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Remarks on the formula for the curvature

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To Professor Noboru Tanaka on his sixtieth birthday

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Let $\omega^1, \ldots, \omega^n$ be a base of 1-forms on a manifold M. Then a Riemann metric, say g, on M has an expression

(1) $$g = g_{jk}\omega^j\omega^k.$$

Where $g = (g_{jk})$ is a $n \times n$ matrix valued function on M. We denote by $<X, Y>_g$ the inner product of tangent vectors X, Y with a common source. We set

(2) $$\xi = \omega(X), \quad \eta = \omega(Y)$$

where ω denotes the $\boldsymbol{R}^n$-valued 1-form $(\omega^1, \ldots, \omega^n)$. We set $<\xi, \eta>_g = g_{jk}\xi^j\eta^k$ so that

(3) $$<X, Y>_g = <\xi, \eta>_g.$$

Write

(4) $$d\omega^j = \frac{1}{2}\beta^j_{kl}\omega^k \wedge \omega^l, \quad \beta^j_{kl} + \beta^j_{lk} = 0.$$

Define a linear map $\beta : \boldsymbol{R}^n \to \mathrm{Hom}(\boldsymbol{R}^n, \boldsymbol{R}^n)$ by

(5) $$(\beta(\xi)\eta)^j = \beta^j_{kl}\xi^k\eta^l.$$

Actually β should be regarded as a map of M into $\mathrm{Hom}(\boldsymbol{R}^n, \mathrm{Hom}(\boldsymbol{R}^n, \boldsymbol{R}^n))$. Then the formula (4) can be rewritten as

(6) $$(d\omega)(X, Y) = \beta(\xi)\eta.$$

We wrote down in [3] the formula for the sectional curvature of g which is expressed by means of g_{jk} and β. In this paper we write down the formula for the curvature tensor. We then write down the O'Neill's formula [4] for the submersion using only g and β.

When we set

(7) $$R(X, Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]},$$

the curvature tensor is given by

(8) $$K(X, Y, U, V)=<R(X, Y)V, U>_g.$$

The sectional curvature of the plane generated by X, Y is

(9) $$(\|X\|_g^2\|Y\|_g^2-<X, Y>_g^2)^{-1}<R(X, Y)Y, X>_g.$$

Now $K(X, Y)_g=<R(X, Y)Y, X>_g$ is expressed by g, β as follows:

(10) $$K(X, Y)_g=K_0(\xi, \eta)_g+K_1(\xi, \eta)_g+K_2(\xi, \eta)_g,$$

where

(11) $$\begin{aligned}K_0(\xi, \eta)_g=&\frac{1}{4}\|\beta^*(\xi)\eta+\beta^*(\eta)\xi\|_g^2-<\beta^*(\xi)\xi, \beta^*(\eta)\eta>_g\\ &-\frac{3}{4}\|\beta(\xi)\eta\|_g^2+\frac{1}{2}<\beta^*(\eta)\xi-\beta^*(\xi)\eta, \beta(\xi)\eta>_g.\end{aligned}$$

Where $\beta^*(\xi)$ denotes the adjoint of $\beta(\xi)$ with respect to the metrc g of $\boldsymbol{R}^n$.

To write down K_1 we define for each x in M and ξ a linear map $L_g(\xi): \boldsymbol{R}^n\to\boldsymbol{R}^n$ by

(12) $$<L_g(\xi)\eta, \gamma>_{g(x)}=\xi^j(E_jg_{kl}(x))\eta^k\gamma^l.$$

Where $E_1, \dots, E_n$ denote the base of tangent vectors dual to $\omega^1, \dots, \omega^n$. We also set for a linear map $L: \boldsymbol{R}^n\to\mathrm{Hom}(\boldsymbol{R}^n, \boldsymbol{R}^n)$

(13) $$\hat{L}(\xi)\eta=L(\eta)\xi,$$

and define

(14) $$2C(\xi)=L_g(\xi)+\hat{L}_g(\xi)-(\hat{L}_g(\xi))^*.$$

Then

$$K_1(\xi, \eta)_g=K_1^1+K_1^2+K_1^3,$$

where

(15) $$\begin{aligned}K_1^1=&\frac{3}{2}<L_g(\eta)\xi-L_g(\xi)\eta, \beta(\xi)\eta>_g,\\ K_1^2=&<\beta^*(\xi)\eta+\beta^*(\eta)\xi, C(\xi)\eta>_g\\ &-<\beta^*(\xi)\xi, C(\eta)\eta>_g-<\beta^*(\eta)\eta, C(\xi)\xi>_g,\\ K_1^3=&<[X, \beta](\eta)\xi, \eta>_g+<[Y, \beta](\xi)\eta, \xi>.\end{aligned}$$

