M^2\rightarrow \boldsymbol{L}^3$ is a timelike isometric immersion with coordinates (X^1, X^2, X^3); Gauss map ψ_1, ψ_2; $a=-h_{12}$ and $b=\frac{1}{2}(h_{11}+h_{12})$ then*

$$(4.3)\qquad \begin{aligned}(b^2-a^2)X^1_t&=(1+\psi_1{}^2+\psi_2{}^2)F_1+2\psi_1\psi_2F_2\\ (b^2-a^2)X^1_s&=2\psi_1\psi_2F_1+(1+\psi_1{}^2+\psi_2{}^2)F_2\\ (b^2-a^2)X^2_t&=-2\psi_1\psi_2F_1+(1-\psi_1{}^2-\psi_2{}^2)F_2\\ (b^2-a^2)X^2_s&=(1-\psi_1{}^2-\psi_2{}^2)F_1-2\psi_1\psi_2F_2\\ (b^2-a^2)X^3_t&=2\psi_1F_1+2\psi_2F_2\\ (b^2-a^2)X^3_s&=2\psi_2F_1+2\psi_1F_2.\end{aligned}$$

PROOF: As in the proof of Theorem 4.1 we have

$$(*)\qquad X^2_s-X^1_t=-(\psi_1{}^2+\psi_2{}^2)(X^1_t+X^2_s)-2\psi_1\psi_2(X^1_s+X^2_t)$$

$$(**)\qquad X^1_s-X^2_t=2\psi_1\psi_2(X^1_t+X^2_s)+(\psi_1{}^2+\psi_2{}^2)(X^1_s+X^2_t).$$

At the same time we can use equations 3.6 to solve for $X^1_t+X^2_s$ and $X^1_s+X^2_t$ in terms of $\psi_{1t}+\psi_{2s}$ and $\psi_{1s}+\psi_{2t}$:

$$(b^2-a^2)(X^1_t+X^2_s)=2F_1$$
$$(b^2-a^2)(X^1_s+X^2_t)=2F_2.$$

Plugging these values into $(*)$ and $(**)$ yields

$$(b^2-a^2)(X^2_s-X^1_t)=-(\psi_1{}^2+\psi_2{}^2)2F_1-2\psi_1\psi_2(2F_2)$$
$$(b^2-a^2)(X^1_s-X^2_t)=2\psi_1\psi_2(2F_1)+(\psi_1{}^2+\psi_2{}^2)2F_2.$$

From the last four equations we get the expressions for X^1_t, X^1_s, X^2_t and X^2_s.

The formulas for X^3_t and X^3_s follow from 3.2. QED

THEOREM 4.3. (*Weierstrass formulas for timelike minimal surfaces.*) *Assume $X: M^2\rightarrow \boldsymbol{L}^3$ is a minimal timelike isometric immersion with no flat points. In a coordinate neighborhood which contains only segments of $s+t$ and $t-s$, there exist functions $P(s+t)$, $p(s+t)$, $Q(t-s)$, and $q(t-s)$ such that $PQ(pq-1)=0$ and*

(4.4)
$$\begin{aligned} X^1_t &= (1+p^2)P+(1+q^2)Q \\ X^1_s &= (1+p^2)P-(1+q^2)Q \\ X^2_t &= (1-p^2)P-(1-q^2)Q \\ X^2_s &= (1-p^2)P+(1-q^2)Q \\ X^3_t &= 2pP+2qQ \\ X^3_s &= 2pP-2qQ. \end{aligned}$$

Conversely, given any P, p, Q and q with $PQ(pq-1)\neq 0$, the above system defines a timelike minimal immersion on such an open set in $\boldsymbol{L}^2$.

NOTES: 1. $X^j(p)=\int_{p_0}^{p}(X^j_t, X^j_s)(dt, ds)$ for any p_0.

2. $X_{t+s}=2P(1+p^2, 1-p^2, 2p)$ and $X_{t-s}=2Q(1+q^2, -(1-q^2), 2q)$.

These are null curves. It is known that every minimal surface is locally the sum of two such curves [B]. This shows that the null curves can be put into a canonical form.

