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On solvability of systems of convolution equations in K'_M

Saleh ABDULLAH

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Abatract

In this note we show that if the $m\times m$ system of convolution equations in K'_M has a solution, then the Fourier transform of the determinant of the matrix of coefficients is slowly decreasing.

Introduction and statement of the results

We will use the notations and results of our paper [1]. By $O'_c(K'^m_M; K'^m_M)$ we denote the space of all $m\times m$ matrices $S=(S_{ij})$, whose entries are convolution operators in K'_M (i. e. $S_{ij}\in O'_c(K'_M; K'_M)$, $1\leq \mathrm{i}, \mathrm{j}\leq m$). For $U=(u_1, u_2, \ldots, u_m)^t\in K'^m_M$, and $\phi=(\varphi_1, \varphi_2, \ldots, \varphi_m)^t\in K^m_M$, the duality bracket $<U, \phi>$ is defined by

$$<U, \phi>=\sum_{j=1}^{m}<u_j, \varphi_j>.$$

For $S=(S_{ji})$, we denote by S^t the matrix $(\check{S}_{ji})$, where $\check{S}_{ji}$ is the symmetry of S_{ji} with respect to the origin. The letter F, as well as $\wedge$, will denote the Fourier transform. In [1], the problem of giving necessary and sufficient conditions for a determined system of convolution equations in K'_M,; the space of distributions of rapid growth, was considered. The main result of [1] is the following

THEOREM 1. *Let $S\in O'_c(K'^m_M; K'^m_M)$, among the following properties of S, the implications* (a)$\rightarrow$(b)$\rightarrow$(c)$\rightarrow$(b) *hold.*

(a) *$det(\hat{S}^t)$ satisfy the following growth condition*

(I) *There exist positive constants C, N and A so that*

$$\sup_{\substack{|z|\leq A\Omega^{-1}[\log(2+|\xi|)] \\ z\in C^n}} |\det (\hat{S}^t)(z+\xi)|\geq C(1+|\xi|)^{-N},\ \xi\in R^n,$$

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where Ω is the Young dual of M.

(b) *The map $S^t * \phi \to \phi$ from $S^t * K_M^m$ into K_M^m is continuous.*

(c) $S * K_M'^m = K_M'^m$.

It was conjectured in [1] that the implication (b)→(a) of the above theorem is true. The aim of this note is to prove that the above conjecture is true. More precisely, the main result of the note is the following

THEOREM 2. *Let $S \in O_c'(K_M'^m; K_M'^m)$, if $S * K_M'^m = K_M'^m$ then det $(\hat{S}^t)$ is slowly decreasing, i. e. it satisfies the growth condition* (I).

As an intermediate step to the proof of theorem 2 we need the following.

THEOREM 3. *Let $S \in O_c'(K_M'^m; K_M'^m)$, if $S * K_M'^m = K_M'^m$ them $S^t * K_M'^m = K_M'^m$.*

Proofs of the resluts.

PROOF OF THEOREM 3. Since $S * K_M'^m = K_M'^m$ there exists an $m \times m$ matrix $E = (E_{ij})$; $E_{ij} \in K_M'$, such that $S * E = I$, where I is the $m \times m$ diagonal matrix with all entries on the main diagonal equal to δ. Moreover, the entire function $\det(FS)$ does not vanish identically. Using Cramar's rule it follows that $\det(S) * E_{ij} = T_{ij} \in O_c'$. By taking the Fourier transform of both sides of the equality one gets

$\det\ (F(S)).\ F(E_{ij}) = F(T_{ij})$. Hence $F(E_{ij}) = \dfrac{F(T_{ij})}{\det(F(S))}$

is a meromorphic function. Moreover, we have

$F(E) = (F(E_{ij})) = \dfrac{1}{\det(F(S))} .\ \mathrm{adj}\ (F(S))$,

hence

$$\det(F(\check{E}^t)) = \frac{1}{\det(F(\check{S}))^m} \det\ (\mathrm{adj}(F(\check{S}))$$
$$= \frac{1}{\det(F(\check{S}))^m} .\ \det\ (F(\check{E}^t))^{m-1} = \frac{1}{\det(F(\check{S}))},$$

where adj denotes the adjoint of the matrix.

Since (b) and (c) of theorem 1 are equivalent, to prove the result we need to show that the map $\check{S} * \phi \to \phi$ from $\check{S} * K_M^m$ into K_M^m is continuous. Since K_M^m is metrizable it suffices to show that it takes bounded sets into bounded sets. By Mackey's theorem the strongly bounded and weakly bounded subsets of K_M^m are the same. Thus we need to show that if B is a weakly bounded subset of $\check{S} * K_M^m$, then the set $A = \{\phi \in K_M^m : \check{S} * \phi \in \mathrm{B}\}$ is weakly bounded in K_M^m. Since $S*E=I$ one has $F(S)F(E)=F(I)$, the identity matrix, and $\operatorname{adj}(F(E)).\ \operatorname{adj}(F(S))=F(I)$. Since the entries of $F(E)=(F(E_{ij}))$ are meromorphic functions, we can calculate $\operatorname{adj}(F(E))$ as matrix of meromorphic functions. Moreover, all the steps in the following set of equalities are well defined. For any $U\in K_M'^m$ and $\phi\in A$ one has