To write down K_2 we set

(16) $$\Delta(\eta, \eta)=\xi^j\eta^kE_kE_j.$$

The above is regarded as a second order differential operator. Then

$$K_2(\xi, \eta)_g = K_2^1 + K_2^2,$$

where

$$\begin{aligned} K_2^1 &= \frac{1}{2}(<(\Delta((\xi, \eta)g + \Delta(\eta, \xi)g)\xi, \eta> \\ &\quad -<(<(\Delta(\xi, \xi)g)\eta, \eta> - <(\Delta(\eta, \eta)g)\xi, \xi>), \\ K_2^2 &= \|C(\xi)\eta\|_2^2 - <C(\xi)\xi, C(\eta)\eta>_g. \end{aligned} \tag{17}$$

It is of some interest to note that the formula has a pattern. Namely, for a pair of bilinear forms $P(\xi, \eta)$, $Q(\xi, \eta)$ valued in a finite dimensional vector space, say E, with a metric g, we set

$$\begin{aligned} K(P, Q;\ \xi, \eta) = &<P(\xi, \eta), Q(\xi, \eta)>_g \\ &-\frac{1}{2}<P(\xi, \xi), Q(\eta, \eta)>_g \\ &-\frac{1}{2}<P(\eta, \eta), Q(\xi, \xi)>_g. \end{aligned} \tag{18}$$

Then $K(\xi, \eta)_g$ is a sum of terms of the form $K(P, Q;\ \xi, \eta)$. To see this let E be $\boldsymbol{R}^n$ with the metric g and set

$$P_0(\xi, \eta) = Q_0(\xi, \eta) = \frac{1}{2}(\beta^*(\xi)\eta + \beta^*(\eta)\xi), \tag{19}$$

$$P_1(\xi, \eta) = -\frac{3}{4}Q_1(\xi, \eta) = \beta(\xi)\eta, \tag{20}$$

$$P_2(\xi, \eta) = \beta^*(\eta)\xi - \beta^*(\xi)\eta),\ \ Q(\xi, \eta) = \frac{1}{2}\beta(\xi)\eta, \tag{21}$$

considered as E valued. Then we see clearly

$$K_0(\xi, \eta)_g = \sum_{j=0}^{2} K(P_j, Q_j;\ \xi, \eta). \tag{22}$$

Similarly, we consider E-valued bilinear forms

$$P_3(\xi, \eta) = L_g(\eta)\xi - L_g(\xi)\eta,\ \ Q_3(\xi)\eta = \frac{3}{2}\beta(\xi)\eta, \tag{23}$$

$$P_4(\xi)\eta = \beta^*(\xi)\eta + \beta^*(\eta)\xi,\ \ Q_4(\xi, \eta) = C(\xi)\eta, \tag{24}$$

$$P_5(\xi, \eta) = [E_j, \beta](\eta)\xi,\ \ Q_5(\xi, \eta) = \xi^j\eta - \eta^j\xi. \tag{25}$$

Noting that $P_5(\xi, \eta)$ is a skew-symmetric form we find that

(26) $K_1(\xi, \eta)_g = \sum_{j=3}^{5} K(P_j, Q_j;\ \xi, \eta).$

To write down K_2 we consider E-valued forms

(27) $P_6(\xi, \eta) = Q_6(\xi, \eta) = C(\xi)\eta.$

We denote by E_1 the vector space of symmetric $n \times n$ matrixes with the standard metric, and consider E_1 valued form

(28) $P_7(\xi, \eta) = \Delta(\xi, \eta)g + \Delta(\eta, \xi)g,\ Q_7(\xi, \eta) = \frac{1}{4}(\xi \otimes \eta + \eta \otimes \xi).$

Then

(29) $K_2(\xi, \eta)_g = \sum_{j=6}^{7} K(P_j, Q_j;\ \xi, \eta).$

Therefore we have the formula:

(30) $K(\xi, \eta)_g = \sum_{j=0}^{7} K(P_j, Q_j;\ \xi, \eta).$

We set for $X_j (j = 1, 2, 3, 4)$

(31) $K(X_1, X_2, X_3, X_4)_g = K(\xi_1, \xi_2, \xi_3, \xi_4)_g.$

Then the muti-linear form $K(\xi_1, \xi_2, \xi_3, \xi_4)_g$ is characterized by the following 4 conditions (cf. Kobayashi-Nomizu [2], p. 198-199)

(32.1) $K(X, Y)_g = K(\xi, \eta, \xi, \eta)_g,$

(32.2) skew-symmmetric in (θ_1, θ_2) as well as in (θ_3, θ_4),

(32.3) $K(\theta_1, \theta_2, \theta_3, \theta_4)_g + K(\theta_1, \theta_3, \theta_4, \theta_2)_g + K(\theta_1, \theta_4, \theta_2, \theta_3)_g = 0.$

It then follows that

(32.4) $K(\theta_1, \theta_2, \theta_3, \theta_4)_g = K(\theta_3, \theta_4, \theta_1, \theta_2)_g.$

The proof of the uniqueness given in the above book actually tells us how to construct $K(\theta_1, \theta_2, \theta_3, \theta_4)_g$ out of $K(\xi, \eta)_g$, where

(33) $K(X, Y)_g = K(\xi, \eta)_g.$

In fact, applying the distribution law to $K(\theta_1, \theta_2 + \theta_4, \theta_1, \theta_2 + \theta_4)_g$, we find by (32. 4) that

(34) $2K(\theta_1, \theta_2, \theta_1, \theta_4)_g = K(\theta_1, \theta_2 + \theta_4)_g - K(\theta_1, \theta_2)_g - K(\theta_1, \theta_4)_g.$

Using $K(\theta_1 + \theta_3, \theta_2, \theta_1 + \theta_3, \theta_4)$ similarly, we find that

$$(35)^{34}\quad \begin{aligned} &K(\theta_1, \theta_2, \theta_3, \theta_4)_g + K(\theta_1, \theta_4, \theta_3, \theta_2)_g \\ &= K(\theta_1+\theta_3, \theta_2, \theta_1+\theta_3, \theta_4)_g - K(\theta_1, \theta_2, \theta_1, \theta_4)_g - K(\theta_3, \theta_2, \theta_3, \theta_4)_g. \end{aligned}$$

Write down $(35)_{34}-(35)_{43}$. We then find by the Jacobi's identity (32.3) that

$$(36)\quad \begin{aligned} 3K(\theta_1, \theta_2, \theta_3, \theta_4)_g = &K(\theta_1+\theta_3, \theta_2, \theta_1+\theta_3, \theta_4)_g - K(\theta_1, \theta_2, \theta_1, \theta_4)_g \\ &- K(\theta_3, \theta_2, \theta_3, \theta_4)_g - K(\theta_1+\theta_4, \theta_2, \theta_1+\theta_4, \theta_3)_g \\ &+ K(\theta_1, \theta_2, \theta_1, \theta_3) + K(\theta_4, \theta_2, \theta_4, \theta_3)_g. \end{aligned}$$

Therefore

$$(37)\quad \begin{aligned} K(\theta_1, \theta_2, \theta_3, \theta_4) = \frac{1}{6}(&K(\theta_1+\theta_3, \theta_2+\theta_4) - K(\theta_1+\theta_4, \theta_2+\theta_3) \\ &+ K(\theta_1, \theta_2+\theta_3) - K(\theta_1, \theta_2+\theta_4) - K(\theta_2, \theta_1+\theta_3) \\ &+ K(\theta_2, \theta_1+\theta_4) + K(\theta_3, \theta_1+\theta_4) - K(\theta_3, \theta_2+\theta_4) \\ &- K(\theta_4, \theta_1+\theta_3) + K(\theta_4, \theta_2+\theta_3) - K(\theta_1, \theta_3) \\ &+ K(\theta_1, \theta_4) + K(\theta_2, \theta_3) - K(\theta_2, \theta_4)). \end{aligned}$$

The referee informed us that the formula (37) is in [1] p. 16.