3. In a neighborhood of flat points the geometry of $X(M^2)$ is well understood [G], [Ma1]. What occurs when the set of flat points has no interior is not known.

PROOF: Since M^2 is minimally immersed, the assumption that there are no flat points is equivalent to $b^2-a^2\neq 0$. Setting $G_j=F_j/(b^2-a^2)$, we can rewrite 4.3 as

(4.5)
$$\begin{aligned} X^1_t &= (1+\psi_1{}^2+\psi_2{}^2)G_1+2\psi_1\psi_2 G_2 \\ X^1_s &= 2\psi_1\psi_2 G_1+(1+\psi_1{}^2+\psi_2{}^2)G_2 \\ X^2_t &= -2\psi_1\psi_2 G_1+(1-\psi_1{}^2-\psi_2{}^2)G_2 \\ X^2_s &= (1-\psi_1{}^2-\psi_2{}^2)G_1-2\psi_1\psi_2 G_2 \\ X^3_t &= 2\psi_1 G_1+2\psi_2 G_2 \\ X^3_s &= 2\psi_2 G_1+2\psi_1 G_2 \end{aligned}$$

Thus, $X^1_t+X^2_s=2G_1$ and $X^1_s+X^2_t=2G_2$. As noted after equation 1.5, $X^j_{ss}=X^j_{tt}$, so that $G_{1_t}-G_{2_s}=0$ and $G_{1_s}-G_{2_t}=0$. Set $P=\frac{1}{2}(G_1+G_2)$ and $Q=\frac{1}{2}(G_1-G_2)$. Then $P_t-P_s=0$ and $Q_s+Q_t=0$. We can conclude that P is a function of $s+t$ and Q is a function of $t-s$. Of course $G_1=P+Q$ and $G_2=P-Q$.

Similarly, by Lemma 3.4, $\psi_{1_t}-\psi_{2_s}=0$ and $\psi_{1_s}-\psi_{2_t}=0$, which yields $\psi_1=\frac{1}{2}(p+q)$ and $\psi_2=\frac{1}{2}(p-q)$ for some $p(s+t)$ and $q(t-s)$.
Substituting these expressions into 4.5 yields 4.4.

To prove the converse we need only check that equations 4.4 satisfy $X^j_{ts}=X^j_{st}$ and $X^j_{ss}=X^j_{tt}$ for $j=1, 2, 3$ and that the metric induced on M is non-degenerate. The first two facts are easily verified and for the last we note that $(X_{s+t}, X_{t-s}) = -8PQ(pq-1)^2$. QED

5. Integrability conditions

In this section it is shown that the mean curvature h and the Gauss map (ψ_1, ψ_2) satisfy a pair of partial differential equations. As in the positive definite cases, these equations are the integrability conditions for the system 4.1.

NOTATION: If $f: M^2 \to \mathbf{R}$ set $\Delta f = f_{ss} - f_{tt}$.

LEMMA 5.1. *If $X: M^2 \to \mathbf{L}^3$ is a timelike isometric immersion then*

$$\text{(5.1a)} \quad \Delta X^1 = \frac{4\lambda^2 h\psi_1}{1+|\psi|^2}$$

$$\text{(5.1b)} \quad \Delta X^2 = \frac{-4\lambda^2 h\psi_2}{1+|\psi|^2}$$

PROOF: To prove 5.1b use 1.3 to calculate

$$X^2_{ss} - X^2_{tt} = \lambda^2 h_{22} e_3^2 - \lambda^2 h_{11} e_3^2 = 2\lambda^2 h e_3^2$$
$$= 2\lambda^2 h \frac{e_3^2}{1+e_3^3} \frac{2}{1+|\psi|^2} = \frac{-4\lambda^2 h\psi_2}{1+|\psi|^2}. \quad \text{QED}$$

To state the next theorem, which gives a pair of partial differential equations which the Gauss map must satisfy, we introduce some additional abbreviations. Set

$$S_1 = \psi_{1_t} - \psi_{2s}$$
$$S_2 = \psi_{1_s} - \psi_{2t}$$
$$T_1 = \psi_{1_t} + \psi_{2s}$$
$$T_2 = \psi_{1_s} + \psi_{2t}$$