$$
\begin{aligned}
(1)\qquad <U,\phi> &= <F(\check{U}),\ F(\phi)> = <\operatorname{adj}(F(E))\operatorname{adj}(F(S))F(\check{U}),\\
&\quad F(\phi)>\\
&= <\operatorname{adj}(F(E))(\operatorname{adj}(F(S)).\ F(S))(F(E)F(\check{U}),\\
&\quad F(\phi)>;\\
&= <\operatorname{adj}(F(E))(\det(F(S)).\ F(I))(F(E)F(\check{U})),\\
&\quad F(\phi)>;\\
&= <\det(F(S))I.\ (\operatorname{adj}(F(E)).\ F(E)).\ F(\check{U}),\ F(\phi)>;\\
&= <(\det(F(\check{S}^t).\ F(I))(\det F(\check{E}^t)F(\check{U})),\ F(\phi)>;\\
&= <\operatorname{adj}(F(\check{S}^t))\det(F(\check{E}^t))F(\check{U}),\ F(\check{S})\ F(\phi)>;\\
&= <\frac{1}{\det(F(\check{S}^t))}\operatorname{adj}(F(\check{S}^t))F(\check{U}),\ F(\check{S})F(\phi)>;\\
&= <F(\check{E}^t)F(\check{U}),\ F(\check{S})F(\phi)>;\\
&= <\check{E}^t*U,\ \check{S}*\phi>.
\end{aligned}
$$

Since $U\in K_M'^m$ and $\check{S}*K_M^m$ is metrizable it follows that $\check{E}^t*U$ is in $(\check{S}*K_M^m)'$. Now, since B is weakly bounded in $\check{S}*K_M^m$ there exists a constant C which depends on U (and on S) such that

$$(2)\qquad |<U,\phi>|=|<\check{E}^t*U,\ \check{S}*\phi>|\leq C,$$

for all ϕ in A. This completes the proof of the theorem.

We remark that since $\check{S}*K_M^m$ is a proper subspace of K_M^m, its dual $(\check{S}*K_M^m)'$ will include elements which are not in $K_M'^m$.

REMARK 1: Although the entries of $\operatorname{adj}(F(E))$ are not necessarily elements in $(F(K_M))'$, all the steps in the set of equalities (1) are well deffined. If one can prove that the entries of $\operatorname{adj}(F(E))$ are in $(F(K_m))'$ the proof of theorem 2 will follow immediately.

PROOF OF THEOREM 2. The idea of the proof is similar to that in the proof of the implication (b)→(a) of theorem 1 of [1]. Thus we will not repeat the unnecessary details. The proof is by contradiction. Suppose det $(F(S^t))$ is not slowly decreasing. Then for every $j\in \boldsymbol{N}$ there exists $\xi_j\in \boldsymbol{R}^n$ so that $|\xi_j|>e^j$, and

$$(3) \qquad \sup_{|z|\leq A_j a_j} |\det(F(S^t))(z+\xi_j)| < (1+|\xi_j|)^{-j}, \quad z \in \boldsymbol{C}^n$$

where $A_j = e^{2j}$ and $a_j = \Omega^{-1}(\log(2+|\xi_j|))$. For each $j \in \boldsymbol{N}$ we let k_j be the greatest integer less than or equal to $\log a_j + 1$. Let φ be a C^∞-function with compact support in the unit ball, $\varphi \geq 0$ and $\check{\varphi}(0) = 1$. For $j \in \boldsymbol{N}$, we define the function φ_j by $\varphi_j(\xi) = a_j \varphi(a_j \xi)$, and the function ψ_j^1 by

$$\psi_j^1(\xi) = e^{i<\xi_j,\xi>}(\varphi_j * \varphi_j * \ldots * \varphi_j)(\xi),$$

where the convolution product is being taken k_j times. Define the function ψ_j by $\psi_j = \psi_j^1 * \psi_j^1$, hence supp ψ_j is contained in the ball $B(0,2)$. Thus

$$\hat{\psi}_j^1(z+\xi_j) = (\hat{\varphi}_j(z))^{k_j}, \quad \hat{\varphi}_j(z) = \hat{\varphi}\left(\frac{z}{a_j}\right),$$

$$\hat{\psi}_j^1(z+\xi_j) = (\hat{\varphi}_j\left(\frac{z}{a_j}\right))^{k_j}, \textit{ and } \hat{\psi}_j^1(\xi_j) = (\hat{\varphi}(0))^{k_j} = 1,$$

(see [1] p. 203).