When $K(X, Y)_g$ is expressed as the sum of $K(P_j, Q_j; \xi, \eta)$ as in (30), $K(\theta_1, \theta_2, \theta_3, \theta_4)_g$ is also expressed as the sum of $K(P, Q; \theta_1, \theta_2, \theta_3, \theta_4)$ defined as in (34), (36) in terms of $K(P, Q; \xi, \eta)$ in stead of $K(\xi, \eta)$. By calculation we find by (34) and (18) that

$$(38)\quad \begin{aligned} 2K(P, Q; \theta_1, \theta_2, \theta_1, \theta_4) = &<P(\theta_1, \theta_2), Q(\theta_1, \theta_4)> \\ &+ <P(\theta_1, \theta_4), Q(\theta_1, \theta_2)> \\ &- \frac{1}{2}<P(\theta_1, \theta_1), Q(\theta_2, \theta_4) + Q(\theta_4, \theta_2)> \\ &- \frac{1}{2}<P(\theta_2, \theta_4) + P(\theta_4, \theta_2), Q(\theta_1, \theta_1)>. \end{aligned}$$

We calculate $K(P, Q; \theta_1, \theta_2, \theta_3, \theta_4)$ when $P=Q$ and skew-symmetric in (ξ, η). By the above we find that

$$(39)\quad K(P, P; \theta_1, \theta_1, \theta_3, \theta_4) = <P(\theta_1, \theta_2), P(\theta_1, \theta_4)>,$$

hence

$$\begin{aligned} &K(P, P; \theta_1+\theta_3, \theta_2, \theta_1+\theta_3, \theta_4) - K(P, P; \theta_1, \theta_2, \theta_1, \theta_4) \\ &- K(P, P; \theta_3, \theta_2, \theta_3, \theta_4) \\ &= <P(\theta_1, \theta_2), P(\theta_3, \theta_4)> - <P(\theta_1, \theta_4), P(\theta_2, \theta_3)>. \end{aligned}$$

Therefore we find by (36)

(40) PROPOSITION. *When $P(\xi, \eta)$ is skew-symmetric in ξ, η,*

$$K(P, P\,;\ \theta_1, \theta_2, \theta_3, \theta_4)=\frac{2}{3}<P(\theta_1, \theta_2),\ P(\theta_3, \theta_4)>$$
$$+\frac{1}{3}<P(\theta_1, \theta_3),\ P(\theta_2, \theta_4)>-\frac{1}{3}<P(\theta_1, \theta_4),\ P(\theta_2, \theta_3)>.$$

Since $K(P, Q: \theta_1, \dots, \theta_4)$ is symmetric and bi-linear in P, Q, it follows by the above that

(41) PROPOSITION. *When $P(\xi, \eta)$ and $Q(\xi, \eta)$ are skew-symmetric in ξ, η,*

$$K(P, Q\,;\ \theta_1, \theta_2, \theta_3, \theta_4)$$
$$=\frac{1}{3}(<P(\theta_1, \theta_2),\ Q(\theta_3, \theta_4)>+<Q(\theta_1, \theta_2),\ P(\theta_3, \theta_4)>)$$
$$+\frac{1}{6}(<P(\theta_1, \theta_3),\ Q(\theta_2, \theta_4)>+<Q(\theta_1, \theta_3),\ P(\theta_2, \theta_4)>)$$
$$-\frac{1}{6}(<P(\theta_1, \theta_4),\ Q(\theta_2, \theta_3)>+<Q(\theta_1, \theta_4),\ Q(\theta_2, \theta_3)>).$$

We next work out the case when P and Q are symmetric. We set first

$$\begin{aligned}(42)\qquad & L(P, Q\,;\ \theta_1, \theta_2\,;\ \theta_3, \theta_4)=K(P, Q\,;\ \theta_1+\theta_2, \theta_3+\theta_4)\\ & -K(P, Q\,;\ \theta_1, \theta_3+\theta_4)-K(P, Q\,;\ \theta_2, \theta_3+\theta_4)-K(P, Q\,;\ \theta_1+\theta_2, \theta_3)\\ & -K(P, Q\,;\ \theta_1+\theta_2, \theta_4)+K(P, Q\,;\ \theta_1, \theta_3)+K(P, Q\,;\ \theta_2, \theta_3)\\ & +K(P, Q\,;\ \theta_1, \theta_4)+K(P, Q\,;\ \theta_2, \theta_4).\end{aligned}$$