THEOREM 5.1. *Let $X: M^2 \to \mathbf{L}^3$ be a timelike isometric immersion. Then the mean curvature h and the Gauss map (ψ_1, ψ_2) satisfy:*

$$\text{(5.2a)} \quad h\Delta\psi_1 = -h_t S_1 + h_s S_2 + \frac{2h}{1+|\psi|^2}(S_1(\psi_1 T_1 - \psi_2 T_2) + S_2(\psi_2 T_1 - \psi_1 T_2))$$

$$\text{(5.2b)} \quad h\Delta\psi_2 = h_t S_2 - h_s S_1 - \frac{2h}{1+|\psi|^2}(S_2(\psi_1 T_1 - \psi_2 T_2) + S_1(\psi_2 T_1 - \psi_1 T_2)).$$

PROOF of 5.2b: Fix $p \in M^2$. We may assume that $h(p) \neq 0$, since if $h(p) = 0$ then $S_j(p) = 0$ and the equation holds. By definition $\Delta\psi_2 = (\psi_{1_s} - \psi_{2t})_t - (\psi_{1_t} - \psi_{2s})_s$. Using 3.4 and 3.5, the right hand side of this equation is

$$\left[-h\frac{2}{(1+e_3^3)^2}(X_s^1+X_t^2)\right]_t-\left[-h\frac{2}{(1+e_3^3)^2}(X_t^1+X_s^2)\right]_s.$$

We now differentiate each summand as a product with 3 terms. Thus,

$$\begin{aligned}\Delta\psi_2=&\frac{h_t}{h}\left[\frac{-2h}{(1+e_3^3)^2}(X_s^1+X_t^2\right]-\frac{h_s}{h}\left[\frac{-2h}{(1+e_3^3)^2}(X_t^1+X_s^2)\right]\\&-\frac{2h}{(1+e_3^3)^2}(X_{st}^1+X_{tt}^2-X_{ts}^1-X_{ss}^2)-\frac{4h(X_t^1+X_s^2)e_{3s}^3}{(1+e_3^3)^3}\\&+\frac{4h(X_s^1+X_t^2)e_{3t}^3}{1+e_3^3)^3}\\=&\frac{h_s}{h}(\psi_{1_t}-\psi_{2s})+\frac{h_t}{h}(\psi_{1s}-\psi_{2t})+\frac{h}{2}(1+|\psi|^2)^2\Delta X^2\\&-2h(1+|\psi|^2)\frac{(X_t^1+X_s^2)}{(1+e_3^3)^2}e_{3s}^3+2h(1+|\psi|^2)\frac{(X_s^1+X_t^2)}{(1+e_3^3)^2}e_{3t}^3.\end{aligned}$$

Via 5.1b and 3.5, we have

$$\begin{aligned}\Delta\psi_2=&\frac{h_s}{h}(\psi_{1_t}-\psi_{2s})+\frac{h_t}{h}(\psi_{1s}-\psi_{2t})-2h^2\lambda^2(1+|\psi|^2)\psi_2\\&+(1+|\psi|^2)e_{3s}^3(\psi_{1_t}-\psi_{2s})-(1+|\psi|^2)e_{3t}^3(\psi_{1s}-\psi_{2t})\\=&-\frac{h_s}{h}S_1+\frac{h_t}{h}S_2-2h^2\lambda^2(1+|\psi|^2)\psi_2-(1+|\psi|^2)(S_2e_{3t}^3-S_1e_{3s}^3)\\=&-\frac{h_s}{h}S_1+\frac{h_t}{h}S_2-2h^2\lambda^2(1+|\psi|^2)\psi_2\\&-(1+|\psi|^2)(S_2(-hX_t^3+bX_t^3+aX_s^3)-S_1(-hX_s^3-aX_t^3-bX_s^3)).\end{aligned}$$

By 3.8a, this becomes

$$\Delta\,\psi_2=-\frac{h_s}{h}S_1+\frac{h_t}{h}S_2-\frac{2}{(1+|\psi|^2)}(S_1{}^2-S_2{}^2)\psi_2-(1+|\psi|^2)(hS_1X_s^3-hS_2X_t^3)-(1+|\psi|^2)(S_2(bX_t^3+aX_s^3)+S_1(aX_t^3+bX_s^3)).$$