Now assume $S * K_M'^m = K_M'^m$. Then, form theorem 2 one has $S^t * K_M'^m = K_M'^m$, i.e. there exists an $m \times m$ matrix $F = (F_{ij})$, $F_{ij} \in K_M'$, such that $S^t * F = I$. Thus with $\tau_\xi \check{\Psi}_j = (\tau_\xi \check{\psi}_j, 0,0,\ldots,0)^t$, one has

$$
\begin{aligned}
(4) \qquad \psi_j(\xi) &= <\delta, \tau_\xi \check{\psi}_j> = <I, \tau_\xi \check{\Psi}_j>; \\
&= <F(\check{S}^t * \check{F}), F(\tau_\xi \check{\Psi}_j)>; \\
&= <F(\check{F}), F(\check{S}). F(\tau_\xi \check{\Psi}_j)>; \\
&= <F(\check{F})F(I), F(\check{S}). F(\tau_\xi \check{\Psi}_j)>; \\
&= <F(\check{F})(\mathrm{adj}(F\check{F}))\mathrm{adj}(F(\check{S}^t)), F(\check{S})F(\tau_\xi \check{\Psi}_j)>; \\
&= <\det(F(\check{F})). F(I)\mathrm{adj}(F(\check{S}^t)), F(\check{S})F(\tau_\xi \check{\Psi}_j)>; \\
&= <\mathrm{adj}(F(\check{S}^t))(\det(F(\check{F}))F(I)), F(\check{S}). F(\tau_\xi \check{\Psi}_j)>; \\
&= <\det(F(\check{F}))F(I), \mathrm{adj}(F(\check{S}))F(\check{S})). F(\tau_\xi \check{\Psi}_j)>; \\
&= <\det(F(\check{F}))F(I), \det(F(\check{S})). F(\tau_\xi \check{\Psi}_j)>.
\end{aligned}
$$

Recall that $F(F) = \dfrac{1}{\det(F(S^t))} \cdot \mathrm{adj}(F(S^t))$, and in all of the above steps the entries of $F(\check{F})$ are considered as meromorphic functions. Hence adj $(F(\check{F}))$ and $\det(F(\check{F}))$ are well defined. Moreover, the duality brackets in all of the above steps are well defined. For example, in the last duality bracket, since $\det(F\check{S}))$ is not identically zero it follows that det $(F\check{F}))$ is equal to $\dfrac{1}{\det(F(\check{S}))}$, which is a continuous linear functional on the subspace $\det(F(\check{S})).\ F(K_M)$ of $F(K_M)$.

From (4) it follows that there exists a positive integer k and a positive constant A_k such that

$$(5)\qquad \begin{aligned}|\phi_j(\xi)| &\le A_k\omega_k(\det(F(\check{S})).\ F(\tau_\xi\Psi_j))\\ &\le A_k\omega_k(\det(F(S^t)F(\tau_\xi\phi_j)).\end{aligned}$$

Inequality (5) corresponds to the first estimate in inequality (12) of [1]. By repeating the estimates (12), (13), (14) and (15) of [1], it follows that $|\phi_j(\xi)|\le e^{-j}$ for large j. Hence

$$(6)\qquad 1=|\hat{\phi}_j(\xi_j)|\le\int|\phi_j(\xi)|d\xi\le 4^n e^{-j}.$$

As j goes to infinity, inequality (6) gives the contradiction which completes the proof of the theorem.

REMARK 2. The question which has been addressed in this note appeared also in [2], where systems of convolution equations in D'_ω were studied, D'_ω is the space of Beurling generalized distributions. In that case $S=(S_{ij})$, $S_{ij}\in E'_\omega$ the space of convolution operators in D'_ω. We remark that theorems 2 and 3 of this note remain valid if the space K'_M is replaced by D'_ω. To prove the analogue of theorem 3, let K_j be a sequence of compact subsets of R^{mn} such that K_j is contained in the interior of K_{j+1}, and the union of all the K_j's is R^{mn}. Since $S*D^m_\omega$ is the inductive limit of the subspaces $S*D^m_\omega(K_j)$, it suffices to show that the map $S*\phi\to\phi$ from $S*D^m_\omega(K_j)$ into $D^m_\omega(K_j)$ is continuous. This could be carried out exactly like the proof of theorem 3. To prove the analogue of theorem 2, i. e. to show that if $S*D^{m'}_\omega=D^{m'}_\omega$ then $\det(F(S^t))$ is slowly decreasing (see definition 2.1 of [2]), we assume the contrary. Let the sequence (φ_j) be as in the proof of the implication (c)=>(a) of theorem 2.1 of [2]. We proceed as in the set of equalities (4), then we establish the following inequality which corresponds to inequality (5).

$$(7)\qquad \begin{aligned}|\varphi_j(x)| &\le A_k\rho_k(\det(F(S^t))F(\tau_x\varphi_j))\\ &\le A_k\rho_k(\tau_{-x}(\det(S^t*\varphi_j)),\end{aligned}$$

where k is a positive integer and A_k is some positive constant. Inequaltiy (7) corresponds to inequality (13) of [2]. We proceed as in the proof of the implication (c)=>(a) of theorem 2.1 of [2]. The contradiction which comes out completes the proof of the assertion.

References

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Department of Mathematics
Kuwait University
P. O. Box 5969
13060 Safat, KUWAIT

Current addres:
Department of Mathematics
Jordan Univ. of Sci. & Tech.
Irbid-JORDAN.