We then find by direct calculation using (18) that, when $P(\xi, \eta)$ and $Q(\xi, \eta)$ are symmetric in ξ, η,

$$\begin{aligned}(43)\qquad & L(P, Q\,;\ \theta_1, \theta_2\,;\ \theta_3, \theta_4)=<P(\theta_1, \theta_3),\ Q(\theta_2, \theta_4)>\\ & +<P(\theta_1, \theta_4),\ Q(\theta_2, \theta_3)>\\ & +<P(\theta_2, \theta_3),\ Q(\theta_1, \theta_4)>+<P(\theta_2, \theta_4),\ Q(\theta_1, \theta_3)>\\ & -2<P(\theta_1, \theta_2),\ Q(\theta_3, \theta_4)>-2<P(\theta_3, \theta_4),\ Q(\theta_1, \theta_2)>.\end{aligned}$$

Note by (37) that

$$\begin{aligned}(44)\qquad 6K(P, Q\,;\ \theta_1, \theta_2, \theta_3, \theta_4)=\ & L(P, Q\,;\ \theta_1, \theta_3\,;\ \theta_2, \theta_4)\\ & -L(P, Q\,;\ \theta_1, \theta_4\,;\ \theta_2, \theta_3).\end{aligned}$$

It then follows by calculation the following:

(45) PROPOSITION. *When $P(\xi, \eta)$ and $Q(\xi, \eta)$ are symmetric in ξ, η,*

$$K(P, Q\,;\ \theta_1, \theta_2, \theta_3, \theta_4)=\frac{1}{2}(<P(\theta_1, \theta_4),\ Q(\theta_2, \theta_3)>$$
$$+<P(\theta_2, \theta_3),\ Q(\theta_1, \theta_4)>-<P(\theta_1, \theta_3),\ Q(\theta_2, \theta_4)>$$
$$-<P(\theta_2, \theta_4),\ Q(\theta_1, \theta_3)>).$$

Therefore we have

(46) PROPOSITION.

$$K(\theta_1, \theta_2, \theta_3, \theta_4)_g = \sum_{j=0}^{7} K(P_j, Q_j\,;\, \theta_1, \theta_2, \theta_3, \theta_4)_g,$$

where, when P_j *and* Q_j *are skew symmetric,* $K(P_j, Q_j\,;\, \theta_1, \theta_2, \theta_3, \theta_4)_g$ *is defined by* (41) *and, when* P_j *and* Q_j *are symmetric,* $K(P_j, Q_j\,;\, \theta_1, \theta_2, \theta_3, \theta_4)_g$ *is defined by* (45).

We now consider a submersion of a Riemann manifold M with a metric g to a Riemann manifold N with a metric g_N. Let X', Y' be tangent vectors to N at a point. To calculate $K(X', Y')_{g_N}$, pick horizontal lifts X, Y of X', Y'. Then by the O'Neill's formula $K(X'Y')_{g_N}$ is the sum of $K(X, Y)_g$ and $\frac{3}{4}\|p_v[X, Y]\|_g^2$, where we extend X, Y to vector fields and p_v denotes the projection to the vertical part.

We see easily

(47) $$[X, Y] = (X\eta^j - Y\xi^j)E_j + \xi^j\eta^k[E_j, E_k].$$

Note that

(48) $$\omega(\xi^j\eta^k[\mathrm{E}_j, \mathrm{E}_k]) = -\beta(\xi)\eta,$$

because $d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y])$. Therefore

(49) $$\omega([X, Y]) = (X\eta - Y\xi) - \beta(\xi)\eta.$$

Let V be a vertical vector field with $\mu = \omega(V)$. Then we see by the above

(50) $$<[X, Y], V>_g = <g(X\eta - Y\xi), \mu> - <\beta(\xi)\eta, \mu>_g.$$

When we set

$$\tilde{\xi} = g\xi,$$

we find that

$$<g(X\eta - Y\xi), \mu> = <X\tilde{\eta} - Y\tilde{\xi}, \mu> + <(Yg)\xi - (Xg)\eta, \mu>.$$

Since X, Y are horizontal, $<\tilde{\eta}, \mu> = <\tilde{\xi}, \mu> = 0$. Hence

(51) $$\begin{aligned} <[X, Y], V>_g = {} & <L_g(\eta)\xi - L_g(\xi)\eta - \beta(\xi)\eta, \mu>_g \\ & + <\tilde{\xi}, Y\mu> - <\tilde{\eta}, X\mu>. \end{aligned}$$

We then find easily an expression of the O'Neill's formula by the above formula. The term is also of the form $K(P, Q\,;\, \xi, \eta)$.

Reference

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