Using 3.6c, 3.6d and 4.1, we arrive at the final equation. QED

Next we assume that $h\neq0$ on M and write the system 4.1 with some additional abbreviations. Set $R=h^{-1}(1+|\psi|^2)^{-2}$ and $Q_j=RS_j$, $j=1,2$. Then 4.1 can be rewritten as

$$\begin{aligned}(5.3)\qquad X_t^1&=-(1+\psi_1{}^2+\psi_2{}^2)Q_1-2\psi_1\psi_2Q_2\\X_s^1&=-2\psi_1\psi_2Q_1-(1+\psi_1{}^2+\psi_2{}^2)Q_2\\X_t^2&=2\psi_1\psi_2Q_1+(\psi_1{}^2+\psi_2{}^2-1)Q_2\\X_s^2&=(\psi_1{}^2+\psi_2{}^2-1)Q_1+2\psi_1\psi_2Q_2\\X_t^3&=-2\psi_1Q_1-2\psi_2Q_2\\X_s^3&=-2\psi_2Q_1-2\psi_1Q_2.\end{aligned}$$

THEOREM 5.2. *Let h, ψ_1, and ψ_2 be functions on M^2 such that $h \neq 0$ and $1+|\psi|^2 \neq 0$. If h, ψ_1 and ψ_2 satisfy 5.2 then the functions X_s^j and X_t^j defined by 5.3 satisfy $X_{st}^j = X_{ts}^j$, for $j=1, 2$.*

PROOF: Note that

$$h^2(1+|\psi|^2)^2 Q_{1x} = h(\psi_{1tx} - \psi_{2sx}) - \left(h_x + \frac{2h}{1+|\psi|^2}(2\psi_2\psi_{2x} - 2\psi_1\psi_{1x})\right) S_1$$

and

$$h^2(1+|\psi|^2)^2 Q_{2x} = h(\psi_{1sx} - \psi_{2tx}) - \left(h_x + \frac{2h}{1+|\psi|^2}(2\psi_2\psi_{2x} - 2\psi_1\psi_{1x})\right) S_2,$$

where x stands for s or t. Also, $h^2(1+|\psi|^2)^2 Q_j = hS_j$, for $j=1, 2$.

We now compute

$$\begin{aligned} &h^2(1+|\psi|^2)^2(X_{st}^1 - X_{ts}^1) \\ &= h^2(1+|\psi|^2)^2(-2\psi_1\psi_2(Q_{1t} - Q_{2s}) + (1+\psi_1^2+\psi_2^2)(Q_{1s} - Q_{2t}) + \\ &Q_1(-2\psi_1\psi_{2t} - 2\psi_2\psi_{1t} + 2\psi_1\psi_{1s} + 2\psi_2\psi_{2s}) - Q_2(2\psi_1\psi_{1t} + 2\psi_2\psi_{2t} - 2\psi_1\psi_{2s} \\ &-2\psi_2\psi_{1s})). \end{aligned}$$

Using 5.2, we see that

$$h^2(1+|\psi|^2)^2(Q_{1t} - Q_{2s}) = \frac{2h}{1+|\psi|^2}(S_1^2 - S_2^2)\psi_1 \text{ and}$$

$$h^2(1+|\psi|^2)^2(Q_{1s} - Q_{2t}) = \frac{2h}{1+|\psi|^2}(S_1^2 - S_2^2)\psi_2.$$

Thus, $h^2(1+|\psi|^2)^2(X_{st}^1 - X_{ts}^1)$ reduces to

$$2h\psi_2(S_1^2 - S_2^2)\left\{\frac{-2\psi_1^2}{1+\psi_2^2-\psi_1^2} + \frac{1+\psi_2^2+\psi_1^2}{1+\psi_2^2-\psi_1^2} - 1\right\} = 0.$$

The proofs that $X_{st}^2 = X_{ts}^2$ and $X_{st}^3 = X_{ts}^3$ are sirmilar. QED

COROLLARY 5.1. [Mi1] *The mean curvature of an isometrically immersed timelike surface $X: M^2 \to \boldsymbol{L}^3$ is constant iff $G: M^2 \to S_1^2$ is harmonic.*

PROOF: S_1^2 can be parametrized using inverse stereographic projection:

$$\pi_1^{-1}(x_1, x_2) = \frac{(2x_1, -2x_2, x_1^2 - x_2^2 + 1)}{(1 + x_2^2 - x_1^2)}.$$

If the induced metric on S_1^2 is denoted by σ then $\sigma_{11} = -\tau^2$, $\sigma_{22} = \tau^2$ and $\sigma_{12} = 0$, where $\tau = \frac{2}{(1+x_2^2-x_1^2)}$. In addition, we see that $\Gamma_{11}^1 = \Gamma_{12}^2 = \Gamma_{22}^1 = \frac{\tau_{x_1}}{\tau}$ and $\Gamma_{22}^2 = \Gamma_{11}^2 = \Gamma_{12}^1 = \frac{\tau_{x_2}}{\tau}$.

A straightforward calculation shows that G is harmonic iff

$$(5.4\text{a}) \quad \Delta\psi_1=\frac{2}{1+|\psi|^2}(S_1(\psi_1 T_1-\psi_2 T_2)+S_2(\psi_2 T_1-\psi_1 T_2))$$

$$(5.4\text{b}) \quad \Delta\psi_2=\frac{-2}{1+|\psi|^2}(S_2(\psi_1 T_1-\psi_2 T_2)+S_1(\psi_2 T_1-\psi_1 T_2)).$$

Note that the Laplacian of ψ_j is $\frac{1}{\lambda^2}\Delta\psi_j$.

If h is constant then these last equations hold by 5.2. Conversely if 5.4 holds $-h_t S_1+h_s S_2=0=h_t S_2+h_s S_1$, so that $h_t=h_s=0$. QED

6. Applications

As a first application, we prove a converse to Theorem 4.1, that is, if equations 4.1 hold for $h\neq 0$ we can find an immersion with precisely the given mean curvature and Gauss map. In the theorem below we require $(\psi_{1_t}-\psi_{2_s})^2-(\psi_{1_s}-\psi_{2_t})^2>0$, that is, that the surface is regular everywhere. We also assume that we have chosen a conformal class of timelike metrics on M^2, and that (t, s) are isothermal coordinates for this class.

THEOREM 6.1. *Let M^2 be a simply connected surface as above, h: $M^2\to\boldsymbol{R}$* a non-vanishing C^∞ function and $G: M^2\to S_1^2$ *a function whose projection satisfies $(\psi_{1_t}-\psi_{2_s})^2-(\psi_{1_s}-\psi_{2_t})^2>0$. If h, ψ_1 and ψ_2 satisfy* 5.2 *then there exists an immersion $X: M^2\to\boldsymbol{L}^3$ such that*

1. *(s, t) are isothermal coordinates for the induced metric g,*
2. *the mean curvature of X is h and its Gauss map is G, and*
3. *$X^j(p)=\int_{p_0}^{p}(X_t^j, X_s^j)\cdot(dt, ds)$ where X_t^j, and X_s^j are given by* 5.3, $j=1, 2, 3$.

PROOF: Given h, ψ_1 and ψ_2, we define X_t^j, X_s^j by 5.3. We know that $X_{st}^j=X_{ts}^j$, so that $X=(X^1, X^2, X^3)$ exists. We can then see that $(X_s, X_s)=\frac{1}{h^2(1+|\psi|^2)^2}\left((\psi_{1_t}-\psi_{2_s})^2-(\psi_{1s}-\psi_{2_t})^2\right)$, $(X_s, X_t)=0$ and $(X_t, X_t)=-(X_s, X_s)$.

From 1.3, $\Delta X=X_{ss}-X_{tt}=2\lambda^2 he_3$, so the Laplacian of X, $\Box X=2he_3$, and X has mean curvature h. From 1.2, we see that $e_3=\left(\frac{2\psi_1}{1+|\psi|^2}, \frac{-2\psi_2}{1+|\psi|^2}, \frac{1-|\psi|^2}{1+|\psi|^2}\right)$, so that the Gauss map is precisely G.

Note that, from 3.8, the sectional curvature of M^2 can be written as

$h^2\left[1+\dfrac{T_2^2-T_1^2}{S_1^2-S_2^2}\right]$, in terms of S_1, S_2, T_1 and T_2. QED

Thus, given h, ψ_1, and ψ_2, satisfying 5.2a and 5.2b, Theorem 6.1 allows us to construct examples of isometric immersions with the given h and G. We now do this for certain surfaces with constant mean curvature.

EXAMPLE 6.1. Let $h=-1$, $\psi_1=t$ and $\psi_2=-s$. We see that ψ_1 and ψ_2 satisfy 5.2a and 5.2b. The timelike immersion X defined by 5.3 is

$$X(t,s)=\left(\frac{2t}{1+s^2-t^2}, \frac{2s}{1+s^2-t^2}, \frac{t^2-s^2+1}{1+s^2-t^2}\right).$$

This is the standard immersion of S_1^2 into $\boldsymbol{L}^3$.

EXAMPLE 6.2. Set $h=-1/2$, $\psi_1=\dfrac{\sinh t}{1+\cosh t}$ and $\psi_2=0$. These satisfy 5.2a and 5.2b. The X obtained is

$$X(t, s)=(\sinh t,\ s,\ \cosh t).$$

This is a hyperbolic cylinder in $\boldsymbol{L}^3$ on which the induced metric is timelike.

EXAMPLE 6.3. The other type of cylinder in $\boldsymbol{L}^3$ can be generated by setting $h=-1/2$, $\psi_1=0$ and $\psi_2=\dfrac{-\cos s}{1+\sin s}$. Here the immersion we obtain is

$$X(t, s)=(t,\ \cos s,\ \sin s).$$

THEOREM 6.2. *Let X, $\bar{X}$ be isometric immersions from a simply connected timelike surface (M^2, g) into $\boldsymbol{L}^3$ with h, $\bar{h}\neq 0$ and Gauss maps G and $\bar{G}$. Then the following statements are equivalent.*

1. *There is a conformal diffeomorphism $\varphi: M\to M$ and an orientation preserving isometry $\tau: \boldsymbol{L}^3\to\boldsymbol{L}^3$ such that for all $p\in M$*

$$(6.1)\qquad \tau\circ X(p)=\bar{X}\circ\varphi(p).$$

2. *There is a conformal diffeomorphism $\varphi: M\to M$ and an orientation preserving isometry $\sigma: S_1^2\to S_1^2$ such that for $p\in M$*

$$(6.2\text{a})\qquad \sigma\circ G(p)=\bar{G}\circ\varphi(p)$$
$$(6.2\text{b})\qquad h(p)=\bar{h}\circ\varphi(p).$$

PROOF: First assume 1. is true. We have $g_{\varphi(p)}(\varphi_*X, \varphi_*Y)=\rho^2(p)g_p(X, Y)$, for some non-zero function ρ on M. Choose coordinates (t, s) which are isothermal with respect to the metric g. It is easy to see that $\bar{X}_*(\varphi_*(\partial/\partial t))$ and $\bar{X}_*(\varphi_*(\partial/\partial s))$ are perpendicular and $g(\bar{X}_*(\varphi_*(\partial/\partial_t), \bar{X}_*(\varphi_*(\partial/\partial t))+g(\bar{X}_*(\varphi_*(\partial/\partial s), \bar{X}_*(\varphi_*(\partial/\partial s))=0$. Because $\tau_*(X_*(\partial/\partial t))=\bar{X}_*(\varphi_*(\partial/\partial t)$ and $\tau_*(X_*(\partial/\partial s))=\bar{X}_*(\varphi_*(\partial/\partial s)$ and τ is an isometry $\tau_*(e_3(p))=\bar{e}_3(\varphi(p))$. If we set $\sigma=\tau_*$ we have 6.2a.

Since $2h=\frac{1}{\lambda^2}g(\Delta X, e_3)$ and $\Delta(\tau\circ X)=\tau_*(\Delta X)$ we have $\bar{h}(\varphi(p))$ $=h(p)$.

Now we assume 2. Extend σ to an orientation preserving isometry of $\boldsymbol{L}^3$. We may assume that σ is the identity, so that $G(p)=\bar{G}(\varphi(p))$ or $\psi_j(p)=\bar{\psi}_j(\varphi(p))$ for $j=1, 2, 3$. From equations 5.3, we find that $X_t^j(p)-\bar{X}_t^j(\varphi(p))=0$ and $X_s^j(p)-\bar{X}_s^j(\varphi(p))=0$, for $j=1,2,3$. Therefore, $X(p)=\bar{X}\circ\varphi(p)+C$ or $\sigma\circ X=\bar{X}\circ\varphi+C$. It follows that we can find an orientation preserving isometry τ so that $\tau\circ X=\bar{X}\circ\varphi$. QED

APPENDIX.

In this appendix we give a sampling of the equations in terms of the other coordinate representation of the Gauss map (μ_1, μ_2).

(3.1 a) $X_t^1+\mu_1X_t^3+X_s^2+\mu_2X_s^3=0.$

(3.1 b) $X_t^2+\mu_2X_t^3+X_s^1+\mu_1X_s^3=0.$

(3.2 a) $X_t^3=-\mu_1(X_t^1-X_s^2)+\mu_2(X_t^2-X_s^1).$

(3.2 b) $X_s^3=-\mu_2(X_t^1-X_s^2)+\mu_1(X_t^2-X_s^1).$

(3.4) $(1+|\mu|^2)(1-e_3^3)=2\,;\ |\mu|^2=\mu_2{}^2-\mu_1{}^2.$

(3.3 a) $\frac{4\lambda^2\mu_1}{(1+|\mu|^2)^2}=-X_t^3(X_t^1-X_s^2)-X_s^3(X_t^2-X_s^1).$

(3.3 b) $\frac{4\lambda^2\mu_2}{(1+|\mu|^2)^2}=-X_t^3(X_t^2-X_s^1)-X_s^3(X_t^1-X_s^2).$

(3.5 a) $\mu_{1_t}-\mu_{2_s}=-\frac{h}{2}(1+|\mu|^2)^2(X_t^1-X_s^2).$

(3.5 b) $\mu_{2_t}-\mu_{1_s}=-\frac{h}{2}(1+|\mu|^2)^2(X_t^2-X_s^1).$

(3.6 a) $\mu_{1_t}+\mu_{2_s}=\frac{(1+|\mu|^2)^2}{2}\,(b(X_t^1-X_s^2)-a(X_t^2-X_s^1)).$

(3.6 b) $\mu_{1_s}+\mu_{2_t}=\frac{(1+|\mu|^2)^2}{2}\,(-a(X_t^1-X_s^2)+b(X_t^2-X_s^1)).$

(3.6 c) $aX_t^3+bX_s^3=\frac{2}{(1+|\mu|^2)^2}\,(-\mu_2(\mu_{1_t}+\mu_{2_s})+\mu_1(\mu_{1_s}+\mu_{2_t})).$

(3.6 d) $bX_t^3+aX_s^3=\frac{2}{(1+|\mu|^2)^2}\,(-\mu_1(\mu_{1_t}+\mu_{2_s})+\mu_2(\mu_{1_s}+\mu_{2_t})).$

(3.7) $\frac{4\lambda^2}{(1+|\mu|^2)^2}=(X_t^1-X_s^2)^2-(X_t^2-X_s^1)^2.$

(3.8 a) $(\mu_{1_t}-\mu_{2_s})^2-(\mu_{2_t}-\mu_{1_s})^2=h^2\lambda^2(1+|\mu|^2)^2.$

(3.8 b) $(\mu_{1_t}+\mu_{2_s})^2-(\mu_{2_t}+\mu_{1_s})^2=(b^2-a^2)\lambda^2(1+|\mu|^2)^2.$

(3.9 a) $-h(\mu_{1t}+\mu_{2s})=b(\mu_{1t}-\mu_{2s})-a(\mu_{2t}-\mu_{1s})$.

(3.9 b) $-h(\mu_{1s}+\mu_{2t})=-a(\mu_{1t}-\mu_{2s})+b(\mu_{2t}-\mu_{1s})$.

(4.1)
$$hX^1_t=\frac{1}{(1+|\mu|^2)^2}(-(1+\mu_1{}^2+\mu_2{}^2)(\mu_{1t}-\mu_{2s})+2\mu_1\mu_2(\mu_{2t}-\mu_{1s})).$$
$$hX^1_s=\frac{1}{(1+|\mu|^2)^2}(-2\mu_1\mu_2(\mu_{1t}-\mu_{2s})+(1+\mu_1{}^2+\mu_2{}^2)(\mu_{2t}-\mu_{1s})).$$
$$hX^2_t=\frac{1}{(1+|\mu|^2)^2}(-2\mu_1\mu_2(\mu_{1t}-\mu_{2s})-(1-\mu_1{}^2-\mu_2{}^2)(\mu_{2t}-\mu_{1s})).$$
$$hX^2_s=\frac{1}{(1+|\mu|^2)^2}((1-\mu_1{}^2-\mu_2{}^2)(\mu_{1t}-\mu_{2s})+2\mu_1\mu_2(\mu_{2t}-\mu_{1s})).$$
$$hX^3_t=\frac{2}{(1+|\mu|^2)^2}(\mu_1(\mu_{1t}-\mu_{2s})-\mu_2(\mu_{2t}-\mu_{1s})).$$
$$hX^3_s=\frac{2}{(1+|\mu|^2)^2}(\mu_2(\mu_{1t}-\mu_{2s})-\mu_1(\mu_{2t}-\mu_{1s})).$$

Setting
$$H_1=\frac{b(\mu_{1t}+\mu_{2s})+a(\mu_{1s}+\mu_{2t})}{(1+|\mu|^2)^2}\ \text{and}\ H_2=\frac{a(\mu_{1t}+\mu_{2s})+b(\mu_{1s}+\mu_{2t})}{(1+|\mu|^2)^2}$$
we have

(4.3)
$$(b^2-a^2)X^1_t=(1+\mu_1{}^2+\mu_2{}^2)H_1-2\mu_1\mu_2H_2.$$
$$(b^2-a^2)X^1_s=2\mu_1\mu_2H_1-(1+\mu_1{}^2+\mu_2{}^2)H_2.$$
$$(b^2-a^2)X^2_t=2\mu_1\mu_2H_1+(1-\mu_1{}^2-\mu_2{}^2)H_2.$$
$$(b^2-a^2)X^2_s=(\mu_1{}^2+\mu_2{}^2-1)H_1-2\mu_1\mu_2H_2.$$
$$(b^2-a^2)X^3_t=-2\mu_1H_1+2\mu_2H_2.$$
$$(b^2-a^2)X^3_s=-2\mu_2H_1+2\mu 1H_2.$$

(5.1 a) $\Delta X^1=\dfrac{4\lambda^2h\mu_1}{1+|\mu|^2}$.

(5.1 b) $\Delta X^2=\dfrac{4\lambda^2h\mu_2}{1+|\mu|^2}$.

Set $U_1=\mu_{1t}-\mu_{2s}$, $U_2=\mu_{2t}-\mu_{1s}$, $V_1=\mu_{1t}+\mu_{2s}$ and $V_2=\mu_{1s}+\mu_{2t}$.

(5.2 a) $\Delta\mu_1=-\dfrac{h_s}{h}U_2-\dfrac{h_t}{h}U_1+\dfrac{2}{1+|\mu|^2}(U_2(\mu_1V_2-\mu_2V_1)+U_1(\mu_1V_1-\mu_2V_2))$

(5.2 b) $\Delta\mu_2=-\dfrac{h_t}{h}U_2-\dfrac{h_s}{h}U_1+\dfrac{2}{1+|\mu|^2}(U_1(\mu_1V_2-\mu_2V_1)+U_2(\mu_1V_1-\mu_2V_2))$.

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Department of Mathematics
Wellesley College
Wellesley, MA 02181